Subcritical turbulence in tokamaks.

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Aspects of subcritical turbulence.

• Subcritical turbulence in neutral fluids turbulence: the 19\textsuperscript{th} C to today.

• Simple plasma models of subcritical turbulence.

• Subcriticality in gyrokinetic simulation.
Transition to turbulence: subcritality

\[ Re = \frac{\rho U D}{\mu} \]

Poiseulle, Hagen and Reynolds, 1800s
What are the interesting questions?

• Transition ‘to’ turbulence: how when do we get into the turbulent state?

• Transition ‘from’ turbulence: how does the turbulent state change as parameters become less unstable (lower Re).
Linear Stability analysis.

Consider infinitesimal disturbances: do they grow or decay?

Rayleigh Bernard

Taylor-Couette
Linear Stability analysis.

Consider infinitesimal disturbances: do they grow or decay?

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Taylor-Couette
Linear transient growth

Non-normal system:
Even in linearly stable system, mode amplitude can be very strongly amplified before eventual decay.

Can do linear analysis for this...

Is the amplitude large enough for nonlinear process to become important?

Nonlinear analysis necessary.
Transient growth: consequences.

• Given a very small initial perturbation, late time state could be large amplitude.
• Linear theory breaks down.
• Nonlinear terms could allow continued growth in linearly stable system...
• Explains turbulence in subcritical systems?
Non-normality in tokamaks.

- MHD with flows.
- Resistive MHD.
- Hasegawa-Wakatani (through adiabaticity parameter)
- Almost everything except simplest MHD model.
Tubulent and laminar manifolds.

Turbulence is structured around simple solutions: periodic orbits..
Edge of chaos.

Edge of chaos is ‘gateway’ to turbulence: dynamics often simpler than fully turbulent state.
‘Plasma Interchange’ model

• Take a 2D slab MHD model with a gravity force opposing a pressure gradient (x -> radial direction).

• Buoyancy instability. (Beyer PRE 2000)

\[
\frac{d\nabla^2 \phi}{dt} = \mathbf{g} \times \nabla(n - n_0) + \nu \nabla^4 \phi
\]

\[
\frac{dn}{dt} = \chi \nabla^2 n
\]
Plasma interchange model equations.

- Go to 1D: 
  \[ n(x, y) = \bar{n}(x) + \tilde{n}(x) \exp(iky) \]

- A simple model consisting of 4 (6) PDEs with one spatial direction.
- Similar to the Hasegawa-Wakatani or Rayleigh-Benard-Couette.
- The control parameter is \( S \), the background shear.

\[
\begin{align*}
\partial_t \bar{n} & = i \partial_x (\bar{n}^* \tilde{\phi} - \tilde{n} \tilde{\phi}^*) \\
\partial_t E & = i \partial_x (\tilde{\phi} \partial_x \tilde{\phi}^* - \tilde{\phi}^* \partial_x \tilde{\phi}) + \partial_{xx} E \\
\partial_t \tilde{\phi} & = -i \tilde{\phi} (Sx + E) + i \tilde{n} + \partial_{xx} \tilde{\phi} \\
\partial_t \tilde{n} & = -i \tilde{n} (Sx + E) + i \tilde{\phi} (\partial_x \bar{n} - 1) + \partial_{xx} \tilde{n}
\end{align*}
\]

\( \bar{n}(L) = \bar{n}(0) \), \( E(L) = E(0) \), \( \tilde{\phi}(L) = \tilde{\phi}(0) \exp\{-itLS\} \) and \( \tilde{n}(L) = \tilde{n}(0) \exp\{-itLS\} \)
Background shear flow.

- Background shear: transform to ‘shearing box’ frame moving with background flow.
Time-varying plane waves...

For zero flow shear, mode growth rate a function of wavevector

-> Instantaneous linear growth rate is a function of time.
Leads to transient linear growth.

Linear dynamics completely solvable for simple model:

Growth then damping...
simpler than ‘Floquet mode’ (Waltz PoP 5 1784) due to slab geometry.

Maximum transient amplification
\[ \frac{A(t)}{A(0)} = \exp\left(\frac{4}{3} S\right) \]

- At small flow shear \( S \), extremely large amplification!
- Scaling is universal for this type of model.
Nonlinear dynamics.

Reproduces many qualitative features of gyrokinetic simulations (McMillan 2009) Turbulence becomes increasingly intermittent and dominated by ‘bursts’ for large $S$. 

S=0.15

S=1.1

![Graphs showing turbulence behavior for S=0.15 and S=1.1](image-url)
Comparison with pipe flow

The dynamics are entirely analogous to pipe flow...

Re small => No turbulence
Re intermediate => Localised puffs
Re large => Expanding turbulence
Re=∞ => Linear instability?

S large => No turbulence
S intermediate => Localised fronts
S small => Expanding turbulence
S=0 => Linear instability
PI model: Finding the edge
PI model: the edge
Travelling wave solution.
Edge in parameter space.
How big a perturbation is needed?

FIG. 11. The nonlinear energy growth optimal ($A=0.69$, purple) and the turbulent seed ($A=0.71$, green) at $S=1.0$. The two states are barely distinguishable and are both strongly localized.

FIG. 12. The amplitudes of the minimum seed (solid) and of the traveling wave solutions (long dashes) as functions of the background shear. Also included is the predicted critical amplitude from the semi-analytical analysis (short dashes) found in section VI. Unsurprisingly the minimal seed triggers turbulence at amplitudes significantly below the edge state. The semi-analytic approach agrees well with the minimal seed, especially for small values of $S$ where the approach is better justified.

To a nonlinearly saturated state. In this section we present a semi-analytic approach to approximating the minimal seed, and hence provide a simple closed form estimate for the critical amplitude, thus developing some insight into the processes occurring during the initial transient.

The approach revolves around the assumption that for turbulence to be sustained, the background shear needs to be ameliorated (this is justified partly by examining the time evolution of $E$ in earlier results). This occurs when $\partial E/\partial x$ is of a similar size to $S$. This is exactly the point where the linear and nonlinear terms in the time evolution of $\tilde{\phi}$ and $\tilde{n}$ become comparable. We assume, for the sake of obtaining a rough estimate, that up to this point we can use the linear time evolution for $\tilde{\phi}$ and $\tilde{n}$ as in section VA.

We consider the evolution of an initial disturbance, $\tilde{\phi}(x,0)$. The wave density is then given by $\tilde{n}=-i\tilde{\phi}$ (in our previous notation this gives $\alpha=-1$), while the remaining two components, $E$ and $\bar{n}$ are taken to be zero. After applying the Fourier transform $\tilde{\Phi}(k,t) = \int_{-\infty}^{\infty} \tilde{\phi}(x,t) e^{ikx} dx$ (31) we know this new field evolves as $\tilde{\Phi}(k,t) = \tilde{\Phi}(k,0) \exp\{(-1-k^2)t + kSt^2 - \frac{1}{3}S^2t^3\}$ (32).
Tokamak simulations

- Gyrokinetic simulations of CYCLONE tokamak parameters using GKW code.
- Base state linearly stable, but avalanche-like bursts observed: turbulent puffs?
Edge of chaos in GK simulations
On the edge.
The edge state, in detail.

Non-zonal potential on outboard mid-plane

Zonally averaged quantities

\[ \frac{d<E>}{dx}, \text{narrow} \]
\[ R/L_T, \text{narrow} \]
\[ d<E>/dx, \text{standard} \]
\[ R/L_T, \text{standard} \]
For the narrow simulation.

Solution is a relative periodic orbit, rather than a travelling wave (due to rational magnetic surfaces, local gyrokinetics has a discrete, rather than continuous translation symmetry)
Scaling of transition amplitude.

Qualitatively, similar to PI model.

Very low amplitudes required for instability at low $S$ (always see turbulence in practice for $S<0.04$?)

Narrow simulation and standard simulation have same behaviour at large shear:

Only need one non-zonal mode for transition ->

Nonlinearity in transition is dominantly drift mode/zonal mode. (not several drift modes).
Summary.

• The edge of chaos allows the transition of nonlinear systems to be quantified/understood.
• Travelling wave solutions dominate the late-time behaviour of the edge.
• Qualitative features are preserved in complex geometry and complicated gyrokinetic model.
• See Pringle, McMillan, Teaca, Arxiv 2017, in press PoP 2017