ENTROPY MEASURE FOR SYSTEMS WITH TOPOLOGICAL CONSTRAINTS

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• RT-1 plasma experiment
- RT-1 produces a laboratory magnetosphere [1]
- A similar device is LDX at MIT
Magnetic Field: $100G \sim 1000G$

$\beta \sim 100\%$ [2]

<table>
<thead>
<tr>
<th>Vacuum vessel</th>
<th>Inner diameter</th>
<th>1.00 m</th>
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<tbody>
<tr>
<td></td>
<td>Height</td>
<td>0.56 m</td>
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<tr>
<td>Floating coil</td>
<td>size</td>
<td>250mm</td>
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<tr>
<td></td>
<td>(cross section)</td>
<td>W=195mm, h=150mm</td>
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<tr>
<td></td>
<td>Coil wire</td>
<td>2160 turn (Bi=2223)</td>
</tr>
<tr>
<td></td>
<td>Coil current</td>
<td>250kAT</td>
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<tr>
<td></td>
<td>Weight</td>
<td>110kg</td>
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<tr>
<td></td>
<td>Operating temperature</td>
<td>20~32K</td>
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<tr>
<td>Lifting magnet</td>
<td>Coil wire</td>
<td>68 turn (copper)</td>
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<td></td>
<td>Coil current</td>
<td>88kAT</td>
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<tr>
<td></td>
<td>Dynamic Range</td>
<td>$&lt;10Hz$ (feedback controlled)</td>
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<td>RF heating (1)</td>
<td>frequency</td>
<td>8.2GHz (klystron)</td>
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<tr>
<td></td>
<td>power</td>
<td>100kW (1sec Pulse)</td>
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<tr>
<td>RF heating (2)</td>
<td>frequency</td>
<td>2.45GHz (magnetron)</td>
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<td></td>
<td>power</td>
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<tr>
<td>Refrigerator</td>
<td>power</td>
<td>50 W (at 20K)</td>
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<tr>
<td></td>
<td>maximum flow</td>
<td>2g/sec</td>
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First Plasma Experiment on RT-1
ENTROPY MEASURE FOR SYSTEMS WITH TOPOLOGICAL CONSTRAINTS

I. Integrable constraints

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INTRODUCTION

- Magnetospheres are ubiquitous in the universe and represent a natural confinement device.

- Creation of density gradients seems to be violating the entropy principle: the magnetic field cannot perform mechanical work.

- Spontaneous confinement is achieved by self-organization of charged particles through the process of inward diffusion.
INTRODUCTION

THE POINT DIPOLE

- $B$
- $\psi$
- $\vartheta$
- $l$
- $\zeta$

Diagram showing the field lines and curves for the point dipole model.
THE POINT DIPOLE

MAGNETIC FIELD

\[ B = \nabla \psi \times \nabla \vartheta \]

FLUX FUNCTION

\[ \psi(r, z) = \frac{r^2}{(r^2 + z^2)^{3/2}} \]

LENGTH OF A FIELD LINE

\[
l(r, z) = \frac{1}{2\psi} \left[ \frac{1}{\sqrt{3}} \log \left( \sqrt{3} \left( \sqrt{1 - (r\psi)^{2/3}} + \sqrt{4 - 3(r\psi)^{2/3}} \right) + \sqrt{1 - (r\psi)^{2/3}} \sqrt{4 - 3(r\psi)^{2/3}} \right) \right] \]
A common understanding is that the particle number per flux-tube volume tends to be homogenized [7,8].

• In magnetospheric plasmas the key normal transport mechanism is diffusion occurring due to $E \times B$ drift:

$$\dot{X}_\perp = \frac{\delta E_\perp}{B} = - \frac{1}{rB} \frac{\partial \delta \varphi}{\partial \vartheta}$$
DIFFUSION ON A FOLIATED PHASE SPACE

- Conservation of adiabatic invariants **foliate phase space**: the motion of particles is constrained on leaves [9-10].

- Random fluctuations in the electromagnetic field break the **weakest topological constraints**. The result is a random process among leaves.

- Entropy shall be maximized on the macroscopic **symplectic leaf**.

- A diffusion operator must be formulated on the corresponding **invariant measure**.

---

IN Variant Measure

The $n$-dimensional **measure** $\text{vol}^n = dx^1 \wedge \cdots \wedge dx^n$ is **invariant** under the flow generated by the vector field $X = \dot{x}^i \partial_i$ when:

$$\mathcal{L}_X \text{vol}^n = \text{div}(X)\text{vol}^n = 0$$

Thanks to Liouville’s theorem, **Hamiltonian systems always have an invariant measure**, the phase space $dp^1 \wedge dq^1 \wedge \cdots \wedge dp^{n/2} \wedge dq^{n/2}$.

The invariant measure is required for a consistent definition of **entropy** [11-12]:

$$S = - \int \text{flogf} \text{vol}^n$$

The invariant measure is also a necessary condition for the **ergodic ansatz** [13].

• Conservation of the magnetic moment implies that we can separate the canonical pair \((\vartheta_c, \mu)\):

\[
dx \wedge dy \wedge dz \wedge dp_x \wedge dp_y \wedge dp_z = m^2 dl \wedge dv_\parallel \wedge d\vartheta \wedge d\psi \wedge [d\vartheta_c \wedge d\mu]
\]

• Dynamical variables are reduced to four: \((l, v_\parallel, \vartheta, \psi)\).

• We consider the guiding center equations of motion:

\[
\begin{align*}
\dot{v}_\parallel &= - \frac{1}{m} (\mu B + e\varphi)_l + v_\parallel v_{E\times B} \cdot k \\
v &= v_\parallel + v_{E\times B} + v_{\nabla B} + v_k
\end{align*}
\]

• The flow \(X = (i, \dot{i}, \dot{\vartheta}, \dot{\psi})\) is measure preserving even if not written with canonically paired variables:

\[
\mathcal{L}_X \text{vol}^4 = \text{div}(X)\text{vol}^4 = 0
\]

Here \(\text{vol}^4 = dl \wedge dv_\parallel \wedge d\vartheta \wedge d\psi\).
CONSTRUCTION OF CANONICAL VARIABLES

- We look for the nondegenerate closed two-form $\omega = d\lambda$ such that:

$$i_x \omega = -dH$$

with $H = \mu B + \frac{m}{2} v_{\parallel}^2 + e\varphi$ the Hamiltonian function. Setting $q = -\partial_l \cdot \partial_\psi$:

$$\omega = dv_{\parallel} \wedge dl + d\psi \wedge (\vartheta + qv_{\parallel})$$

- The symplectic potential is [14]:

$$\lambda = v_{\parallel} dl + \psi d\eta$$

with $\eta = \vartheta + qv_{\parallel}$.

FOKKER-PLANCK EQUATION ON A FOLIATED PHASE SPACE

• In order to build a statistical mechanics on the foliation, we consider the following hypothesis:

1. Charge neutrality $\langle E \rangle = 0$ which implies $\langle v_{E \times B} \rangle = 0$.

2. Ergodic Hypothesis on the symplectic leaf:

$$\mathcal{E} = -d\varphi = \frac{m}{e} D_{\parallel}^{1/2} \Gamma_{\parallel} dl + D_{\perp}^{1/2} \Gamma_{\perp} d\theta + D_{\vartheta}^{1/2} \Gamma_{\vartheta} d\vartheta$$

with $dW = \Gamma dt$ a Wiener process.

• Under this assumptions, we can convert the guiding center equations of motion in a system of SDEs that are associated to a FPE.

• The $E \times B$ drift velocity is then written as

$$dX_{\perp} = -\frac{1}{rB} \frac{\partial \delta \varphi}{\partial \vartheta} dt = \frac{D_{\perp}^{1/2}}{rB} dW_{\perp}$$

Here $dW_{\perp} = \Gamma_{\perp} dt$ is the normal component of the Wiener process.
Depending on how we define the **stochastic integral**, the stochastic equation of motion has a different solution:

\[
\int_{t_0}^{t} dX_t = ms-\lim_{n \to \infty} \sum_{i=1}^{n} \frac{D_{1/2}}{rB} (t_{i-1} + \alpha \Delta t_i)[W_i - W_{i-1}]
\]

with \(\alpha \in [0,1]\), \(W_i = W(t_i)\), \(\Delta t_i = t_i - t_{i-1}\) and \(ms-\lim\) the limit taken in \(L^2\).

- The case \(\alpha = 0\) is known as the **Ito integral**.
- The case \(\alpha = 1/2\) is known as the **Stratonovich integral**.

There are two important consequences of the definition:

- Ordinary calculus does not hold. This affects the equations of motion in the new frame, which turn out to be dependent on \(\alpha\).

- The FPE (Fokker-Planck Equation) corresponding to a given SDE (Stochastic Differential Equation) depends on \(\alpha\).
STOCHASTIC INTEGRATION

First consequence [15]:

\[ df(X,t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \left( \frac{1}{2} - \alpha \right) G^2 \frac{\partial^2 f}{\partial x^2} dt \]

\[ (df)_I = (df)_x - \alpha \frac{\partial f}{\partial x} \frac{\partial G}{\partial x} G dt \]

Second consequence [15]:

\[ df(X^i,t) = F^i dt + J^{ik} dW_k \]

\[ \frac{\partial P}{\partial t} = \frac{\partial}{\partial x^i} \left( -F^i + \frac{1}{2} \frac{\partial}{\partial x^j} J^{jk} J^{ik} - \alpha J^{jkl} \frac{\partial J^{ik}}{\partial x^j} \right) P \]

where \( P \) is the probability distribution of \( X \).

(a) Converges to the Wiener process with Itō’s definition of the stochastic integral.

(b) Converges to the Wiener process with Stratonovich’s definition of the stochastic integral.

(c) \( \delta \phi \propto \Gamma = dW/dt \)
\[ X_{\perp}(t + \Delta t) - X_{\perp}(t) = \int_t^{t+\Delta t} \frac{dW}{rB} = \frac{W_{t+\Delta t} - W_t}{rB(\Psi(t))} \rightarrow \alpha = 0 \]
FOKKER-PLANCK EQUATION ON A FOLIATED PHASE SPACE

\[
\begin{align*}
    dL &= \left\{ v_\parallel + \mathcal{C}_l + \left( \frac{1}{2} - \alpha \right) D_\perp \left[ (\partial_\psi + q \partial_l)q + q (\partial_\psi + q \partial_l) \ln(rB) \right] \right\} dt + q D_\perp^{1/2} dW_\perp \\
    dV_\parallel &= -\left( \frac{\mu}{m} \frac{\partial B}{\partial l} + \gamma v_\parallel - \mathcal{C}_\parallel \right) dt + D_\parallel^{1/2} dW_\parallel - D_\perp^{1/2} v_\parallel q_l dW_\perp \\
    d\Theta &= \left[ \frac{\mu}{e} (\partial_\psi + q \partial_l)B + \mathcal{C}_\vartheta - \frac{m}{e} v_\parallel^2 q_l \right] dt - D_\vartheta^{1/2} dW_\vartheta - \frac{m}{e} q D_\parallel^{1/2} dW_\parallel \\
    d\Psi &= \left[ D_\perp \left( \frac{1}{2} - \alpha \right) (\partial_\psi + q \partial_l) \ln(rB) + \mathcal{C}_\psi \right] dt + D_\perp^{1/2} dW_\perp \\
    \frac{\partial P}{\partial t} &= -\frac{\partial}{\partial l} \left\{ v_\parallel + \mathcal{C}_l + \left( \frac{1}{2} - \alpha \right) D_\perp \left[ (\partial_\psi + q \partial_l)q + q (\partial_\psi + q \partial_l) \ln(rB) \right] \right\} P \\
        &+ \frac{\partial}{\partial v_\parallel} \left[ \left( \frac{\mu}{m} \frac{\partial B}{\partial l} + \gamma v_\parallel - \mathcal{C}_\parallel \right) P \right] - \frac{\partial}{\partial \vartheta} \left[ \frac{\mu}{e} (\partial_\psi + q \partial_l)B + \mathcal{C}_\vartheta - \frac{m}{e} v_\parallel^2 q_l \right] P \\
        &- \frac{\partial}{\partial \psi} \left[ D_\perp \left( \frac{1}{2} - \alpha \right) (\partial_\psi + q \partial_l) \ln(rB) + \mathcal{C}_\psi \right] P + \frac{1}{2} D_\perp \frac{\partial^2}{\partial l^2} (q^2 P) + \frac{1}{2} D_\parallel \frac{\partial^2}{\partial v_\parallel^2} P + \frac{1}{2} D_\vartheta \frac{\partial^2}{\partial \vartheta^2} P + \frac{m^2}{2e^2} D_\parallel \frac{\partial^2}{\partial \vartheta^2} q^2 P \\
        &- \frac{m}{e} D_\parallel \frac{\partial^2}{\partial v_\parallel^2 \partial \vartheta} qP + D_\perp \frac{\partial^2}{\partial l \partial \psi} (qP) - \alpha D_\perp \frac{\partial}{\partial l} \left[ (\partial_\psi + q \partial_l)q \right] P - D_\perp \frac{\partial^2}{\partial l \partial v_\parallel} v_\parallel q_l q_l P + \frac{1}{2} D_\perp \frac{\partial^2}{\partial v_\parallel^2} (v_\parallel q_l)^2 P \\
        &- D_\perp \frac{\partial^2}{\partial \psi \partial v_\parallel} v_\parallel q_l P + \alpha D_\perp \frac{\partial}{\partial v_\parallel} \left[ v_\parallel (q \partial_l + \partial_\psi) q_l - v_\parallel q_l^2 \right] P
\end{align*}
\]

\[ P dl \wedge d\psi \wedge d\vartheta \wedge dv_\parallel \wedge [d\mu \wedge d\vartheta_c] = PB dx \wedge dy \wedge dz \wedge dv_\parallel \wedge [d\mu \wedge d\vartheta_c] \]
CHANGE OF COORDINATES TO THE LABORATORY FRAME

• The creation of the density gradient is the result of the inhomogeneous Jacobian $B$ linking symplectic leaf and laboratory frame [15]:

$$dl \wedge d\psi \wedge d\vartheta \wedge dv_{\parallel} = B \ dx \wedge dy \wedge dz \wedge dv_{\parallel}$$

• The density $\rho(x, y, z)$ in the laboratory frame $(x, y, z)$ is:

$$\rho = Bu = B \int Pdv_{\parallel} \wedge [d\mu \wedge d\vartheta_c]$$

where $u$ is the density on the symplectic leaf.

• As $u$ becomes flat due to inward diffusion, $\rho$ becomes inhomogeneous.

• The magnetic field cannot perform mechanical work but alters the topology of a mechanical system.

SYMPLECTIC LEAF DENSITY $u$

Time Evolution

(a) $\psi_{2.5}$

STEP 1

(b) $\psi_{2.5}$

STEP 2

(c) $\psi_{2.5}$

STEP 3

(d) $\psi_{2.5}$

STEP 4
LABORATORY FRAME DENSITY $\rho$

Time Evolution
PARALLEL TEMPERATURE $T_\parallel$

Time Evolution
PERPENDICULAR TEMPERATURE $T_\perp$

Time Evolution

![Graphs showing time evolution of perpendicular temperature](image-url)
TEMPERATURE ANISOTROPY $T_{\perp}/T_{\parallel}$

Time Evolution
$D_\perp$ and $\rho$

$10^{-2}D_\perp$

$D_\perp$

$10^{-1}D_\perp$

$10D_\perp$
$D_\perp$ and $\rho_M$
$D_{||}$ and $\rho$

- $0.25D_{||}$
- $0.5D_{||}$
- $D_{||}$
- $2D_{||}$
**ENTROPY**

- Correct entropy measure on symplectic leaf [16]:
  \[ \Sigma = \int \sigma dt = - \int P \log P \, dl \wedge d\psi \wedge d\theta \wedge dv_\parallel \wedge [d\mu \wedge d\theta_c] \]

- Wrong entropy measure in laboratory coordinates:
  \[ \tilde{S} = - \int PB \log(PB) \, dx \wedge dy \wedge dz \wedge dv_\parallel \wedge [d\mu \wedge d\theta_c] \]

ENTROPY MEASURE FOR SYSTEMS WITH TOPOLOGICAL CONSTRAINTS

II. Non-integrable constraints

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Non-integrable Constraints

- Microscopic systems arise in Canonical Hamiltonian form.
- Macroscopic systems result from the reduction of negligible degrees of freedom that impart topological constraints.
- In a dipole magnetic field, self-organization is driven by adiabatic invariants.

- The reduction process may destroy the canonical form.
- The statistical mechanics in the presence of non-integrable topological constraints (e.g. a Beltrami field) is not understood.

ENERGY = H

Almost Poisson Algebra
FAILURE OF JACOBI IDENTITY

Almost Hamiltonian Mechanics

Nonholonomic Systems [18-19]

THE PROBLEM OF STATISTICAL MECHANICS

- There are 3 main problems with almost Hamiltonian systems:
  1. Symmetries of the Hamiltonian $H$ do not give rise to conservation laws.
  2. There are no canonical pairs: $\Omega_{(p,q)} dp \land dq \neq 0$.
  3. In general, there is no invariant measure: $\mathcal{L}_x g\text{vol}^n \neq 0 \quad \forall g: \mathcal{M} \to \mathbb{R}$.

- The 2 principal consequences are:
  1. Integration of the equations of motion is difficult [18].
  2. There are no grounds for the ergodic hypothesis [13] and the definition of differential entropy breaks down [11-12,16] with violation of the second law of thermodynamics [16]:

\[
S = -\int f \log f \text{vol}^n
\]

Hamilton’s canonical equations of motion

\[ X = \left( \begin{array}{c} \dot{p} \\ \dot{q} \end{array} \right) = J_c(dH) = \left( \begin{array}{c} -H_q \\ H_p \end{array} \right) \]

Hamilton’s non-canonical equations of motion

\[ X = J(dH) \]

\[ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \quad \forall \; f, g, h \in C^\infty \]

\[ \iff J^{im} J^j_m \partial_i \wedge \partial_j \wedge \partial_k = 0 \]

Almost Hamiltonian equations of motion

\[ X = J(dH) \]

\[ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \neq 0 \quad \forall \; f, g, h \in C^\infty \]

\[ \iff J^{im} J^j_m \partial_i \wedge \partial_j \wedge \partial_k \neq 0 \]
Motion of a charged particle in an e-m field

\[ m \frac{dv}{dt} = q(v \times B + E) \]

Canonical Hamiltonian form

- 6D phase space: \( \omega^2 = dp^i \wedge dx^i = J_c^{-1} \)
- Charged particle Hamiltonian: \( H = \frac{1}{2m} (p - qA)^2 + q\varphi \)
- equations of motion: \( i_x \omega^2 = -dH \) with \( X = (\dot{p}, \dot{x}) \)

Reduction \( m \to 0 \) (non canonical Hamiltonian form)

- 3D phase space: \( \omega^2 = dA_i \wedge dx^i = d\mathcal{A} = \mathcal{B} \)
- Non inertial Hamiltonian: \( H = q\varphi \)
- Equations of motion: \( i_x d\mathcal{A} = -d\varphi \) with \( X \times B + E = 0 \) and \( X = (\dot{x}, \dot{y}, \dot{z}) \)

Reduction \( v_{||} \to 0 \)

\[ X = J(dH) = w \times \nabla H = -\frac{B}{B^2} \times E \]

\[ \text{Jacobian} = \frac{B \cdot \nabla \times B}{B^4} \neq 0 \Rightarrow \text{almost Hamiltonian} \]
To formulate the statistical mechanics of 3D almost Hamiltonian systems, we repair the almost Poisson bracket
\[ \{f, g\} = \langle df, J(dg) \rangle \]
where \( f, g \in C^\infty(\mathcal{M}) \) and \( J \in \Lambda^2 T\mathcal{M} \), by introducing a new degree of freedom and by operating a time reparametrization.

Almost Poisson
\[ X_3 = J(dH) \]

Conf. Poisson (Invariant Measure)
\[ X_4 = \tilde{J}(dH) \]

Poisson
\[ Y_4 = \mathcal{J}(dH) \]

\[ \rho^\infty(x) = \int f^\infty ds \]

\[ f^\infty = g(x, s)P^\infty \]

\[ S = -\int P\log P \]

\[ P^\infty = \frac{1}{Z} e^{-\beta H} \]
**Def 1.** A bivector $J$ defines an **almost Poisson bracket** that is antisymmetric, bilinear, and satisfies the Leibniz condition.

**Def 2.** A bivector $J$ such that $\mathcal{L}_x g \text{vol}^n = 0 \quad \forall H$ for some Jacobian $g \neq 0$ defines a **measure preserving almost Poisson bracket**.

**Def 3.** A bivector $J$ such that $\mathcal{F} = r^{-1} J$ satisfies the Jacobi identity with $r: \mathcal{M} \rightarrow \mathbb{R} - \{0\}$ defines a **conformally Poisson bracket**.

**Def 4.** A bivector $J$ satisfying the Jacobi identity defines a **Poisson bracket**.

- Note that $4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$.

**Thm 1.** Let $J \in \Lambda^2 T\mathcal{M}^n$ be not measure preserving. Let $x^{n+1} = s$ and $x^{n+2} = u$ be 2 new variables. Let the functions $a^{in+2} \in C^\infty, i = 1, \ldots, n + 1,$ be such that $\Sigma_{i=1}^{n+1} \partial_i a^{in+2} = 0$ and $\partial_{n+2} a^{in+2} = 0 \quad \forall i = 1, \ldots, n + 1$. Then, the $n + 1 + \delta_{n,2m}$ dimensional almost Poisson operator:

$$\mathcal{F} = J + x^{n+1} \frac{\partial J^{ij}}{\partial x^l} \partial_j \wedge \partial_l + \delta_{n,2m} a^{in+2} \partial_i \wedge \partial_{n+2}$$

is measure preserving.
**Thm 2.** Let $J \in \Lambda^2 TM^n$ with $n = 4$. If $J$ is measure preserving, then $J$ is conformal.

**Cor 1.** A 3D almost Poisson bracket which is not measure preserving can always be extended to a conformal bracket.

**Proof:**

Thm1: extend $J$ to a 4D measure preserving operator $\tilde{J}$.
Thm2: $\tilde{J}$ is conformal with $\tilde{J} = r J$.
The equations $Y = J(dH)$ are Hamiltonian.

---

**Time Reparametrization**

- In a conformal system, the equations of motion take the form:

  \[
  \begin{align*}
  \dot{p}^i &= -r^{-1} H q^i \\
  \dot{q}^i &= r^{-1} H p^i
  \end{align*}
  \]

- The same energy gradient produces different forces depending on the position in space. The issue is solved by introducing the proper time:

  \[
  \frac{d \tau}{dt} = r
  \]
POISSONIZATION OF $E \times B$ DRIFT

- Consider the motion $X = J(dH) = \frac{E \times B}{B^2}$ of a charged particle:
  \[
  \begin{align*}
  \frac{B}{B^2} \rightarrow J &= (\cos z + \sin z)\partial_z \wedge \partial_y + (\cos z - \sin z)\partial_x \wedge \partial_z \\
  E &= -q\nabla \phi \rightarrow H = \frac{1}{2}(x^2 + y^2 + z^2)
  \end{align*}
  \]

- This system is not Hamiltonian: the Jacobi identity is not satisfied $\frac{B \cdot \nabla \times B}{B^4} \neq 0$.

- Poissonization gives:
  \[
  Y = \mathcal{F}(dH) = \frac{\tilde{J}}{r}(dH) = \frac{\{J + [B + s\nabla \times (B/B^2)] \wedge \partial_s\}}{1 + s\frac{B \cdot \nabla \times B}{B^4}}(dH)
  \]

\[\frac{dt}{d\tau} = r = 1 + s\frac{B \cdot \nabla \times B}{B^4}\]
\[s = \frac{mv_\parallel}{\sqrt{2}} + o(m^2)\]
\[\tau = t + \sqrt{2}m\ell + o(m^2)\]
Lemma 1. Let $X = dx/dt$ and $Y = dy/d\tau$ be two vector fields. Let $g$ be the Jacobian of the coordinate change $vol^m_Y = g^{-1}vol^m_X$. If

$$\mathcal{L}_X vol_X^n = \mathcal{L}_Y vol_Y^n = 0,$$

then,

$$g = \frac{dt}{d\tau}.$$

If $x = (x, y, z, s)$ and $y = (p^1, q^1, p^2, q^2)$ then:

$$g = r^{-1} = \frac{1}{1 + s\frac{B \cdot \nabla \times B}{B^4}}.$$
CONCLUDING REMARKS

SUMMARY OF PART I

• Inward diffusion is an example of transport occurring in a system with a constrained topology.

• The integrability of the constraint enables the construction of statistical mechanics on the symplectic leaf.

SUMMARY OF PART II.

• Reduction of a microscopic system may destroy the Hamiltonian form.

• Almost Poisson operators do not satisfy the Jacobi identity and therefore do not guarantee the existence of an invariant measure that is required to formulate statistical mechanics.

• By extension, a measure preserving bracket can be recovered.

OPEN PROBLEMS

• Necessary and sufficient conditions for homogeneous fluctuations: the Jacobi identity may be a necessary condition.
MAIN REFERENCES