WEIGHTED BMO SPACES ASSOCIATED TO OPERATORS

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Abstract. Let X be a metric space equipped with a metric d and a nonnegative Borel measure μ satisfying the doubling property and let $\{\mathcal{A}_t\}_{t>0}$, be a generalized approximations to the identity, for example $\{\mathcal{A}_t\}$ is a holomorphic semigroup e^{-tL} with Gaussian upper bounds generated by an operator L on $L^2(X)$. In this paper, we introduce and study the weighted BMO space $\mathrm{BMO}_{\mathcal{A}}(X,w)$ associated to the the family $\{\mathcal{A}_t\}$. We show that for these spaces, the weighted John-Nirenberg inequality holds and we establish an interpolation theorem in scale of weighted L^p spaces. As applications, we prove the boundedness of some singular integrals with non-smooth kernels on the weighted BMO space $\mathrm{BMO}_{\mathcal{A}}(X,w)$.

Contents

1. Introduction

The BMO (bounded mean oscillation) function spaces introduced by John and Nirenberg on Euclidean spaces and by Coifmann and Wiess on spaces of homogeneous type played an important role in modern harmonic analysis, see [13, 6]. It is known that the BMO spaces are good substitutions to L^∞ spaces in studying the boundedness of Calderón-Zygmund operators and in interpolation theory. Although Calderón-Zygmund operators are not bounded on L^∞, they map continuously from L^∞ to BMO. Moreover, it is also known that if a linear operator T is bounded on $L^p, 1 \leq p < \infty$ and

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bounded from L^∞ to BMO then by interpolation T is bounded on L^q for all $p < q < \infty$.

In practical, there are large classes of operators which do not fall within the scope of the Calderón-Zygmund theory of singular integral operators. In these cases, the classical BMO space are no longer suitable spaces for the study of the endpoint estimates at $p = \infty$ for such singular integral operators. Note that the weighted estimates for singular integrals with non-smooth kernels have been studied intensively by many authors, see for example [3, 2, 7, 18, 19] and the references therein. Although the weighted endpoint estimates at $p = 1$ have been investigated thoughtfully via the scale of weighted Hardy spaces associated to operators, see for example [2, 18, 19], the weighted endpoint estimates at $p = \infty$ are less well-known and the aim of this paper is to fill in this gap.

In this paper, given a family of operators $\{\mathcal{A}_t\}_{t>0}$ which is a generalized approximations to the identity (See its definition in Section 2) and a suitable weight w, we develop the theory of weighted BMO space $\mathrm{BMO}_{\mathcal{A}}(X, w)$ associated to $\mathcal{A}_t$. An important example of the family $\{\mathcal{A}_t\}$ is when $\{\mathcal{A}_t\} = I - (I - e^{-tL})^M$ for some positive integer M in which e^{-tL} is a holomorphic semigroup generated by an operator L on $L^2(X)$, assuming that L satisfies Gaussian heat kernel upper bounds and has a bounded L^2 holomorphic functional calculus. We show that the new weighted spaces $\mathrm{BMO}_{\mathcal{A}}(X, w)$ retain some important properties, similarly to the classical weighted BMO spaces. It turns out that the spaces $\mathrm{BMO}_{\mathcal{A}}(X, w)$ play an essential role in obtaining the weighted endpoint estimates at $p = \infty$ for singular integrals such as the spectral multipliers and holomorphic functional calculi. While it is not clear whether or not the weighted endpoint estimates at $p = 1$ in [2, 18, 19] can interpolate to obtain the weighted L^p estimates for singular integrals, our interpolation theorem for the weighted spaces $\mathrm{BMO}_{\mathcal{A}}(X, w)$ implies the weighted L^p norm inequalities for these operators, see Theorem 4.3. More precisely, the new results in this article are the following:

- (i) The introduction of weighted BMO space associated to generalized approximations of identity $\mathrm{BMO}_{\mathcal{A}}(X, w)$ (Section 3.1);
- (ii) The weighted John-Nirenberg inequality (Lemma 3.6), and the equivalence of $\mathrm{BMO}^p_{\mathcal{A}}(X, w)$ for $1 \le p < \infty$ (Theorem 3.5);
- (iii) An interpolation theorem concerning $\mathrm{BMO}_{\mathcal{A}}(X, w)$ (Theorem 4.3);
- (iv) Applications to some singular integrals with non-smooth kernels (Section 5).

We note that under suitable conditions on the operator L and the weight w, the dual space of the weighted Hardy space $H^1_L(X, w)$ associated to the operator L introduced in [18] (see also [2, 19]) should be the weighted BMO spaces $\mathrm{BMO}_{L^*}(X, w)$ in this paper. However, we do not try to address this issue in this article.

Throughout the paper, we shall write $A \lesssim B$ if there is a universal constant C so that $A \le CB$. Likewise, we shall write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

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2. Preliminaries

We first recall the definition of the spaces of homogeneous type in [6]. Let (X, d, μ) be a metric space with a metric d and a nonnegative Borel measure μ satisfying the doubling property

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty$$

for any $x \in X$ and $r > 0$, where the constant $C \geq 1$ is independent of x and r and $B(x, r) := \{y : d(x, y) < r\}$.

It can be verified that the doubling property implies that there exist $c, n > 0$ so that

$$\mu(B(x, \lambda r)) \leq c\lambda^n \mu(B(x, r)) \tag{1}$$

for all $\lambda \geq 1$ and $x \in X$. The value of parameter n is a measure of the dimension of the space. Moreover, there also exist c and $N, 0 \leq N \leq n$, so that

$$\mu(B(y, r)) \leq c\Big(1 + \frac{d(x, y)}{r}\Big)^N \mu(B(x, r)) \tag{2}$$

for all $x, y \in X$ and $r > 0$. For further details on the spaces of homogeneous type, we refer to [6].

To simplify notation, for a measurable subset E in X, we write $V(E)$ instead of $\mu(E)$. We will often use B for $B(x_B, r_B)$. Also given $\lambda > 0$, we will write λB for the λ-dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$ and denote $V(x, r) = \mu(B(x, r))$ for all $x \in X$ and $r > 0$. For each ball $B \subset X$ we set

$$S_0(B) = B \text{ and } S_j(B) = 2^j B \backslash 2^{j-1} B \text{ for } j \in \mathbb{N}.$$

Recall that the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{V(B)} \int_B |f(x)| d\mu(x).$$

We now give a simple covering lemma which states that we can cover a given ball by a finite overlapping family of balls with smaller radii. This will be used frequently in the sequel.

Lemma 2.1. *For any ball $B(x_B, lr)$ in X, with $l \geq 1$ and $r > 0$, then there exists a corresponding family of balls $\{B(x_1, r), \ldots, B(x_k, r)\}$ such that*

(a) $B(x_j, r/3) \subset B(x_B, lr)$, for all $j = 1, \ldots, k$;
(b) $B(x_B, lr) \subset \cup_{j=1}^k B(x_j, r)$;
(c) $k \leq Cl^n$;
(d) $\sum_{j=1}^k \chi_{B(x_j, r)} \leq C$, where C is independent of l and r.

The proof of this lemma is just a consequence of Vitali covering lemma and doubling property (1). Hence we omit details here.

2.1. **Approximations to the identity.** Let $\{\mathcal{A}_t\}_{t>0}$ be a family of linear operators. Suppose that for each $t > 0$, the operator $\mathcal{A}_t$ has an associated kernel $a_t(x, y)$ in the sense that

$$\mathcal{A}_t f(x) = \int_X a_t(x, y) f(y) d\mu(y)$$

for every function $f \in \cup_{p\geq 1} L^p(X)$.

Assume that the kernel $a_t(x, y)$ of $\mathcal{A}_t$ satisfies the Gaussian upper bound

$$|a_t(x, y)| \leq \frac{C}{V(x, t^{1/m})} \exp\Big(- c \frac{d(x, y)^{m/(m-1)}}{t^{1/(m-1)}}\Big), \tag{3}$$

for all $t > 0$ and $x, y \in X$ where m is a positive constant, $m \geq 2$.

The decay of the kernel $a_t(x, y)$ guarantees that $\mathcal{A}_t$ is bounded on $L^p(X)$ for all $p \in (1, \infty)$. More precisely, we have the following proposition, see [9].

Proposition 2.2. *For each* $p \in [1, \infty]$, *we have*

$$|\mathcal{A}_t f(x)| \lesssim M f(x)$$

for all $t > 0$ *and* $f \in L^p(X)$, μ*-a.e.*

2.2. **Muckenhoupt weights.** We recall the definition and basic properties of Muckenhoupt weights in [17]. Throughout this article, we shall denote $w(E) := \int_E w(x)d\mu(x)$ and $V(E) = \mu(E)$ for any measurable set $E \subset X$. For $1 \leq p \leq \infty$ let p' be the conjugate exponent of p, i.e. $1/p + 1/p' = 1$.

We first introduce some notation. We use the notation

$$\fint_B h(x) d\mu(x) = \frac{1}{V(B)} \int_B h(x) d\mu(x).$$

A weight w is a non-negative measurable and locally integrable function on X. We say that $w \in A_p$, $1 \leq p < \infty$, if there exists a constant C such that for every ball $B \subset X$,

$$\Big(\fint_B w(x) d\mu(x)\Big)\Big(\fint_B w^{-1/(p-1)}(x) d\mu(x)\Big)^{p-1} \leq C. \tag{4}$$

When $p = 1$, (4) is understood that there is a constant C such that for every ball $B \subset X$,

$$\fint_B w(y) d\mu(y) \leq C w(x) \text{ for a.e. } x \in B.$$

We set $A_\infty = \cup_{p\geq 1} A_p$.

For $1 < q \leq \infty$, we say that the weight w belongs to the reverse Hölder class RH_q if there is a constant C such that for any ball $B \subset X$,

$$\Big(\fint_B w^q(y) d\mu(y)\Big)^{1/q} \leq C \fint_B w(x) d\mu(x). \tag{5}$$

When $q = \infty$, (5) is understood that there is a constant C such that for any ball $B \subset X$,

$$w(x) \leq C \fint_B w(y) d\mu(y) \text{ for a.e. } x \in B.$$

Let $w \in A_\infty$, for $1 \le p < \infty$, the weighted spaces $L^p_w(X)$ are defined by

$$L^p_w(X) := \Big\{ f : \int_X |f(x)|^p w(x) d\mu(x) < \infty \Big\}$$

and we set

$$\|f\|_{L^p_w(X)} = \Big(\int_X |f(x)|^p w(x) d\mu(x) \Big)^{1/p}.$$

We recall some of the standard properties of classes of Muckenhoupt weights in the following lemma, see for example [17].

Lemma 2.3. *The following properties hold:*

(i) $A_1 \subset A_p \subset A_q$ *for* $1 < p \le q < \infty$.
(ii) $RH_\infty \subset RH_q \subset RH_p$ *for* $1 < p \le q < \infty$.
(iii) *If* $w \in A_p, 1 < p < \infty$, *then there exists* $1 < r < p < \infty$ *such that* $w \in A_r$.
(iv) *If* $w \in RH_q, 1 < q < \infty$, *then there exists* $q < p < \infty$ *such that* $w \in RH_p$.
(v) $A_\infty = \cup_{1 \le p < \infty} A_p \subset \cup_{1 < q \le \infty} RH_q$

Lemma 2.4. *For any ball* B, *any measurable subset* E *of* B *and* $w \in A_p, p \ge 1$, *there exists a constant* $C_1 > 0$ *such that*

$$C_1 \Big(\frac{V(E)}{V(B)} \Big)^p \le \frac{w(E)}{w(B)}.$$

If $w \in RH_r, r > 1$. *Then, there exists a constant* $C_2 > 0$ *such that*

$$\frac{w(E)}{w(B)} \le C_2 \Big(\frac{V(E)}{V(B)} \Big)^{\frac{r-1}{r}}.$$

From the first inequality of Lemma 2.4, if $w \in A_1$ then there exists a constant $C > 0$ so that for any ball $B \subset X$ and $\lambda > 1$, we have

$$w(\lambda B) \le C \lambda^n w(B).$$

3. Weighted BMO spaces associated to operators

3.1. **Definition of** $\mathrm{BMO}_{\mathcal{A}}(X, w)$**.** Throughout this paper, we assume that the family of the operators $\{\mathcal{A}_t\}_{t \ge 0}$ satisfies the Gaussian upper bounds (3) and these operators commute, i.e. $\mathcal{A}_s \mathcal{A}_t = \mathcal{A}_t \mathcal{A}_s$ for all $s, t > 0$. Note that we do not assume the semigroup property $\mathcal{A}_s \mathcal{A}_t = \mathcal{A}_{s+t}$ on the family $\{\mathcal{A}_t\}_{t \ge 0}$.

Following [11], we now define the class of functions that the operators $\{\mathcal{A}_t\}_{t \ge 0}$ act upon. A function $f \in L^1_{loc}(X)$ is said to be a function of type (x_0, β) if f satisfies

$$\Big(\int_X \frac{|f(x)|^2}{(1 + d(x_0, x))^\beta V(x_0, 1 + d(x_0, x))} d\mu(x) \Big)^{1/2} \le c < \infty. \tag{6}$$

We denote $M_{(x_0,\beta)}$ the collection of all functions of type (x_0, β). If $f \in M_{(x_0,\beta)}$, the norm of f is defined by

$$||f||_{M_{x_0,\beta}} = \inf\{c : (6) \text{ holds}\}.$$

For a fixed $x_0 \in X$, one can check that $M_{(x_0,\beta)}$ is a Banach space under the norm $||f||_{M_{x_0,\beta}}$. For any $x_1 \in X$, $M_{(x_0,\beta)} = M_{(x_1,\beta)}$ with equivalent norms. Denote by

$$\mathcal{M} = \cup_{x_0 \in X} \cup_{0<\beta<\infty} M_{(x_0,\beta)}.$$

Definition 3.1. *A function $f \in \mathcal{M}$ is said to be in $BMO_{\mathcal{A}}(w)$ with $w \in A_\infty$, the space of functions of bounded mean oscillation associated to $\{\mathcal{A}_t\}_{t\geq 0}$ and w, if there exists some constant c such that for any ball B,*

$$\frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})f(x)| d\mu(x) \leq c, \tag{7}$$

*where $t_B = r_B^m$ (m is a constant in (3)) and r_B is the radius of B.
The smallest bound c for which (7) is satisfied is then taken to be the norm of f in this space and is denoted by $||f||_{BMO_{\mathcal{A}}(X,w)}$.*

Remark: The space $(\mathrm{BMO}_{\mathcal{A}}(X,w), \|\cdot\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)})$ is a seminormed vector space, with the seminorm vanishing on the space $\mathcal{K}_{\mathcal{A}}$, defined by

$$\mathcal{K}_{\mathcal{A}} = \{f \in \mathcal{M} : \mathcal{A}_t f(x) = f(x) \text{ for almost all } x \text{ and for all } t > 0\}.$$

In this paper, $\mathrm{BMO}_{\mathcal{A}}(X,w)$ space is understood to be modulo $\mathcal{K}_{\mathcal{A}}$.

The following result gives a sufficient condition for the $\mathrm{BMO}(X,w)$ to be contained in $\mathrm{BMO}_{\mathcal{A}}(X,w)$. The proof for the unweighted case was given in [15] (see also [11]).

Proposition 3.2. *Suppose that $w \in A_1$ and $\mathcal{A}_t(1) = 1$ for all $t > 0$, i.e., $\int_X a_t(x,y)d\mu(y) = 1$ for almost all $x \in X$. Then the inclusion* $\mathrm{BMO}(X,w) \subset \mathrm{BMO}_{\mathcal{A}}(X,w)$ *holds where*

$$\mathrm{BMO}(X,w) := \{f \in L^1_{\mathrm{loc}} : \|f\|_{\mathrm{BMO}(X,w)} := \sup_B \frac{1}{w(B)} \int_B |f - f_B| d\mu < \infty\}.$$

Proof. Let $f \in \mathrm{BMO}(X,w)$. For any ball B, due to $\mathcal{A}_t(1) = 1$, we have

$$\begin{aligned}
\frac{1}{w(B)} \int_B &|f(x) - \mathcal{A}_{t_B} f(x)| d\mu(x) \\
&= \frac{1}{w(B)} \int_B \Big| f(x) - \int_X a_{t_B}(x,y) f(y) d\mu(y) \Big| d\mu(x) \\
&= \frac{1}{w(B)} \int_B \Big| \int_X a_{t_B}(x,y)(f(x) - f(y)) d\mu(y) \Big| d\mu(x) \\
&= \frac{1}{w(B)} \int_B \int_X \Big| a_{t_B}(x,y)(f(x) - f(y)) \Big| d\mu(y) d\mu(x).
\end{aligned}$$

This, along with (3), gives

$$\begin{aligned}
&\frac{1}{w(B)}\int_B |f(x)-\mathcal{A}_{t_B}f(x)|d\mu(x)\\
&\qquad \leq \frac{C}{V(B)w(B)}\int_B\int_X \exp\Big(-c\frac{d(x,y)^{m/(m-1)}}{t_B^{1/(m-1)}}\Big)\\
&\qquad\qquad\qquad \times |f(x)-f(y)|d\mu(y)d\mu(x)\\
&\qquad = \frac{C}{V(B)w(B)}\int_B\int_{2B}\ldots + \sum_{j\geq 2}\frac{1}{V(B)w(B)}\int_B\int_{S_j(B)}\ldots\\
&\qquad = I + \sum_{j\geq 2} I_j.
\end{aligned}$$

Let us estimate I first. We have

$$\begin{aligned}
I &\lesssim \frac{1}{V(B)w(B)}\int_B\int_{2B}|f(x)-f_B|d\mu(y)d\mu(x)\\
&\quad + \frac{1}{V(B)w(B)}\int_B\int_{2B}|f(y)-f_{2B}|d\mu(y)d\mu(x)\\
&\quad + \frac{1}{V(B)w(B)}\int_B\int_{2B}|f_{2B}-f_B|d\mu(y)d\mu(x)\\
&\lesssim \|f\|_{\mathrm{BMO}(X,w)}.
\end{aligned}$$

For the term $I_j, j\geq 2$, we have

$$\begin{aligned}
I_j &\lesssim \frac{1}{V(B)w(B)}\int_B\int_{2^jB}\Big|\exp(-c2^{jm/(m-1)})(f(x)-f(y))\Big|d\mu(y)d\mu(x)\\
&\lesssim \frac{\exp(-c2^{jm/(m-1)})}{V(B)w(B)}\Big(\int_B\int_{2^jB}|f(x)-f_B|d\mu(y)d\mu(x)\\
&\quad + \int_B\int_{2^jB}|f(y)-f_{2^jB}|d\mu(y)d\mu(x)\\
&\quad + \int_B\int_{2^jB}|f_{2^jB}-f_B|d\mu(y)d\mu(x)\Big)\\
&\lesssim \|f\|_{\mathrm{BMO}(X,w)}.
\end{aligned}$$

These estimates on I and $I_j, j\geq 2$ give $\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}\leq \|f\|_{\mathrm{BMO}(X,w)}$. This completes our proof. □

Proposition 3.3. *For $t>0, K>1$ and $w\in A_1$ we have for a.e. $x\in X$*

$$|(\mathcal{A}_tf(x)-\mathcal{A}_{Kt}f(x))|\lesssim (1+\log K)\frac{w(B(x,t^{1/m}))}{V(x,t^{1/m})}\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}.$$

Before coming to the proof, we would like to mention that the same estimates as in Proposition 3.3 was obtained in [11] under the extra assumption of semigroup property on the family $\{\mathcal{A}_t\}$. While the argument in [11] relies on Christ's covering lemma, our argument uses Lemma 2.1.

Proof. For any $s,t>0$ such that $t\leq s\leq 2t$, we have

$$|\mathcal{A}_tf(x)-\mathcal{A}_sf(x)|\leq |\mathcal{A}_t((I-\mathcal{A}_s)f(x))|+|\mathcal{A}_s((I-\mathcal{A}_t)f(x))| := I_1+I_2.$$

We first estimate I_1. The Gaussian upper bound (3) of $\mathcal{A}_t$ and the fact that $t \approx s$ gives that

$$\begin{aligned}
I_1 &\lesssim \frac{1}{V(x,t^{1/m})}\int_X \exp\Big(-c\frac{d(x,y)^{m/(m-1)}}{t^{1/(m-1)}}\Big)|(I-\mathcal{A}_s)f(y)|d\mu(y)\\
&\lesssim \frac{1}{V(x,s^{1/m})}\int_{B(x,s^{1/m})} \exp\Big(-c\frac{d(x,y)^{m/(m-1)}}{s^{1/(m-1)}}\Big)|(I-\mathcal{A}_s)f(y)|d\mu(y)\\
&\quad+\sum_{j\geq 2}\frac{1}{V(x,s^{1/m})}\int_{S_j(B(x,s^{1/m}))} \exp\Big(-c\frac{d(x,y)^{m/(m-1)}}{s^{1/(m-1)}}\Big)|(I-\mathcal{A}_s)f(y)|\\
&= I_{11}+\sum_{j\geq 2} I_{1j}.
\end{aligned}$$

For the term I_{11}, since $t \approx s$ and $w \in A_1$, we have

$$I_{11} \leq \|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}\frac{w(B(x,s^{1/m}))}{V(x,s^{1/m})} \leq C\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}\frac{w(B(x,t^{1/m}))}{V(x,t^{1/m})}.$$

For $j \geq 2$, using Lemma 2.1 we can cover the annulus $S_j(B(x,s^{1/m}))$ by a finite overlapping family of at most $C2^{jn}$ balls $B(x_k^j, s^{1/m})$. Using $w \in A_1$, we can dominate the term I_{1j} as follows.

$$\begin{aligned}
I_{1j} &\lesssim \frac{1}{V(x,s^{1/m})}\int_{S_j(B(x,s^{1/m}))} \exp\Big(-c\frac{d(x,y)^{m/(m-1)}}{s^{1/(m-1)}}\Big)|(I-\mathcal{A}_s)f(y)|d\mu(y)\\
&\lesssim \frac{1}{V(x,s^{1/m})}\int_{S_j(B(x,s^{1/m}))} e^{-c2^{j/(m-1)}}|(I-\mathcal{A}_s)f(y)|d\mu(y)\\
&\lesssim \sum_k \frac{1}{V(x,s^{1/m})}\int_{B(x_k^j,s^{1/m})} e^{-c2^{j/(m-1)}}|(I-\mathcal{A}_s)f(y)|d\mu(y)\\
&\lesssim \sum_k \frac{w(B(x_k^j,s^{1/m}))}{V(x,s^{1/m})}e^{-c2^{j/(m-1)}}\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}\\
&\lesssim \frac{w(B(x,2^j s^{1/m}))}{V(x,s^{1/m})}e^{-c2^{j/(m-1)}}\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}\\
&\lesssim 2^{jn}\frac{w(B(x,2^j s^{1/m}))}{V(x,s^{1/m})}e^{-c2^{j/(m-1)}}\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}\\
&\lesssim 2^{jn}e^{-c2^{j/(m-1)}}\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}\frac{w(x,t^{1/m})}{V(x,t^{1/m})}.
\end{aligned}$$

This implies

$$I_1 \leq C\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}\frac{w(B(x,t^{1/m}))}{V(x,t^{1/m})}.$$

A similar argument also gives

$$I_2 \leq C\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}\frac{w(B(x,t^{1/m}))}{V(x,t^{1/m})}.$$

Therefore, we have

$$|(\mathcal{A}_t f(x) - \mathcal{A}_{t+s} f(x))| \lesssim \|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}\frac{w(B(x,t^{1/m}))}{V(x,t^{1/m})} \tag{8}$$

for all $t \le s \le 2t$.

In general case, taking $l \in \mathbb{N}$ such that $2^l \le K < 2^{l+1}$, we can write

(9)
$$
\begin{aligned}
|(\mathcal{A}_t f(x) - \mathcal{A}_{Kt} f(x))| &\le \sum_{k=1}^{l} |\mathcal{A}_{2^{l-1}t} f(x) - \mathcal{A}_{2^l t} f(x)| + |\mathcal{A}_{2^l t} f(x) - \mathcal{A}_{Kt} f(x)| \\
&\lesssim \sum_{k=1}^{l} \|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)} \frac{w(B(x, 2^{l-1}t^{1/m}))}{V(x, 2^{l-1}t^{1/m})}.
\end{aligned}
$$

Since $w \in A_1$, we have

$$
\frac{w(B(x, 2^k t^{1/m}))}{V(x, 2^k t^{1/m})} \le C \frac{w(B(x, t^{1/m}))}{V(x, t^{1/m})}
$$

for all $k \ge 0$.

This together with (9) gives

$$
|(\mathcal{A}_t f(x) - \mathcal{A}_{Kt} f(x))| \lesssim (1 + \log K)\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)} \frac{w(B(x, t^{1/m}))}{V(x, t^{1/m})}.
$$

This completes the proof. □

3.2. **John-Nirenberg inequality on $\mathbf{BMO}_{\mathcal{A}}(X, w)$.** In this section, we will show that functions in the new class of weighted BMO spaces $\mathrm{BMO}_{\mathcal{A}}(X, w)$ satisfy the John-Nirenberg inequality. The unweighted version was obtained in [11]. Here, we extend to the weighted BMO spaces associated to the family of operators $\{\mathcal{A}_t\}_{t>0}$.

Definition 3.4. *For $w \in A_1$ and $p \in [1, \infty)$, the function $f \in \mathcal{M}$ is said to be in $\mathrm{BMO}^p_{\mathcal{A}}(X, w)$, if there exists some constant c such that for any ball B,*

$$
\Big(\frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B}) f(x)|^p w^{1-p}(x) d\mu(x)\Big)^{1/p} \le c. \tag{10}
$$

where $t_B = r_B^m$ and r_B is the radius of B.
The smallest bound c for which (10) holds is then taken to be the norm of f in this space and is denoted by $||f||_{\mathrm{BMO}^p_{\mathcal{A}}(X,w)}$.

Similar to the classical case, it turns out that the spaces $\mathrm{BMO}^p_{\mathcal{A}}(X, w)$ are equivalent for all $1 \le p < \infty$. More precisely, we have the following result.

Theorem 3.5. *For $w \in A_1$ and $p \in [1, \infty)$, the spaces $\mathrm{BMO}^p_{\mathcal{A}}(X, w)$ coincide and their norms are equivalent.*

Before coming to the proof Theorem 3.5 we need the following result.

Theorem 3.6. *For $w \in A_1$ and $f \in \mathrm{BMO}_{\mathcal{A}}(X, w)$, there exist positive constants c_1 and c_2 such that for any ball B and $\lambda > 0$ we have*

(11)
$$
w\{x \in B : |(f(x) - A_{t_B} f(x)) w^{-1}(x)| > \lambda\} \le c_1 w(B) \exp\Big(-\frac{c_2 \lambda}{\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}}\Big).
$$

Proof. Let us recall that if $w \in A_\infty$, there exist $C > 0$ and $\delta > 0$ such that for any ball B and any measurable subset $E \subset B$ we have

$$
\frac{w(E)}{w(B)} \le C\Big(\frac{\mu(E)}{\mu(B)}\Big)^{\delta}.
$$

So, to prove (11), it suffices to show that

(12)

$$\mu\{x\in B:|(f(x)-\mathcal{A}_{t_B}f(x))w^{-1}(x)|>\lambda\}\le c_1\mu(B)\exp\Big(-\frac{c_2\lambda}{\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}}\Big).$$

The proof of (12) is similar to that of Theorem 3.1 in [11] in which Proposition 2.6 in [11] is replaced by Proposition 3.3. However, for the sake of completeness, we sketch out the proof here.

Without the loss of generality, we may assume that $\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}=1$. Then we need to claim that for all balls B, we have

$$\mu\{x\in B:|(f(x)-\mathcal{A}_{t_B}f(x))w^{-1}(x)|>\lambda\}\le c_1e^{-c_2\lambda}\mu(B). \tag{13}$$

If $\alpha<1$, (13) holds for $c_1=e$ and $c_2=1$. Hence, we consider the case $\alpha\ge 1$.

For any ball $B\subset X$, we set

$$f_0=[(f(x)-\mathcal{A}_{t_B}f(x))w^{-1}(x)]\chi_{10B}.$$

Then, using the fact that $w\in A_1$, we have

$$\begin{aligned}\|f_0\|_{L^1}&\le\int_{10B}|(I-\mathcal{A}_{t_B})f(x)|w^{-1}(x)d\mu(x)\\&\le\frac{V(10B)}{w(10B)}\int_{10B}|(I-\mathcal{A}_{t_B})f(x)|d\mu(x)\\&\lesssim V(B)\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}=V(B).\end{aligned}$$

Taking $\beta>1$, we set

$$F=\{x:M(f_0)(x)\le\beta\}\ \text{ and }\ \Omega=X\setminus F.$$

Then we can pick a family of balls $\{B_{1,i}\}_{i=1}^\infty$ so that

(i) $\cup_iB_{1,i}=\Omega$;

(ii) there exists $\kappa>0$ so that $\sum_i\chi_{B_{1,i}}\le\kappa$;

(iii) there exists c_0 such that $c_0B_{1,i}\cap F\neq\emptyset$ for all i.

See [6, Chaptier].

For $x\in B\setminus[\cup_iB_{1,i}]$, by (i), we have

$$|(I-\mathcal{A}_{t_B})f(x)|w^{-1}(x)=|f_0(x)|\chi_F(x)\le M(f_0)(x)\chi_F(x)\le\beta.$$

Moreover, from (ii)-(iii) and the fact that the Hardy-Littlewood maximal function M is of weak type $(1,1)$, we have

$$\begin{aligned}\sum_i\mu(B_{1,i})\lesssim\mu(\Omega)&\lesssim\frac{1}{\beta}\|f_0\|_{L^1}\\&\le\frac{c_3}{\beta}V(B).\end{aligned}$$

By using argument as in [11, pp. 24-25], we can prove that for $B_{1,i}\cap B\neq\emptyset$, we have

$$|(\mathcal{A}_{t_{B_{1,i}}}-\mathcal{A}_{t_B})f(x)|w^{-1}(x)\le c_4\beta$$

for all $x\in B_{1,i}$.

On each $B_{1,i}$, repeat the argument above with the function

$$f_{1,i}=[(I-\mathcal{A}_{t_{B_{1,i}}})f(x)w^{-1}(x)]\chi_{10B_{1,i}}$$

and the same value β. Then we can pick the family of balls $\{B_{2,m}\}_{m=1}^{\infty}$ such that

(a) for any $x \in B_{1,i} \setminus [\cup_m B_{2,m}]$, $|(I - \mathcal{A}_{t_{B_{1,i}}})f(x)|w^{-1}(x) \le \beta$;
(b) $\sum_m \mu(B_{2,m}) \le \frac{c_3}{\beta} V(B_{1,i})$;
(c) for any $B_{2,m} \cap B_{1,i} \neq \emptyset$, $|(\mathcal{A}_{t_{B_{2,m}}} - \mathcal{A}_{t_{B_{1,i}}})f(x)|w^{-1}(x) \le c_4\beta$, for all $x \in B_{2,m}$.

We now abuse the notation $\{B_{2,m}\}$ for the family of all families $\{B_{2,m}\}$ corresponding to different $B'_{1,i}s$. Then, for all $x \in B \setminus [\cup_m B_{2,m}]$ we have

$$\begin{aligned}|(I - \mathcal{A}_{t_B})f(x)|w^{-1}(x) \le& |(I - \mathcal{A}_{t_{B_{1,i}}})f(x)|w^{-1}(x)\\ &+ |(\mathcal{A}_{t_{B_{1,i}}} - \mathcal{A}_{t_B})f(x)|w^{-1}(x)\\ \le& 2c_4\beta\end{aligned}$$

and

$$\sum_m \mu(B_{2,m}) \le \Big(\frac{c_3}{\beta}\Big)^2 V(B).$$

In the consequence, for each $K \in \mathbb{N}_+$ we can find a family of balls $\{B_{K,m}\}_{m=1}^{\infty}$ so that

$$|(I - \mathcal{A}_{t_B})f(x)|w^{-1}(x) \le Kc_4\beta \quad \text{for all} \quad x \in B \setminus [\cup_m B_{k,m}]$$

and

$$\sum_m \mu(B_{K,m}) \le \Big(\frac{c_3}{\beta}\Big)^K V(B).$$

If $Kc_4\beta \le \alpha \le (K+1)c_4\beta$ for all $K \in \mathbb{N}_+$, we have

$$\begin{aligned}\mu\{x \in B : |(f(x) - \mathcal{A}_{t_B}f(x))|w^{-1}(x) > \lambda\} &\le \sum_m \mu(B_{K,m}) \le \Big(\frac{c_3}{\beta}\Big)^K V(B)\\ &\le \sqrt{\beta}\exp\Big(-\frac{\alpha\log\beta}{4c_4\beta}\Big)V(B)\end{aligned}$$

provided $\beta > c_3^2$.

If $\alpha < c_4\beta$, then

$$\mu\{x \in B : |(f(x) - \mathcal{A}_{t_B}f(x))|w^{-1}(x) > \lambda\} \lesssim e^{-\frac{\alpha}{c_4\beta}} V(B).$$

Hence, this completes our proof. □

Proof of Theorem 3.5: For $f \in \mathrm{BMO}^p_{\mathcal{A}}(X,w)$, using Hölder's inequality, we have, for all balls B,

(14)
$$\begin{aligned}&\frac{1}{w(B)}\int_B |(I - \mathcal{A}_{t_B})f(x)|d\mu(x)\\ &\quad\le \frac{1}{w(B)}\Big(\int_B |(I - \mathcal{A}_{t_B})f(x)|^p w^{1-p}(x)d\mu(x)\Big)^{1/p}\Big(\frac{1}{w(B)}\int_B w(x)d\mu(x)\Big)^{1/p'}\\ &\quad\le \|f\|_{\mathrm{BMO}^p_{\mathcal{A}}(X,w)}.\end{aligned}$$

This implies that $\mathrm{BMO}^p_{\mathcal{A}}(X,w) \subset \mathrm{BMO}_{\mathcal{A}}(X,w)$.

Conversely, by Lemma 3.6, we have for any $f \in \mathrm{BMO}_{\mathcal{A}}(X,w)$, the ball B and $p \in [1,\infty)$,

$$\begin{aligned}
&\frac{1}{w(B)}\int_B |(I-\mathcal{A}_{t_B})f(x)|^p w^{1-p}(x)d\mu(x)\\
&\quad= \frac{1}{w(B)}\int_B |(I-\mathcal{A}_{t_B})f(x)w^{-1}(x)|^p w(x)d\mu(x)\\
&\quad= \frac{p}{w(B)}\int_0^\infty \lambda^{p-1} w\{x\in B: |(I-\mathcal{A}_{t_B})f(x)w^{-1}(x)|>\lambda\}d\lambda\\
&\quad\le c_p\frac{1}{w(B)}\int_0^\infty \lambda^{p-1}w(B)\exp\Big(-c_2\frac{\lambda}{\|f\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)}}\Big)d\lambda\\
&\quad\le c_p\|f\|^p_{\mathrm{BMO}_{\mathcal{A}}(X,w)}.
\end{aligned}$$

The proof is complete.

□

4. An Interpolation Theorem

In this section, we study the interpolation of the weighted BMO space $\mathrm{BMO}_{\mathcal{A}}(X,w)$ in general setting of spaces of homogeneous type. Firstly, We review the concept of the sharp maximal operator $M^\sharp_{\mathcal{A}}$ associated to the family $\{\mathcal{A}_t\}_{t>0}$ defined on $L^p(X), p>0$ as well as its basic properties in [15],

$$M^\sharp_{\mathcal{A}}f(x) = \sup_{x\in B}\Big(\frac{1}{\mu(B)}\int_B |(I-\mathcal{A}_{t_B})f(x)|d\mu(x)\Big),$$

where $t_B = r_B^m$.
We recall the following results in [15].

Theorem 4.1. *Let $0<p<\infty$ and $w\in A_\infty$. For every $f\in L^1_0(X)$ with $Mf\in L^p_w(X)$, we have*

(i) $M^\sharp_{\mathcal{A}}f(x)\le CMf(x)$.
(ii) $||Mf||_{L^p_w(X)} \le C||M^\sharp_{\mathcal{A}}f||_{L^p_w(X)}$ *if* $\mu(X)=\infty$.
(iii) $||Mf||_{L^p_w(X)} \le C(||M^\sharp_{\mathcal{A}}f||_{L^p_w(X)}+||f||_{L^1})$ *if* $\mu(X)<\infty$.

In what follows, the operator T is said to be bounded from wL^∞ to $BMO_{\mathcal{A}}(X,w)$ if there exists c such that for all $f\in L^\infty(X)$,

$$\|T(fw)\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)} \lesssim \|f\|_{L^\infty}.$$

We recall an interpolation theorem for the classical weighted BMO in [5].

Theorem 4.2. *Let T be a linear operator which is bounded on $L^2(\mathbb{R}^n)$. Assume that T and T^* are bounded from wL^∞ to $\mathrm{BMO}(X,w)$ for all $w\in A_1$. Then T is bounded on $L^p_w(\mathbb{R}^n)$ for all $1<p<\infty$ and $w\in A_p$.*

It is interesting that our weighted $\mathrm{BMO}_{\mathcal{A}}(X,w)$ can be considered as a good substitution the classical weighted BMO in the sense of interpolation. By adapting the arguments in [5] to our situation, we will establish an interpolation theorem concerning the our weighted BMO spaces $\mathrm{BMO}_{\mathcal{A}}(X,w)$ which generalizes Theorem 4.2 to the range of weights and to the weighted BMO spaces associated to the family $\{\mathcal{A}_t\}_{t>0}$.

Theorem 4.3. *Assume that T is a linear operator which is bounded on $L^2(X)$. Assume also that T is bounded from wL^∞ to $\mathrm{BMO}_{\mathcal{A}}(X,w)$ and T^* is bounded from wL^∞ to $\mathrm{BMO}_{\mathcal{A}^*}(X,w)$ for all $w \in A_1 \cap RH_s$ with $1 \le s < \infty$. Then T is bounded on $L^p_w(X)$ for all $s < p < \infty$, $w \in A_{p/s}$.*

Proof. For the sake of simplicity we assume that $\mu(X) = \infty$, the case that $\mu(X) < \infty$ can be treated in the same way. When $w \equiv 1$, the operator T is bounded from L^∞ to $\mathrm{BMO}_{\mathcal{A}}(X,w)$. Due to [11, Theorem 5.2], T is bounded on $L^p(X)$ for $p \in (2,\infty)$. By duality, one gets that T is bounded on $L^p(X)$ for $p \in (1,\infty)$.

Now for $w \in A_1 \cap RH_s$ and $f \in L^\infty(X)$, we have

$$\begin{aligned} w^{-1}(x) M^\sharp_{\mathcal{A}}(T^*(wf))(x) &= \sup_{B \ni x} \frac{1}{V(B)} \int_B |(I - \mathcal{A}_{t_B}) T^*(wf)(y)| d\mu(y) w^{-1}(x) \\ &\le \sup_{B \ni x} \frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B}) T^*(wf)(y)| d\mu(y) \\ &\le c\|T^*(wf)\|_{\mathrm{BMO}_{\mathcal{A}}(X,w)} \le c\|f\|_{L^\infty} \end{aligned}$$

for all $x \in X$. This implies that the operator $w^{-1}M^\sharp_{\mathcal{A},T^*w}$ is bounded on $L^\infty(X)$, where $M^\sharp_{\mathcal{A},T^*w}$ is defined by $M^\sharp_{\mathcal{A},T^*w} f = M^\sharp_{\mathcal{A}}(T^*(wf))$. On the other hand due to Proposition 4.1 and the L^2-boundedness of T^*, $M^\sharp_{\mathcal{A},T^*}$ is bounded on $L^2(X)$. This together with the interpolation, see for example [4], gives

$$u^{2/p-1} M^\sharp_{\mathcal{A}}(T^* u^{1-2/q}) : L^q \to L^q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

This implies

$$M^\sharp_{\mathcal{A},T^*} : L^q(w^{2-q}) \to L^q(w^{2-q}).$$

Using Theorem 4.1, we have

$$T^* : L^q(w^{2-q}) \to L^q(w^{2-q}).$$

Let $g \in L^q(w^{2-q})$ and $f \in L^p(w^{2-p})$. We have

$$\int_X |(Tf)g| d\mu = \int_X |f w^{1-2/q} (T^*g) w^{2/q-1}| d\mu \le \|T^*g\|_{L^q(w^{2-q})} \|f\|_{L^p(w^{2-p})}.$$

By duality, $T : L^p(w^{2-p}) \to L^p(w^{2-p})$, or, $w^{2/p-1} T w^{1-2/p} : L^p \to L^p$.

On the other hand, for $f \in L^p$ and $g \in L^q$, we have

$$\int_X |T(fw^{2/q-1}) w^{1-2/q} g| d\mu = \int_X |f \times w^{2/q-1} T^*(w^{1-2/q} g)| d\mu \le c\|f\|_{L^p} \|g\|_{L^q},$$

and hence $w^{1-2/q} T w^{2/q-1} : L^p \to L^p$.

Since we can interchange T and T^*, we can show that for $\frac{1}{p} + \frac{1}{q} = 1$, p near 1, and $w, v \in A_1$,

$$w^{1-2/q} T w^{2/q-1} : L^p \to L^p \text{ and } v^{2/q-1} T v^{1-2/q} : L^q \to L^q.$$

By interpolation, we obtain

$$w^{\alpha(t)} v^{\beta(t)} T(w^{-\alpha(t)} v^{-\beta(t)}) : L^{1/t} \to L^{1/t} \text{ for } \frac{1}{q} \le t \le \frac{1}{p}$$

for all $v, w \in A_1 \cap RH_s$, where $\alpha(t) = t - \frac{1}{q}$ and $\beta(t) = t - \frac{1}{p}$.

This gives $T : L^{p_0}(u) \to L^{p_0}(u)$ whenever

$$u = w^{p_0\alpha(1/p_0)}v^{p_0\beta(1/p_0)}, \quad w, v \in A_1 \cap RH_s \text{ and } p < p_0 < q. \tag{15}$$

Take $p_0 = (q + s) - qs/p$ and $r_0 = \frac{pq}{q-p}$. For any $u \in A_{p_0/s}$, by Jones Factorization, there exist $u_1, u_2 \in A_1$ such that $u = u_1u_2^{1-p_0/s}$, see [14]. Setting $u_1 = w_1^s$ and $u_2 = w_2^s$, then $w_1, w_2 \in A_1 \cap RH_s$. Hence, we can pick $\delta > 0$ so that $u_1^{1+\delta}, u_2^{1+\delta} \in A_1$. For p close enough to 1, $r_0 < 1 + \delta$ and hence $u_1^{r_0} = w_1^{r_0 s}, u_2^{r_0} = w_2^{r_0 s} \in A_1$. This implies $w_1^{r_0}, w_2^{r_0} \in A_1 \cap RH_s$. Due to (15), T is bounded on $L^{p_0}(v)$, here $v = (w_1^{r_0})^{p_0\alpha(1/p_0)}(w_2^{r_0})^{p_0\beta(1/p_0)} = w_1^s w_2^{s(1-p_0)} = u$. Applying Theorem 4.9 in [3], T is bounded on $L^p_w(X)$ for all $s < p < \infty$ and $w \in A_{p/s}$. This completes our proof. □

5. Applications to boundedness of singular integrals

Let X be a space of homogeneous type (X, d, μ). Let T be a bounded linear operator from $L^2(X)$ to $L^2(X)$ with kernel k such that for every f in $L^\infty(X)$ with bounded support,

$$Tf(x) = \int_X k(x, y)f(y)d\mu(y),$$

for μ-almost all $x \notin suppf$. We will consider the following conditions:

(H1) There exists a class of approximation to the identity $\{\mathcal{A}_t\}_{t>0}$ satisfying (3) such that the operators $(T-\mathcal{A}_tT)$ and $(T-T\mathcal{A}_t)$ have associated kernels $K_t^1(x, y)$ and $K_t^2(x, y)$ respectively and there exist positive constants α and c_1, c_2 such that

$$\max\{|K_t^2(x, y)|, |K_t^1(x, y)|\} \le c_2 \frac{1}{V(x, d(x, y))} \frac{t^{\alpha/m}}{d(x, y)^\alpha}$$

when $d(x, y) \ge c_1 t^{1/m}$.

(H2) There exists a class of approximation to the identity $\{\mathcal{A}_t\}_{t>0}$ satisfying (3) such that the operators $(T-\mathcal{A}_tT)$ and $(T-T\mathcal{A}_t)$ have associated kernels $K_t^1(x, y)$ and $K_t^2(x, y)$ so that there exist $1 < p_0 < \infty$ and $\delta > 0$ such that for any ball $B \subset X$ we have

$$\Big(\int_{S_j(B)} |K^i_{r_B^m}(z, y)|^{p_0} d\mu(y)\Big)^{1/p_0} \lesssim 2^{-j\delta} V(2^jB)^{1/p_0-1} \tag{16}$$

for all $z \in B$, all $j \ge 2$ and $i = 1, 2$.

It was proven in [7] that if T is an operator satisfying (H1) or (H2) above, then T bounded on $L^p(X)$ for $1 < p < 2$. Note that condition (H2) does not require the regularity assumption on space variables. This allows us to obtain L^p-boundedness of certain singular integrals with nonsmooth kernels such as the holomorphic functional calculi and spectral multipliers of L, see Subsections 5.1 and 5.2.

We now prove the following theorems:

Theorem 5.1. *Let T be an operator satisfying (H1). Then for any $w \in A_1$, T and T^* are bounded from $wL^\infty(X)$ to $\mathrm{BMO}_{\mathcal{A}}(X, w)$ and from $wL^\infty(X)$ to $\mathrm{BMO}_{\mathcal{A}^*}(X)$. Then, by interpolation, T is bounded on $L^p_w(X)$ for all $p \in (1, \infty)$ and $w \in A_p$.*

Proof. For $f \in L^\infty$, we claim that

$$\frac{1}{w(B)}\int_B |(I-\mathcal{A}_{t_B})T(fw)(x)|d\mu(x) \le C\|f\|_{L^\infty}$$

for any ball $B \subset X$.

Set $f = f_1 + f_2$ where $f_1 = f\chi_{cB}$ with $c = \max\{c_1, 4\}$. We have

$$\begin{aligned}\frac{1}{w(B)}\int_B |(I-\mathcal{A}_{t_B})T(fw)(x)|d\mu(x) &\le \frac{1}{w(B)}\int_B |(I-\mathcal{A}_{t_B})T(f_1w)(x)|d\mu(x)\\ &\quad + \frac{1}{w(B)}\int_B |(I-\mathcal{A}_{t_B})T(f_2w)(x)|d\mu(x)\\ &= I_1 + I_2.\end{aligned}$$

Let us estimate I_1 first. Since $w \in A_1$ then there exists $r > 1$ such that $w \in RH_r$. Using the L^p boundedness of T and the Hardy-Littlewood maximal function, we have

$$\begin{aligned}I_1 &\le c\frac{1}{w(B)}\int_B M(T(f_1w))(x)d\mu(x)\\ &\le c\frac{1}{w(B)}\|T(f_1w)\|_{L^r}V(B)^{1/r'}\\ &\le c\|f\|_{L^\infty}\frac{1}{w(B)}\Big(\int_{cB} w^r(x)d\mu(x)\Big)^{1/r}V(B)^{1/r'}\\ &\le c\|f\|_{L^\infty}\frac{1}{w(B)}\frac{w(B)}{V(B)}V(B)^{1/r}V(B)^{1/r'} = c\|f\|_{L^\infty}.\end{aligned}$$

For the second term, by (b) we have

$$\begin{aligned}I_2 &\le \frac{1}{w(B)}\int_B\int_{X\backslash cB} K^1_{t_B}(x,y)(f_2w)(y)|d\mu(y)d\mu(x)\\ &\le \frac{1}{w(B)}\int_B\int_{X\backslash cB}\frac{1}{V(x,d(x,y))}\frac{r_B^\alpha}{d(x,y)^\alpha}(f_2w)(y)|d\mu(y)d\mu(x)\\ &\le c\|f\|_{L^\infty}\frac{1}{w(B)}\int_B\int_{X\backslash cB}\frac{1}{V(x,d(x,y))}\frac{r_B^\alpha}{d(x,y)^\alpha}w(y)|d\mu(y)d\mu(x).\end{aligned}$$

Since $c > 4$, we have

$$\begin{aligned}I_2 &\le c\|f\|_{L^\infty}\sum_{j\ge 2}\frac{1}{w(B)}\int_B\int_{S_j(B)}\frac{1}{V(x,d(x,y))}\frac{r_B^\alpha}{d(x,y)^\alpha}w(y)|d\mu(y)d\mu(x)\\ &\le c\|f\|_{L^\infty}\sum_{j\ge 2}2^{-j\alpha}\frac{V(B)}{w(B)}\int_B\frac{w(2^jB)}{V(2^jB)}\\ &\le c\|f\|_{L^\infty}.\end{aligned}$$

The boundedness of T^* can be treated similarly. This completes our proof. □

Theorem 5.2. *Let T be an operator satisfying (H2). Then for any $w \in A_1 \cap RH_{p_0'}$, T and T^* are bounded from $wL^\infty(X)$ to $\mathrm{BMO}_{\mathcal{A}}(X,w)$ and from $wL^\infty(X)$ to $\mathrm{BMO}_{\mathcal{A}^*}(X)$. Then, by interpolation, T is bounded on $L^p_w(X)$ for all $p \in (p_0', \infty)$ and $w \in A_{p/p_0'}$.*

Proof. For $f \in L^\infty$ and $w \in A_1 \cap RH_{p_0'}$, we will claim that

$$\frac{1}{w(B)} \int_B |(I - A_{t_B})T(fw)(x)| d\mu(x) \le C\|f\|_{L^\infty}$$

for balls $B \subset X$.

Using the decomposition $f = \sum_{j\ge 2} f_j + f_0$ where $f_0 = f\chi_{2B}$ and $f_j = f\chi_{S_j(B)}$, We have

$$\begin{aligned}\frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})T(fw)(x)| d\mu(x) &\le \frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})T(f_0 w)(x)| d\mu(x)\\ &\quad + \sum_{j\ge 2} \frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})T(f_j w)(x)| d\mu(x)\\ &= I_0 + \sum_{j\ge 2} I_j.\end{aligned}$$

Since $w \in RH_{p_0'}$, using the L^p boundedness of T and the Hardy-Littlewood maximal function, we have

$$\begin{aligned}I_0 &\lesssim \frac{1}{w(B)} \int_B M(T(f_0 w))(x) d\mu(x)\\ &\lesssim \frac{1}{w(B)} \|T(f_1 w)\|_{L^{p_0'}} V(B)^{1/p_0}\\ &\lesssim \|f\|_{L^\infty} \frac{1}{w(B)} \Big(\int_{2B} w^{p_0'}(x) d\mu(x)\Big)^{1/p_0'} V(B)^{1/p_0}\\ &\lesssim \|f\|_{L^\infty} \frac{1}{w(B)} \frac{w(B)}{V(B)} V(B)^{1/p_0} V(B)^{1/p_0'} = c\|f\|_{L^\infty}.\end{aligned}$$

For $j \ge 2$, by (H2) and Hölder's inequality, we have

$$\begin{aligned}I_j &\le \frac{1}{w(B)} \int_B \int_{S_j(B)} |K^1_{t_B}(x,y)(f_j w)(y)| d\mu(y) d\mu(x)\\ &\le \frac{1}{w(B)} \int_B \Big(\int_{S_j(B)} |K^1_{t_B}(x,y)|^{p_0} d\mu(y)\Big)^{1/p_0} \Big(\int_{S_j(B)} |f_j(y)w(y)|^{p_0'} d\mu(y)\Big)^{1/p_0'} d\mu(x)\\ &\lesssim \frac{V(B)}{w(B)} 2^{-j\delta} V(2^j B)^{1/p_0 - 1} \|f\|_{L^\infty} \Big(\int_{2^j B)} |w(y)|^{p_0'} d\mu(y)\Big)^{1/p_0'}\\ &\lesssim \frac{V(B)}{w(B)} 2^{-j\delta} V(2^j B)^{1/p_0 - 1} \|f\|_{L^\infty} \frac{w(2^j B)}{V(2^j B)} V(2^j B)^{1/p_0'}\\ &\lesssim 2^{-j\delta} \frac{V(B)}{w(B)} \frac{w(2^j B)}{V(2^j B)} \|f\|_{L^\infty}.\end{aligned}$$

Since $w \in A_1$,

$$\frac{V(B)}{w(B)} \frac{w(2^j B)}{V(2^j B)} \le C.$$

Therefore,

$$\sum_{j\ge 2} I_j \lesssim \sum_j 2^{-j\delta} \|f\|_{L^\infty} \lesssim \|f\|_{L^\infty}$$

provided $\delta > 0$.

This yields that T is bounded from $wL^\infty(X)$ to $\mathrm{BMO}_{\mathcal{A}}(X,w)$. The boundedness of T^* can be treated similarly. This completes our proof. □

5.1. **Holomorphic functional calculi.** We now give some preliminary definitions of holomorphic functional calculi as introduced by A. McIntosh [16]. Let $0 \le \nu < \pi$. We define the closed sector in the complex plane $\mathbb{C}$

$$S_\nu = \{z \in \mathbb{C} : |\arg z| \le \nu\}$$

and denote the interior of S_ν by S^0_ν.

Let $H(S^0_\nu)$ be the space of all holomorphic functions on S^0_ν. We define the following subspaces of $H(S^0_\nu)$:

$$H_\infty(S^0_\nu) = \{b \in H(S^0_\nu) : ||b||_\infty < \infty\},$$

where $||b||_\infty = \sup\{|b(z)| : z \in S^0_\nu\}$, and

$$\Psi(S^0_\nu) = \{\psi \in H(S^0_\nu) : \exists s > 0, |\psi(z)| \le c|z|^s(1 + |z|^{2s+1})^{-1}\}.$$

Let L be a linear operator of type ω on $L^2(X)$ with $\omega < \pi/2$; hence L generates a holomorphic semigroup $e^{-zL}, 0 \le |Arg(z)| < \pi/2 - \omega$. Assume the following two conditions.

Assumption (a): The holomorphic semigroup $e^{-zL}, 0 \le |Arg(z)| < \pi/2 - \omega$, is represented by the kernel $p_z(x,y)$ which satisfies the Gaussian upper bound

$$|p_z(x,y)| \le c_\theta \frac{1}{V(x,|z|^{1/m})} \exp\Big(-\frac{d(x,y)^{m/(m-1)}}{c|z|^{1/(m-1)}}\Big) \tag{17}$$

for $x, y \in X, |Arg(z)| < \pi/2 - \theta$ for $\theta > \omega$.

Assumption (b): The operator L has a bounded H_∞-calculus on $L^2(X)$. That is, there exists $c_{\nu,2} > 0$ such that $b(L) \in \mathcal{L}(L^2, L^2)$, and for $b \in H_\infty(S^0_\nu)$,

$$||b(L)f||_2 \le c_{\nu,2}||b||_\infty||f||_2$$

for any $f \in L^2(X)$.

We have the following result.

Theorem 5.3. *Let L satisfy conditions (a) and (b) and let $f \in H_\infty(S^0_\nu)$. Then for any $w \in A_1$, $f(L)$ and $[f(L)]^*$ are bounded from $wL^\infty(X)$ to $\mathrm{BMO}_{\mathcal{A}}(X,w)$ and from $wL^\infty(X)$ to $\mathrm{BMO}_{\mathcal{A}^*}(X)$ where $\mathcal{A}_t = e^{-tL}$. Then, by interpolation, $g(L)$ is bounded on $L^p_w(X)$ for all $p \in (1,\infty)$ and $w \in A_p$.*

Note that in the similar condition, it was proved in [7] that the functional calculus $f(L)$ is of weak type $(1,1)$ and hence bounded on $L^p(X)$ for all $1 < p < \infty$. Moreover, the weighted estimates for the functional calculus $f(L)$ were investigated in [15] in which the author proved that $f(L)$ is bounded on $L^p_w(X)$ for all $1 < p < \infty$ and $w \in A_p$. Here, in Theorem 5.3, we prove the weighted endpoint estimates for the functional calculus $f(L)$ and then by the interpolation theorem we regain the weighted estimates for $f(L)$.

Proof. By the convergence lemma in [16], we can assume that $f \in \Psi(S^0_\nu)$. Then, it was proved in [7] that $g(L)$ and $[g(L)]^*$ satisfy **(H1)** with $\mathcal{A}_t = e^{-tL}$. Hence, the desired result follows directly from Theorem 5.1. □

5.2. **Spectral multipliers.** Let L be a non-negative self-adjoint operator on $L^2(X)$ and the operator L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ whose kernels $p_t(x,y)$ satisfies Gaussian upper bound:

$$|p_t(x,y)| \leq \frac{C}{V(x,t^{1/m})}\exp\Big(-c\frac{d(x,y)^{m/(m-1)}}{t^{1/(m-1)}}\Big) \tag{18}$$

for all $x,y\in X$ and $t>0$.

By the spectral theorem, for any bounded Borel function $F:[0,\infty)\to\mathbb{C}$, one can define the operator

$$F(L)=\int_0^\infty F(\lambda)dE(\lambda) \tag{19}$$

which is bounded on $L^2(X)$. We have the following result.

Theorem 5.4. *Let L be a non-negative self-adjoint operator satisfying (GE). Suppose that $n>s>n/2$ and for any $R>0$ and all Borel functions F such that* $\mathrm{supp}F\subset[0,R]$,

$$\int_X |K_{F(\sqrt[m]{L})}(x,y)|^2 d\mu(x) \leq \frac{C}{V(y,R^{-1})}\|\delta_R F\|^2_{L^q} \tag{20}$$

for some $q\in[2,\infty]$. Then for any Borel function F such that

$$\sup_{t>0}\|\eta\delta_t F\|_{W^q_s}<\infty,$$

where $\delta_tF(\lambda)=F(t\lambda)$, $\|F\|_{W^q_s}=\|(I-d^2/dx^2)^{s/2}F\|_{L^q}$, the operator $F(L)$ and $F(L)^*=\overline{F}(L)$ is bounded from wL^∞ to $\mathrm{BMO}_{\mathcal{A}}(X,w)$ for all $w\in A_1\cap RH_{r_0'}$, where $\mathcal{A}_t=I-(I-e^{-tL})^M$ for $M>\frac{s}{m}$ and $r_0=n/s$. Hence by interpolation, $F(L)$ is bounded on $L^p_w(X)$ for $w\in A_{p/r_0}$ and $p\in(r_0,\infty)$.

Note that under the condition as in Theorem 5.4, it was proved in [8] that the spectral multiplier $F(L)$ is of weak type $(1,1)$ and hence bounded on $L^p_w(X)$. The weighted estimates for $F(L)$ was studied in [1, 10]. The main contribution in Theorem 5.4 is the weighted endpoint estimates for the spectral multipliers $F(L)$.

Proof. From the proof of Theorem 4.5 in [1], we get that **(H2)** holds for $T:=F(L)$ and the family $\mathcal{A}_t:=I-(I-e^{-tL})^M$ for $M>\frac{s}{m}$ and all $p_0<r_0'$. Hence, using Theorem 5.2 and letting $p_0\to r_0'$, we get the desired result. □

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