GLOBAL MIRROR CURVE AND ITS IMPLICATION

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ABSTRACT. The remodeling conjecture recasts all genus Gromov-Witten theory of a toric Calabi-Yau 3-fold in terms of complex geometry of its mirror curve. We illustrate how to construct a family of mirror curves, over the global moduli of the toric Calabi-Yau 3-fold’s stringy Kähler moduli space. With this construction, the remodeling conjecture then reveals many properties of the Gromov-Witten invariants, such as the modularity and the crepant transformation property.

1. INTRODUCTION

1.1. Mirror symmetry for a toric Calabi-Yau 3-orbifold. Let $X$ be a toric Calabi-Yau 3-orbifold. The mirror symmetry predicts its Gromov-Witten invariants from its mirror B-model. Usually the mirror of $X$ is a non-compact Calabi-Yau hypersurface, which can be further reduced to an affine curve in $(\mathbb{C}^*)^2$, called the mirror curve. In this survey we only consider the mirror curve as its B-model.

The mirror B-model of $X$ predicts both closed and open Gromov-Witten invariants [2, 3, 6–8, 16]. In [16], the B-model for the closed higher genus invariants is from the BCOV holomorphic anomaly equation [5]. The Bouchard-Klemm-Mariño-Pasquetti’s remodeling conjecture [6, 7, 17] predicts all genus open-closed Gromov-Witten invariants from another viewpoint on the B-model, the Eynard-Orantin’s topological recursion [9]. This prediction from the topological recursion is called the remodeling conjecture. This conjecture is proved later in [10, 13, 14].

More precisely, there is a certain type of Lagrangian submanifolds, the Aganagic-Vafa branes in $X$. In case such branes are not gerby, they are all homeomorphic to $S^1 \times \mathbb{R}^2$. We fix such a Lagrangian $L \subset X$, and consider the open Gromov-Witten potential

$$F_{g,n}^X(L; \tau; \tilde{X}_1, \ldots, \tilde{X}_n),$$

which is a generating function parametrizing the number of holomorphic maps from a genus $g$ bordered Riemann surface with $n$
boundary components to $\mathcal{X}$ while the boundary lands on $\mathcal{L}$. The Kähler parameter $\tau$ records the extended Kähler class of $\mathcal{X}$. In case that $\mathcal{X}$ is a smooth manifold, by the divisor equation the power of $e^{\tau}$ records the homology class of the image of this map, while the power of $\tilde{X}_1$ records the winding number of each boundary component into $\mathcal{L} \cong \mathbb{S}^1 \times \mathbb{R}^2$.

The mirror curve of $\mathcal{X}$ is an affine curve $C_q = \{ H_q(X,Y) = 0 \}$ in $(\mathbb{C}^*)^2$. Here $H_q(X,Y) = 0$ is the equation for $C_q$. The conjecture of Aganagic-Klemm-Vafa [2, 3] predicts

$$F_{X,\mathcal{L}}^{X,\mathcal{L}}(\tau; \tilde{X}) = \int_X \log Y \frac{dX}{X},$$

under certain explicit open-closed mirror map

$$\tau = \log q + O(q), \quad \tilde{X} = X(1 + O(q)).$$

One should understand this integral as anti-derivative and $\log Y$ is a function of $X$ near a particular point on $\tilde{C}_q$ with $X = 0$, where $\tilde{C}_q$ is a compactification of $C_q$.

The Eynard-Orantin’s topological recursion starts from a choice of Lagrangian subspace of $H^1(\tilde{C}_q; \mathbb{C})$ where the symplectic pairing is the cohomology pairing $(\alpha, \beta) \rightarrow \int_{\tilde{C}_q} \alpha \cup \beta$. Then one can recursively and uniquely constructs a meromorphic and symmetric $n$-form $\omega_{g,n}$ on $(\tilde{C}_q)^n$. Then the BKMP remodeling conjecture says under the same open-closed mirror map

$$F_{g,n}^{X,\mathcal{L}}(\tau; \tilde{X}_1, \ldots, \tilde{X}_n) = \int_{X_1} \ldots \int_{X_n} \omega_{g,n}.$$  

1.2. **String Kähler moduli and global mirror symmetry.** The topological recursion on the mirror curve as the B-model automatically carries many interesting properties. For example, the modularity of the recursion algorithm was already addressed in [9] when such algorithm was proposed.

There are many “phases” of A-model theories. If $\mathcal{X}$ is a smooth manifold, then the Gromov-Witten theory of $\mathcal{X}$ is a *theory at a large radius limit*. In general there are many limit points on the stringy Kähler moduli space $M_K$ of $\mathcal{X}$. When $\mathcal{X}$ is a toric Calabi-Yau 3-fold, $M_K$ can be identified with its secondary toric variety. Around each torus fixed point $s_i$ of $M_K$, we can associate a toric Calabi-Yau 3-orbifold $\mathcal{X}_i$ depending on the GIT stability condition. To one of these torus fixed point $s_0$, $\mathcal{X}_0 = \mathcal{X}$ itself, while at other points they
are McKay equivalent toric Calabi-Yau 3-orbifolds, related by being a partial crepant resolution pairs of a same singular toric variety.

The A-model theory around each torus fixed point $s_i$ in $\mathcal{M}_K$ is the orbifold Gromov-Witten theory of the toric Calabi-Yau 3-fold $\mathcal{X}_i$. \textit{A priori} there is no reason these theories about $\mathcal{X}_i$ could patch together globally over $\mathcal{M}_K$. However this desired global behavior is more accessible from the B-model.

The mirror B-model considered in this paper is an affine curve $C_q$ together with its compactification $\bar{C}_q$, where $q$ is the complex parameter. We will see that $q \in \mathcal{M}_K$ and construct a family of mirror curves $\mathcal{C}$, inside a family of toric surfaces $S_q$ over $\mathcal{M}_K$. At each $q$ the fiber of this family is indeed $\bar{C}_q$ while $C_q = \bar{C}_q \setminus (\partial S_q)$.

The global mirror curve $\mathcal{C}$ implies the existence of a global B-model over the stringy Kähler moduli space $\mathcal{M}_K$. The modularity of the B-model generating function is automatic given such a global mirror curve $\mathcal{C}$. The B-model theory near each limit point $s_i$ are related by analytic continuation, since at every point in $\mathcal{M}_K$ the B-model theory is well-defined. Translated back into the A-model Gromov-Witten theory, one obtains the modularity of the Gromov-Witten theory and the crepant resolution conjecture.

1.3. \textbf{The structure of this paper.} We will illustrate the construction of $\mathcal{C}$ and discuss its implication by a main example $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$. In Section 2 we state the BKMP remodeling conjecture for both $\mathcal{X}$ and its orbifold phase $\mathcal{X}' = \mathbb{C}^3/\mathbb{Z}_3$. Then we explain how to construct a global mirror curve $\mathcal{C}$ for this example in Section 3, and we will also explain the crepant resolution conjecture and the modularity of the Gromov-Witten theory from mirror symmetry.

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$M = N^\vee$. We also choose $e_3^\vee = (0, 0, 1) \in M$ and let $M' = M/\langle e_3^\vee \rangle \cong \mathbb{Z}^2$
and $N' = \ker(e_3^\vee) \subset N$.

For $X = \mathcal{O}_{\mathbb{P}^2}(-3)$, let

$$P = \text{Conv}((0, 0), (3, -1), (0, 1)) \subset N',$$

and its triangulation is given in the Figure 1.

The 1-(resp. 2-, 3-)cones in the fan data of $X$ are cones from the origin in $N_\mathbb{R}$ over vertices (resp. edges, faces) of the triangulated $P \times \{1\} \subset N'_\mathbb{R} \times \{1\} \subset N_\mathbb{R}$. We write down the generators of 1-cones here:

$$b_1 = (0, 0, 1), \quad b_2 = (1, 0, 1), \quad b_3 = (0, 1, 1), \quad b_4 = (3, -1, 1).$$

By toric geometry, the fan data prescribes a torus action $G \cong \mathbb{C}^*$ on $\mathbb{C}^4$:

$$t \cdot (Z_1, Z_2, Z_3, Z_4) = (tZ_1, tZ_2, tZ_3, t^{-3}Z_4).$$

The smooth variety $X$ is defined as the following quotient

$$X = (\mathbb{C}^4 \setminus ((0, 0, 0) \times \mathbb{C})) / G.$$

The moment map $\tilde{\mu}$ for action of the maximal compact subgroup $G_\mathbb{R}$ is

$$(Z_1, Z_2, Z_3, Z_4) \mapsto |Z_1|^2 + |Z_2|^2 + |Z_3|^2 - 3|Z_4|^2.$$

Then $X$ is also obtained as a symplectic quotient

$$\mathcal{X} = \tilde{\mu}^{-1}(r)/G_\mathbb{R}, \quad r > 0.$$

The parameter $r$ is the Kähler parameter, which is the symplectic area of the base $\mathbb{P}^2$.

We denote the 3-dimensional torus $T = N \otimes_\mathbb{Z} \mathbb{C}^*$, which acts on and is also open and dense in $\mathcal{X}$. Let $T'$ be the 2-dimensional subtorus which acts trivially on its canonical bundle. Let $T'_\mathbb{R}$ the maximal compact subgroup of $T'$ one may consider its moment map $\mu' : \mathcal{X} \to M'_\mathbb{R} \cong \mathbb{R}^2$. The one-dimensional $T'$-invariant subvariety $\mathcal{X}^1$ of $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$ is the union of three $\mathbb{P}^1$ in $\mathbb{P}^2$ and the fibers over three
The toric graph of $\mathcal{X}$ and the image of an outer Aganagic-Vafa brane $\mathcal{L}$.

torus fixed point in $\mathbb{P}^2$. The image of $\mathcal{X}^1$ under $\mu'$ is the toric graph, shown in Figure 2.

An Aganagic-Vafa brane is an Lagrangian submanifold in the pre-image of a non-vertex point in the toric graph. Precisely, we define an Aganagic-Vafa brane $\mathcal{L}$ below.

$$|Z_1|^2 - |Z_2|^2 = |Z_3|^2 = c > 0, \quad \text{Arg}(Z_1 \ldots Z_4) = \text{const.}$$

This Lagrangian brane $\mathcal{L}$ is homeomorphic to $\mathbb{R}^2 \times S^1$. It is outer since its image under $\mu'$ is the point $a$ on a non-compact leg of the toric graph, which is also illustrated in Figure 2. We label the unique $\mathbb{T}$-fixed point on the 1-dimensional $\mathbb{T}$-invariant subvariety that $\mathcal{L}$ intersects by $p_0$. Let $\iota_0 : p_0 \hookrightarrow \mathcal{X}$ be the embedding.

### 2.2. The Gromov-Witten theory of $\mathcal{X}$

We define the closed Gromov-Witten primary correlators where $\gamma_1, \ldots, \gamma_n \in H^*(\mathcal{X}; \mathbb{C})$

$$\langle \gamma_1, \ldots, \gamma_n \rangle^{\mathcal{X}}_{g,n,\beta} = \int_{[\bar{M}_{g,n}(\mathcal{X};\beta)]^{\text{vir}}} \text{ev}_1^*\gamma_1 \cup \cdots \cup \text{ev}_n^*\gamma_n.$$

Here $\bar{M}_{g,n}(\mathcal{X};\beta)$ is the moduli space of stable maps from genus $g$, $n$-marked points to $\mathcal{X}$ in homology class $\beta \in H_2(\mathcal{X}; \mathbb{Z}) \cong \mathbb{Z}$, $[\bar{M}_{g,n}(\mathcal{X};\beta)]^{\text{vir}}$ is its virtual fundamental class, and $\text{ev}_i$ is the $i$-th evaluation map. Similarly the notion $\langle \gamma_1, \ldots, \gamma_n \rangle^{\mathcal{X},\mathbb{T}}_{g,n,\beta}$ is for the equivariant Gromov-Witten theory where $\gamma_i \in H^*_{\mathbb{T}}(\mathcal{X}; \mathbb{C})$. Replacing $\mathbb{T}$ by other groups.
acting on \( \mathcal{X} \) is self-evident. In the rest of this section, we only fix notations in the non-equivariant setting while the equivariant invariants are completely parallel. The descendant correlators are

\[
\langle \tau_{a_1}(\gamma_1), \ldots, \tau_{a_n}(\gamma_n) \rangle_{g,n,\beta}^{\mathcal{X}} = \langle \gamma_1 \psi_{a_1}^{a_1}, \ldots, \gamma_n \psi_n^{a_n} \rangle_{g,n,\beta}^{\mathcal{X}}
\]

\[
= \int_{(\bar{\mathcal{M}}_{g,n}(X;\beta))^{\vir}} \text{ev}_1^{*} \gamma_1 \cup \psi_{a_1}^{a_1} \cup \cdots \cup \text{ev}_n^{*} \gamma_n \cup \psi_n^{a_n}.
\]

The psi-class \( \psi_i \) is the first Chern-class of the \( i \)-th tautological line bundle on \( \bar{\mathcal{M}}_{g,n}(X;\beta) \).

The double brackets are

\[
\langle \gamma_1 \psi_{a_1}^{a_1}, \ldots, \gamma_n \psi_n^{a_n} \rangle_{g,n,\beta}^{\mathcal{X}} = \sum_{\beta \geq 0, \ell \geq 0} \frac{1}{\ell!} \langle \gamma_1 \psi_{a_1}^{a_1}, \ldots, \gamma_n \psi_n^{a_n}, \tau, \ldots, \tau \rangle_{g,n+n+\ell,\beta}^{\mathcal{X}}.
\]

So whenever double brackets appear they are functions of \( \tau \in H^2(\mathcal{X};\mathbb{C}) \).

In the equivariant setting the notion \( \langle \ldots \rangle_{g,n}^{\mathcal{X},T} \) is a function of \( \tau \in H^2_T(\mathcal{X};\mathbb{C}) \). Here we do not need to introduce Novikov variables to deal with the issue of convergence (see [14, Remark 3.2]).

We define the genus \( g \) free energy of \( \mathcal{X} \) as

\[
F_{\mathcal{X}}^g(Q) = \langle \rangle_{g,0}^{\mathcal{X}}.
\]

This is a power series in \( Q = e^\tau \).

Open GW invariants for \((\mathcal{X}, \mathcal{L})\) count holomorphic maps

\[
u: (\Sigma, x_1, \ldots, x_\ell, \partial \Sigma) = \bigsqcup_{j=1}^n R_j \rightarrow (\mathcal{X}, \mathcal{L})
\]

where \( \Sigma \) is a bordered Riemann surface with interior marked points \( x_i \) and \( R_j \cong S^1 \) are connected components of \( \partial \Sigma \). These invariants depend on the following data:

- the topological type \((g, n)\) of the coarse moduli of the domain, where \( g \) is the genus of \( \Sigma \) and \( n \) is the number of connected components of \( \partial \Sigma \),
- the degree \( \beta' = u_* [\Sigma] \in H_2(\mathcal{X}, \mathcal{L}; \mathbb{Z}) \),
- the winding numbers \( \mu_1, \ldots, \mu_n \in H_1(\mathcal{L}; \mathbb{Z}) \cong \mathbb{Z}, \)
- the framing \( f \in \mathbb{Z} \) of \( \mathcal{L} \).

We call the pair \((\mathcal{L}, f)\) a framed Aganagic-Vafa Lagrangian brane. We write \( \bar{\mu} = (\mu_1, \ldots, \mu_n) \). Let \( \bar{\mathcal{M}}_{(g,n),\ell}(\mathcal{X}, \mathcal{L} | \beta', \bar{\mu}) \) be the compactified moduli space parameterizing stable maps described above. Evaluation at the \( i \)-th marked point \( x_i \) gives a map \( \text{ev}_i: \bar{\mathcal{M}}_{(g,n),\ell}(\mathcal{X}, \mathcal{L} | \beta', \bar{\mu}) \rightarrow \mathcal{X} \).
The framing \( f \) specifies a subtorus \( T'_f = \ker((0, 1) - f(1, 0)) \), where \((0, 1), (1, 0) \in M' \) are characters for \( T' \). For \( \gamma_1, \ldots, \gamma_n \in H^*_T(X; \mathbb{C}) \), we define by localization

\[
\langle \gamma_1, \ldots, \gamma_n \rangle_{g, \beta, \mu}^{X, (L, f)} := \int_{[\hat{X}(q; n), \mu]} \prod_{i=1}^{g} \ev^*_i \gamma_i \text{vir} \in \mathbb{C} \Lambda_{q, \beta, \mu}^{\ell} \]

where \( T_R \) and \( (\hat{T}_R) \) are the corresponding real sub-torus of \( T' \) and \( T_f \) that preserves the Lagrangian \( L, H^*_T(\text{pt}) = \mathbb{C}[v], \beta \in H_2(X; \mathbb{Z}) \cong \mathbb{Z} \) and \( \beta' = \beta + \sum \mu_i \in H_2(X; \mathbb{Z}) \).

Then we define the open Gromov-Witten potential by

\[
P_{g, \mu}^{X, (L, f)}(\tau; \hat{X}_1, \ldots, \hat{X}_n) = \sum_{\beta = \mu}^{\beta' > 0} \sum_{\ell > 0} \frac{\langle \tau^\ell \rangle_{g, \beta, \mu}^{X, (L, f)}}{\ell!} \hat{X}_1^{\mu_1} \ldots \hat{X}_n^{\mu_n}.
\]

This potential does depend on the choice of \( f \), and is in degree 0 of \( v \). Mirror symmetry predicts these \( P_{g, \mu}^{X, (L, f)} \) from the mirror curve of \( X \). The free energy \( F_{g}^{X} \) is the special case for \( n = 0 \), involves only closed invariants and does not depend on \( f \).

2.3. Mirror curve as the B-model. The mirror curve of \( X \) is the following

\[
\{U_1U_2U_3U_4 = q, U_1 + U_2 + U_3 + U_4 = 0\}/\mathbb{C}^*.
\]

Here \( q \) is the complex parameter. The overall \( \mathbb{C}^* \) action rescales \( U_1, \ldots, U_4 \) simultaneously.

One can rewrite the mirror curve in an equation

\[
C_q = \{H_q(X, Y) = X + Y + 1 + qX^3Y^{-1} = 0\} \subset (\mathbb{C}^*)^2.
\]

We will see that these specific choice of coordinates are related to the phase (location) of \( L \). Each term of \( H_q \) corresponds to an integer point in the defining polytope \( P \).

When \( |q| \) is small, the curve \( C_q \) is a genus 1 curve with three punctures (see Figure 3). The affine curve \( C_q \) allows a natural compactification \( \bar{C}_q \) in \( S_P \), the toric surface associated to the defining polytope. In this particular example \( S_P \cong \mathbb{P}^2/\mathbb{Z}_3 \), which is a singular toric Fano surface. The curve \( \bar{C}_q \) is a compact Riemann surface of genus 1. There are three puncture points in \( C_q \), which are the intersection \( \bar{C}_q \cap S_P \). When \( q = 0 \), \( \bar{C}_q \) degenerates into a compact nodal curve \( \bar{C}_0 \), while the curve \( C_q \) also denegerates into a nodal curve \( C_0 \subset \bar{C}_0 \). We
FIGURE 3. The mirror curve $C_q$ of $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$.

denote a very small ball $B$ containing 0 such that when $q \in B \setminus \{0\}$ the curve $C_q$ and $\bar{C}_q$ are smooth.

One of the punctures in $C_q$ for $q \in B$ has coordinates $(X, Y) = (0, -1)$. We label this point as a open large radius limit $x_q$, which corresponds to the large radius in the open parameter.

The choice of framing in the A-model is interpreted of changing coordinates in the B-model. Let

$$X = \hat{X}\hat{Y}, \ Y = \hat{Y}.$$ 

The mirror curve equation becomes

$$H_q = \hat{X}\hat{Y} - f + \hat{Y} + 1 + q\hat{X}^3\hat{Y}^{1-3f}.$$ 

We introduce the Seiberg-Witten form

$$\Phi = \log \frac{\hat{Y}d\hat{X}}{\hat{X}}.$$ 

This form is multi-valued – it is well-defined on the universal cover of $C_q$. The B-model genus 0 disk potential is defined as

$$\mathcal{F}_{0,1}(X; q) = \int_X \log \frac{\hat{Y}d\hat{X}}{\hat{X}}.$$ 

Let $U_q$ be a small neighborhood of $x_q$ in $\bar{C}_q$. We should understand this integral as an anti-derivative, where $\log \hat{Y}$ is expanded in terms of $\hat{X}$ in $U_q$. One should discard the anti-derivative from the degree 0 term in the expansion of $\log \hat{Y}$, and obtain a power series in $\hat{X}$ with no degree 0 term. There is a constant ambiguity while taking $\log$ – however it is in the discarded part and does not contribute. We use the notation $\overset{\sim}{=}$ to denote that the degree-0 term in $\hat{X}$ is discarded.
The higher genus B-model theory is defined via the Eynard-Orantin topological recursion. It starts from a spectral curve, which contains the following data

- an affine curve $C_q$ and its compactification $\bar{C}_q$;
- two holomorphic Morse functions $\hat{X}$ and $\hat{Y}$ on $C_q$ and meromorphic on $\bar{C}_q$ – their critical points do not coincide;
- a fundamental bidifferential form $\omega_{0,2}$ (Bergman kernel), which is a meromorphic symmetric form on $\bar{C}_q$.

We explain $\omega_{0,2}$ a little bit here. It is uniquely determined by a Lagrangian subspace $A \subset H_1(\bar{C}_q; \mathbb{C})$. The symplectic pairing on this space is the cohomology pairing (PD is Poincaré pairing)

$$(a, b) = \int_{\bar{C}_q} \text{PD}(a) \cup \text{PD}(b).$$

The fundamental form $\omega_{0,2}$ is uniquely characterized by $A$ and its pole behavior:

- For any cycle $A \in A$,
  $$\int_{z \in A} \omega_{0,2}(z_1, z_2) = 0.$$
- The only pole of $\omega_{0,2}$ is the double pole at the diagonal and normalized at
  $$\omega_{0,2} = \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{holomorphic part}.$$

We let $X = e^{-x}, Y = e^{-y}, \hat{X} = e^{-\hat{x}}, \hat{Y} = e^{-\hat{y}}$. Near each ramification point of $\hat{x}$, we denote $\bar{p}$ to be the point such that $\hat{x}(\bar{p}) = \hat{x}(p)$ and $\bar{p} \neq p$.

The Eynard-Orantin’s topological recursion produces a meromorphic symmetry $n$-form on $\bar{C}_q^n$ recursively as below.

$$(2)$$

$$\omega_{g,n}(p_1, \ldots, p_n) = \sum_{d\xi_{p, r=0}} \text{Res}_{p=r} \frac{\int_{\xi=p} B(p_n, \xi)}{2(\Phi(p) - \Phi(\bar{p}))} (\omega_{g-1,n+1}(p, \bar{p}, p_1, \ldots, p_{n-1})$$

$$+ \sum_{g_1 + g_2 = g, I+J = \{1, \ldots, n-1\}} \omega_{g_1, |I|+1}(p, p_1) \omega_{g_2, |J|+1}(\bar{p}, p_1)) .$$

Here the sum symbol $\sum$ excludes the case $(g_1, |I|) = (0, 1), (0, n - 1), (g_1, 1)$ or $(g, n - 1)$.

The resulting form $\omega_{g,n}$ (for $2g-2+n > 0$) is smooth away from the ramification point $d\hat{x} = 0$. In particular they are holomorphic in $U_q^n$,
where $U_q$ is the small open neighborhood around $x_q$ in each copy of $\bar{C}_q$.

We still need to specify $A$ as a Lagrangian subspace of $H_1(\bar{C}_q; \mathbb{C})$ to completely write down the mirror curve as a spectral curve and produce higher genus invariants.

2.4. The remodeling conjecture. The BKMP remodeling conjecture [6, 7, 17] predicts $F_{g,n}(\mathcal{L}, f)$ by $\omega_{g,n}$. Both sides are related by a change of variables called the mirror map.

We explicitly write down the mirror map here for $X$. The cohomology $H^*_T(f(X); \mathbb{C})$ is a 2-dimensional $\mathbb{C}$-vector space – the equivariant parameter $v$ and any lift of the hyperplane class form a basis. We let $H$ be the equivariant lift of the hyperplane class such that $\iota_0^*H = 0$. (Recall that $\iota_0 : p_0 \to X$ is the embedding of the $T$-fixed point “closest” to $L$)

$$\tau = (\log q - 3 \sum_{d>0} \frac{(-1)^{d-1}(3d - 1)!}{(d!)^3} q^d)H,$$

$$\log \tilde{X} = \log \hat{X} + \sum_{d>0} \frac{(-1)^{d-1}(3d - 1)!}{(d!)^3} q^d.$$

These mirror maps have geometric interpretation. There exists an cycle $\Lambda \in H_1(C_q; \mathbb{Z})$ such that

$$\tau = \tau H = \left(\frac{1}{2\pi i} \int_{A} \Phi \right) H.$$  

We denote the image of this cycle in $H_1(\bar{C}_q; \mathbb{Z})$ by $\bar{\Lambda}$. It spans a Lagrangian subspace of $H_1(\bar{C}_q; \mathbb{C})$. Therefore the mirror curve is then equipped with a spectral curve structure.

We define B-model open potentials

$$\tilde{F}_{0,2}^X(q; \tilde{X}_1, \tilde{X}_2) = \int_{\tilde{X}_1} \int_{\tilde{X}_2} \left( \omega_{0,2} - \frac{d\tilde{X}_1 d\tilde{X}_2}{(\tilde{X}_1 - \tilde{X}_2)^2} \right),$$

$$\tilde{F}_{g,n}^X(q; \tilde{X}_1, \ldots, \tilde{X}_n) = \int_{\tilde{X}_1} \cdots \int_{\tilde{X}_n} \omega_{g,n}, \ 2g - 2 + n > 0.$$  

Similarly to $\tilde{F}_{0,1r}$, we consider the expansion of $\omega_{g,n}$ in $U_q^\omega$, and the resulting integrals (anti-derivatives) are power series in $\tilde{X}_1, \ldots, \tilde{X}_n$ with no degree 0 term. Notice $\omega_{0,2}$ has diagonal pole so we need to subtract the principal part first.

**Theorem 2.1.** We have the following mirror symmetry statements, where $q \in B$ and $\tilde{X} \in U_q$.
GLOBAL MIRROR CURVE AND ITS IMPLICATION

- **Disk mirror theorem** [2, 3], proved in [15]:
  \[ F^X_{\tau,0,1}(\tilde{X}; \hat{X}) = F^X_{\tau,0,1}(q; \hat{X}). \]

- **Higher genus mirror symmetry, a.k.a. the BKMP remodeling conjecture** [6, 7, 17], proved in [10, 14]:
  \[ F^X_{g,n}(\tau; \tilde{X}_1, \ldots, \tilde{X}_n) = \hat{F}^X_{g,n}(q; \hat{X}_1, \ldots, \hat{X}_n). \]

- For \( g > 1 \), the free energy
  \[ F^X_g(\tau) = \frac{1}{2 - 2g} \sum_{p_0} \text{Res}_{p=p_0} \omega_{g,1}(p) \int \Phi(p). \]
  Here \( \int \Phi \) is the anti-derivative of \( \Phi \), which we regard as a local function around each ramification point (the ambiguity does not affect the residue).

2.5. **The remodeling conjecture for** \( \mathbb{C}^3/\mathbb{Z}_3 \). We let \( X' = \mathbb{C}^3/\mathbb{Z}_3 \), the quotient stack. The orbifold \( X' \) is obtained by the same polytope \( P \) in Section 2.1 while there is no further triangulation inside the polytope. It is given by the GIT quotient at a different stability condition.

\[ X' = (\mathbb{C}^3 \times \mathbb{C}^\ast) / G, \]

where the torus \( G = \mathbb{C}^\ast \) acts by

\[ t \cdot (Z_1, \ldots, Z_4) = (tZ_1, tZ_2, tZ_3, t^{-3}Z_4). \]

It is also a symplectic quotient

\[ X' = \tilde{\mu}^{-1}(r)/G_\mathbb{R}, \]

where \( r < 0 \). The Aganagic-Vafa brane \( \mathcal{L}' \) is in the pre-image of the point \( a' \) for the moment polytope \( \mu'^{1}_{G} \) in the toric graph as in Figure 4. We should consider the Chern-Ruan orbifold cohomology \( H^*_{CR}(X'; \mathbb{C}) \) for the extended Kähler classes. In particular, \( H^2_{CR}(X'; \mathbb{C}) \) is generated by an age 1 element. The open-closed Gromov-Witten potentials are defined as

\[ F^X_{g,n}(\mathcal{L}', f)(\tau'; \hat{X}_1', \ldots, \hat{X}_n') = \sum_{\mu=(\mu_1, \ldots, \mu_n)} \sum_{\ell \geq 0} \frac{(\tau'^{\ell})^{\mu_1, \ldots, \mu_n} }{\ell!} \hat{X}_1^{\mu_1} \ldots \hat{X}_n^{\mu_n} \in \mathbb{Q} \]

for \( \tau' \in H^2_{CR, T'}(X'; \mathbb{C}) \). When \( n = 0 \), this is usually written as \( F^X_g \), and it involves only closed Gromov-Witten invariant which do not depend on \( f \).

The mirror curve is also explicitly given by

(4) \[ H^q_q(X', Y') = 1 + X^{3}Y^{q-1} + Y' + q'X'; \]
while the framed mirror curve equation is also given by a simple change of variables as below.

\[ X' = \hat{X}'\hat{Y}'^{-f}, \ Y' = \hat{Y}', \]
\[ H'_q = 1 + \hat{X}'^3\hat{Y}'^{-1-3f} + \hat{Y}' + q'\hat{X}'\hat{Y}'^{-f}; \]

We denote this mirror curve by \( C'_q \), and its compactification by \( \bar{C}'_q \). When \(|q'|\) is very small, \( C'_q \) is also a 3-punctured curve of genus 1, while \( \bar{C}'_q \) is a compact Riemann surface of genus 1. When \( q' = 0 \), \( \bar{C}'_0 \) is not singular, unlike the mirror curve of \( X = \mathcal{O}_{\mathbb{P}^2}(-3) \). We denote a small neighborhood \( B' \) of 0 such that when \( q' \in B' \), \( C'_q \) and \( \bar{C}'_q \) are smooth.

There is also a distinguished point \( s'_q \) in \( C'_q \subset \bar{C}'_q \), given by \( X' = 0 \) and \( Y' = -1 \) for \( q' \in B' \). We use \( U'_q \) to denote an small open neighborhood of \( t'_q \) in \( \bar{C}'_q \). Then we define open B-model disk potential as below

\[ \tilde{F}^{X'_0,1} = " \int_{X'} \Phi', \]

where \( \Phi' = \log \hat{Y}' \frac{d\hat{X}'}{\hat{X}'} \). We consider this integral as an anti-derivative of \( \Phi' \) expanded in \( U'_q \), and define \( \tilde{F}^{X'}_{0,1} \) by discarding degree-0 terms in \( \hat{X}' \).

To construct higher genus B-model open potential, one also runs the Eynard-Orantin topological recursion. The cohomology \( \text{H}^2_{X'}(X; \mathbb{C}) \)
is a 2-dimensional $\mathbb{C}$-vector space. Let $1_1$ be the generator of the age 1 elements. We also have a cycle $A' \in H_1(C'_q; \mathbb{C})$ such that the integral

$$\tau' = \left( \frac{1}{2\pi \sqrt{1}} \int_{A'} \Phi' \right) 1_1 = q' \left( \sum_{k \geq 0} \frac{\Gamma(2/3)^3}{\Gamma(\frac{2}{3} - k)^3(3k)!} 3q'^3k \right) 1_1.$$ 

The open mirror map is trivial for $X' = C^3/\mathbb{Z}_3$:

$$\tilde{X}' = \hat{X}' .$$

The Lagrangian subspace $A'$ spanned by $\bar{A}'$, the image of $A'$ in $H_1(\bar{C}'_q; \mathbb{C})$, is the last piece of information to make $C'_q$ and $\bar{C}'_q$ into a spectral curve. Then the Eynard-Orantin topological recursion produces $\omega_{g,n}'$. We define

$$\tilde{F}_{0,2}(q'; \tilde{X}'_1, \tilde{X}'_2) = \int_{\tilde{X}'_1} \int_{\tilde{X}'_2} \left( \omega'_{0,2} - \frac{d\tilde{X}'_1 d\tilde{X}'_2}{(\tilde{X}'_1 - \tilde{X}'_2)^2} \right),$$

$$\tilde{F}_{g,n}(q; \tilde{X}'_1, \ldots, \tilde{X}'_n) = \int_{\tilde{X}'_1} \ldots \int_{\tilde{X}'_n} \omega'_{g,n}, \quad 2g - 2 + n > 0 .$$

These integrals are understood as anti-derivatives for the relevant differential forms in $(U'_q)^n \subset \mathbb{C}^n$.

**Theorem 2.2.** We have the following mirror symmetry statements under the open-closed mirror map, where $q' \in B'$ and $\tilde{X}' \in U'_q$:

- **Disk mirror theorem, proved in [12]:**
  $$F_{0,1}(L', f)(\tau'; \tilde{X}') = \tilde{F}_{0,1}(q'; \tilde{X}') .$$

- **Higher genus mirror symmetry, a.k.a. the BKMP remodeling conjecture [7], proved in [13]:**
  $$F_{g,n}(L', f)(\tau; \tilde{X}'_1, \ldots, \tilde{X}'_n) = \tilde{F}_{g,n}(q'; \hat{X}'_1, \ldots, \hat{X}'_n) .$$

- **When $g > 1$, the free energy**
  $$F_g(\tau) = \frac{1}{2 - 2g} \sum \text{Res}_{p_0=0} \omega_{g,1}'(p) \int \Phi'(p) .$$

### 3. The construction of the global mirror curve

#### 3.1. Family of mirror curves.

The mirror curve equations (1) and (4) are the same after a simple change of variables:

$$q = q'^{-3}, \quad X = X' q', \quad Y = Y'. $$
So \( C_q \) and \( C'_q \) should form a family of affine curves. Here we give a toric construction such that \( \bar{C}_q \) and \( \bar{C}'_q \) form a family of compact curves over the weighted projective line \( \mathbb{P}(1, 3) \).

Recall that \( X = \mathcal{O}_{\mathbb{P}^2}(-3) \). Its fan is the cone over the defining polytope \( P \), as shown in Figure 1.

Its secondary stacky fan \( \mathcal{G} \) is a complete fan in \( \mathbb{R} \). The generators of its 1-cones are

\[
\begin{align*}
&b_1 = 1, \quad b_2 = 1, \quad b_3 = 1, \quad b_4 = -3. \\
&\tilde{b}_4 = (0, 0, 1), \quad \tilde{b}_5 = (1, 1, 0), \quad \tilde{b}_6 = (-2, 1, 0), \quad \tilde{b}_7 = (1, -2, 0).
\end{align*}
\]

The top dimensional cones are spanned by \( \tilde{b}_i \) where \( i \) ranges from the following index sets

\[
\{4, 5, 6\}, \quad \{4, 6, 7\}, \quad \{4, 5, 7\}, \quad \{5, 1, 2\}, \quad \{5, 1, 3\}, \\
\{6, 1, 2\}, \quad \{6, 2, 3\}, \quad \{7, 2, 3\}, \quad \{7, 1, 3\}, \quad \{1, 2, 3\}.
\]

The 2-cones are faces of 3-cones. We denote the toric orbifold associated to the fan \( \tilde{\mathcal{G}} \) by \( \tilde{M}_K \).

We now define the following extended secondary fan \( \tilde{\mathcal{G}} \) as a complete fan in \( N_K \otimes \mathbb{R} \approx \mathbb{R}^3 \), where \( N_K = \mathbb{Z}^3 \). The generators of its 1-cones in \( N_K \) are

\[
\begin{align*}
&\tilde{b}_4 = (0, 0, 1), \quad \tilde{b}_5 = (1, 1, 0), \quad \tilde{b}_6 = (-2, 1, 0), \quad \tilde{b}_7 = (1, -2, 0).
\end{align*}
\]

The fiber \( \pi^{-1}(s) \) for \( s \neq s_{LRL} \) is a toric orbifold defined by the stacky fan given by \( \tilde{b}_5, \tilde{b}_6, \tilde{b}_7 \) (on \( \mathbb{R}^2 \)). It is isomorphic to \( \mathbb{P}^2/\mathbb{Z}_3 \). Over the smooth torus fixed point, the fiber \( \pi^{-1}(s_{LRL}) \) is three \( \mathbb{P}^2 \) intersecting along three \( \mathbb{P}^1 \) with normal crossing singularities. If one intersects the fan \( \tilde{\mathcal{G}} \) by a vertical plane, at different horizontal position, we get the fan of each fiber toric surface. See Figure 7.
The third coordinates of the generators $\tilde{b}_i$ are the same if they are in the same color. The rays $\tilde{b}_5, \tilde{b}_6, \tilde{b}_7$ form the toric graph of $\mathbb{C}^3/\mathbb{Z}_3$. There is an obvious fan map $\tilde{\mathcal{S}} \to \mathcal{S}$. 

One can also understand $\tilde{\mathcal{S}}$ in the following way. The fan $\mathcal{S}$ lives inside $G^\vee_K = \mathbb{R}$, and $\mathcal{X}_r = \tilde{\mu}^{-1}(r)/\mathbb{R}_\mathbb{Z}$ (here $\mathcal{X} \cong \mathcal{X}_r,$ $r > 0$ and $\mathcal{X}' \cong \mathcal{X}_r,$ $r < 0$). The intersection $\pi^{-1}(r) \cap (\tilde{\mathcal{S}}(2) \cup \tilde{\mathcal{S}}(1) \cup \tilde{\mathcal{S}}(0))$ is precisely the toric graph of $\mathcal{X}_r$.

We understand $X, Y, q$ as characters in $\text{Hom}(T_K, \mathbb{C}^*) = N^\vee_K$, where $T_K$ is the open dense 3-torus in $\mathcal{M}_K$, and $N_K \cong \mathbb{Z}^3$ is the lattice that $\tilde{b}_i$ belong to. Then $X, Y, q$ corresponds to $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ in $N^\vee_K$ respectively. They are sections of a line bundle $L = \mathcal{O}_{\mathcal{M}_K}(\sum_{i=1}^6 D_i)$ (here each $D_i$ is the toric divisor corresponding to each $\tilde{b}_i$). We define a section $H \in \mathcal{H}^0(L)$

$$H = X + Y + 1 + qX^3Y^{-1}.$$ 

We define the compactified global mirror curve $\mathcal{C} = \mathcal{H}^{-1}(0) \subset \tilde{\mathcal{M}}_K$. It is parametrized over $\mathcal{M}_K$ by $\pi_\mathcal{C} = \pi_\mathcal{C} : \mathcal{C} \to \mathcal{M}_K$. For any $s \in \mathcal{M}_K$, the fiber $\pi_\mathcal{C}^{-1}(s)$ is a compact (possibly singular) curve. Let $\mathcal{M}_{K,0}$ be the part of $\mathcal{M}_K$ where $\pi_\mathcal{C}^{-1}(\mathcal{M}_{K,0})$ is smooth. As shown in Figure...
We have a family of toric surfaces given by $\pi$. When $s \neq s_{\text{LRL}}$, the fiber $\pi^{-1}(s) \cong \mathbb{P}^2/\mathbb{Z}_3$, given by the stacky fan spanned by $b_5, b_6, b_7$. Over $s_{\text{LRL}}$, the toric surface degenerates to a normal crossing of three $\mathbb{P}^2$, as shown by the “fan” and the polytope. The first rows are polytopes and the second rows are fans for fiber toric surfaces at different points in $M_K$.

$\mathbb{M}_K, s_{\text{LRL}} \notin \mathbb{M}_K, 0$ since the fiber is a nodal curve (three $\mathbb{P}^1$ with nodal singularities), while $s_{\text{orb}} \notin \mathbb{M}_K, 0$ since itself is a stacky point. There is another point other than $s_{\text{LRL}}$ not in $\mathbb{M}_K, 0$, where the fiber has one nodal singularity. This point is called the conifold point $s_{\text{con}}$. Thus $\mathbb{M}_K, 0 = \mathbb{M}_K \setminus \{s_{\text{LRL}}, s_{\text{orb}}, s_{\text{con}}\}$.

By our notation, $C_q$ and $C'_q$, are identified with $C_s$ when $q = q'^{-3}$, where $C_s = \pi_C^{-1}(s)$, $q(s) = q$, and $q'(s) = q'$.

### 3.2. Open crepant resolution conjecture for disk potentials

The open crepant resolution conjecture (CRC) for disk potentials is a direct consequence of the global mirror curve $\mathcal{C}$. A CRC result should relate Gromov-Witten invariants around the large radius point to orbifold Gromov-Witten invariants around the orbifold points. The CRC for disk potentials, by its name, should relate $F^X_{0,1}(C, t)$ and $F^{X'}_{0,1}(C', t)$.

We pick a path $\gamma : [0, 1] \to \mathcal{M}_{K, 0}$ such that $\gamma(0) = s_{\text{LRL}}$ and $\gamma(1) = s_{\text{orb}}$. We also pick a lift of this $\gamma$ to $\tilde{\gamma} : [0, 1] \to \mathcal{C}$ such that $\tilde{\gamma}(0) = \tilde{x}_0$, $\tilde{\gamma}(1) = \tilde{x}'_0$, and $\pi \circ \tilde{\gamma} = \gamma$.

The function $\log \tilde{Y} = \log \tilde{Y}'$ is a well-defined analytic function from a small tubular neighborhood of $\tilde{\gamma}([0, 1])$ in $\mathcal{C}$ to $\mathcal{C}/(2\pi\sqrt{-1})$. By the
Figure 8. Over $\mathcal{M}_{K}$, we have a family of compactified mirror curves $\mathcal{C}$. At $s_{\text{con}}$ and $s_{\text{LRL}}$ the mirror curves are singular. As before, the sharp ends in the mirror curve picture are the punctures on the mirror curve. After compactification, they become compact curves in $\pi^{-1}(s)$. All puncture points are smooth.

The disk mirror theorem

$$
\dot{X} \frac{d}{dX} F_{0,1}^{X,(L,f)} = \log \dot{Y}, \quad \dot{X}' \frac{d}{dX'} F_{0,1}^{X',(L',f)} = \log \dot{Y}'
$$

up to degree-0 terms in $\dot{X}$ or $\dot{X}'$. Since we know the degree-0 term of $\log Y'$'s expansion in terms of $\dot{X}$ is $\log(-1)$ (the deg-0 term of the expansion $\log Y'$ in $\dot{X}'$ is also $\log(-1)$), one can analytically continue $\dot{X} \frac{d}{dX} F_{0,1}^{X,(L,f)}$, considered as a function near $s_0$, along $\tilde{y}$. The resulting holomorphic function near $s'_0$ differs with $\dot{X}' \frac{d}{dX'} F_{0,1}^{X',(L',f)}$ by an integral multiple of $2\pi \sqrt{-1}$.

3.3. Modular invariance of fundamental normalized differentials of the second kind. The mirror curve $C_q$ (and its compactification $\bar{C}_q$) is a spectral curve. The genus of the compactified mirror curve $\bar{C}_q$ is 1. We fix two sets of Torelli markings $(\bar{A}, \bar{B}), (\bar{A}', \bar{B}')$ on $\bar{C}_q$, such that

$$(\bar{A}, \bar{B}) = (\bar{A}', \bar{B}') = 1, \quad (\bar{A}, \bar{B}) = (\bar{B}, \bar{B}) = (\bar{A}', \bar{A}') = (\bar{B}', \bar{B}') = 0.$$

They differ by an SL(2; $\mathbb{Z}$) transformation

$$
\begin{pmatrix}
\bar{A} \\
\bar{B}
\end{pmatrix} =
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\bar{A}' \\
\bar{B}'
\end{pmatrix}
$$
and \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{Z}) \). Let \( \omega \) be the non-trivial holomorphic form on \( \tilde{C}_q \) given by the Torelli marking \((\tilde{A}, \tilde{B})\), i.e.

\[
\int_{\tilde{A}} \omega = 1.
\]

The period \( \theta \) is given by

\[
\theta = \int_{B} \omega.
\]

We know \( \text{Im} \theta > 0 \). It depends on the choice of cycles \( A, B \) and the parameter \( q \). Similarly,

\[
\int_{A'} \omega' = 1, \quad \theta' = \int_{B'} \omega'.
\]

We have

\[
\theta' = \frac{\theta a - c}{d - \theta b}, \quad \omega' = J \omega,
\]

where \( J = (d - \theta b)^{-1} \).

Define the modified cycles

\[
\tilde{A}(\theta) = \tilde{A} - \kappa \tilde{B}(\theta), \quad \tilde{B}(\theta) = \tilde{B} - \theta \tilde{A},
\]

\[
\tilde{A}'(\theta') = \tilde{A}' - \kappa \tilde{B}'(\theta'), \quad \tilde{B}'(\theta') = \tilde{B}' - \theta' \tilde{A}.
\]

Here

\[
\kappa(\theta, \tilde{\theta}) = \frac{1}{\theta - \tilde{\theta}}
\]

is a function of \( \theta \) (not holomorphic). As a convention, we denote the fundamental differential associated to the A-cycle \( \tilde{A} \) by \( \omega_{0,2} \), and the fundamental differential associated to the modified A-cycles \( \tilde{A}(\theta) \) by \( \tilde{\omega}_{0,2} \). We also denote the fundamental differential associated to \( \tilde{A}' \) by \( \eta_{0,2} \), while the fundamental differential associated to \( \tilde{A}'(\theta') \) by \( \tilde{\eta}_{0,2} \).

By direct calculation, Eynard-Orantin show that in [9]

\[
\tilde{\omega}_{0,2} = \omega_{0,2} + 2\pi \sqrt{-1} \kappa(\theta, \tilde{\theta}) \theta.
\]

They also show that

\[
\eta_{0,2} = \omega_{0,2} + 2\pi \sqrt{-1} \tilde{\kappa}(\theta) \theta,
\]

where \( \tilde{\kappa} = bJ \).

The fact that

\[
J \kappa(\theta', \tilde{\theta}') J + \kappa(\theta) = \frac{1}{\theta - \tilde{\theta}}
\]

implies

\[
\tilde{\eta}_{0,2} = \tilde{\omega}_{0,2}.
\]
**Proposition 3.1** (Eynard-Orantin). Given any Torelli marking \((\bar{A}, \bar{B})\), the modified fundamental differential \(\tilde{\omega}_{0,2}\) given by the modified Torelli marking \((\bar{A}(\theta), \bar{B}(\theta))\) is independent of the choice of \((\bar{A}, \bar{B})\).

This property implies that given a fixed spectral curve, we have a preferred choice of the fundamental differential \(\tilde{\omega}_{0,2}\) independent of the choice of the A-cycles. Moreover, under the limit \(\text{Im} \theta \to \infty\), \(\tilde{\omega}_{0,2} \to \omega_{0,2}\). Notice the parameter \(\theta\) and \(\omega_{0,2}\) depends on the choice of the A-cycle.

From the explicit expression of the Eynard-Orantin recursion (Equation (2)), for any spectral curve, we can define its modified B-model invariants \(\tilde{\omega}_{g,n}\) based on this modified fundamental differential \(\tilde{\omega}_{0,2}\), with

\[
\lim_{\text{Im} \theta \to \infty} \tilde{\omega}_{g,n} = \omega_{g,n}.
\]

### 3.4. Modularity.

The monodromies of the Gauss-Manin connection on the local system \(H^1(\mathcal{C}_s; \mathbb{C}) \cong H_1(\mathcal{C}_s; \mathbb{C})\) over \(\mathcal{M}_{K,0}\) (as computed in [1]) gives the modular group \(\Gamma\) of this local system. It is a normal subgroup of the symplectic group \(\text{SL}(2; \mathbb{Z})\) of index 3.

Over \(\mathcal{M}_{K,0}\), we have a smooth family of mirror curves, and the coordinates \(X, Y\) are well defined. So \(X, Y\) are invariant under the action of the modular group \(\Gamma\). If we use the modified fundamental differential \(\tilde{\omega}_{0,2}\) to define the higher genus B-model invariants \(\tilde{\omega}_{g,n}\), then they are all well-defined global invariants on \(\mathcal{C}/\text{divid} \langle \text{s.al} \rangle_0 \mathcal{M}_{K,0}\). In other words, if one uses Torelli-marking-sensitive coordinate \(\theta\) to express these \(\tilde{\omega}_{g,n}\), they are invariant under the action of the modular group \(\Gamma\).

Using the mirror map (3) we define the open potential in the holomorphic polarization under A-model flat coordinates when \(2g - 2 + n > 0\).

\[
\tilde{\tilde{F}}_{g,n}^{X,(L,f)}(\tilde{X}_1, \ldots, \tilde{X}_n, \tau) = \int_{\tilde{X}_1} \ldots \int_{\tilde{X}_n} \tilde{\omega}_{g,n}.
\]

The A-model coordinate \(Q = e^\tau\) is well-defined around the LRL point, and is related to B-model coordinate \(q\) around the LRL point under the closed mirror map. The open potential \(\tilde{\tilde{F}}_{g,n}^{X,(L,f)}\) has non-holomorphic dependence on \(s\) (\(q\) or \(\theta\)), in contrast to the name “holomorphic polarization”. Under the holomorphic limit

\[
\lim_{\text{Im} \theta \to \infty} \tilde{\omega}_{g,n} = \omega_{g,n}.
\]
With the BKMP remodeling conjecture (Theorem 2.1), for $2g - 2 + n > 0$ and $n \geq 1$

$$\lim_{\text{Im} \theta \to \infty} \tilde{F}^X_{g,n}(\mathcal{L}, f) = F^X_{g,n}(\mathcal{L}, f).$$

If one defines

$$\tilde{F}^X_g = \frac{1}{2 - 2g} \sum_{d \chi(p_0) = 0} \text{Res}_{p=p_0} \tilde{\omega}_{g,1}(p) \int \Phi(p),$$

then for $g \geq 2$

$$\lim_{\text{Im} \theta \to \infty} \tilde{F}^X_g = F^X_g.$$

The potential $\tilde{F}^X_{g,n}(\mathcal{L}, f)$ and $\tilde{F}^X_g$ are globally defined over $\mathcal{M}_K$, although their expansions in $Q = e^\tau$ are only defined around $s_{\text{LRL}}$ since $Q$ is a flat coordinate around $s_{\text{LRL}}$. Their dependence on $s \in \mathcal{M}_K$ is not holomorphic.

**Theorem 3.2.** The Gromov-Witten potential $F^X_g$ can be completed into an analytic function $\tilde{F}^X_g$, which under the mirror map (3) is globally defined on $\mathcal{M}_K$. When $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$, $\mathcal{M}_K$ is a modular curve, the function $\tilde{F}^X_g$ is a function of $\theta$ and modular invariant.

**Remark 3.3.** In the unstable cases $(g, n) = (0, 0), (0, 1), (0, 2), (1, 0)$, the theorem also holds but we need to treat these cases separately. We did not very clearly spell out what this “anti-holomorphic completion” is, as it should be stronger than (5). Indeed, $\tilde{F}^X_g$ can be written as a polynomial in $\text{Im} \theta$ with holomorphic coefficients $[9, 11]$. The lowest order of $\text{Im} \theta$ is $2 - 2g$, and each coefficient in non-holomorphic terms are given by combinations of $F^X_g$, $g' < g$ and their derivatives in a graph sum formula.

**Remark 3.4.** One could use the modularity property to compute higher genus Gromov-Witten invariants for certain toric Calabi-Yau 3-(orbi)fold, thanks to the complete structure theorem of almost holomorphic modular forms. See [1, 4, 18] for numerical calculations and closed formulae for some $\tilde{F}^X_g$ and $F^X_g$.

**References**


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