GLOBAL MIRROR CURVE AND ITS IMPLICATION

BOHAN FANG

ABSTRACT. The remodeling conjecture recasts all genus Gromov-Witten theory of a toric Calabi-Yau 3-fold in terms of complex geometry of its mirror curve. We illustrate how to construct a family of mirror curves, over the global moduli of the toric Calabi-Yau 3-fold's stringy Kähler moduli space. With this construction, the remodeling conjecture then reveals many properties of the Gromov-Witten invariants, such as the modularity and the crepant transformation property.

1. INTRODUCTION

1.1. **Mirror symmetry for a toric Calabi-Yau 3-orbifold.** Let $\mathcal{X}$ be a toric Calabi-Yau 3-orbifold. The mirror symmetry predicts its Gromov-Witten invariants from its mirror B-model. Usually the mirror of $\mathcal{X}$ is a non-compact Calabi-Yau hypersurface, which can be further reduced to an affine curve in $(\mathbb{C}^*)^2$, called the *mirror curve*. In this survey we only consider the mirror curve as its B-model.

The mirror B-model of $\mathcal{X}$ predicts both closed and open Gromov-Witten invariants [2, 3, 6–8, 16]. In [16], the B-model for the closed higher genus invariants is from the BCOV holomorphic anomaly equation [5]. The Bouchard-Klemm-Mariño-Pasquetti's remodeling conjecture [6,7,17] predicts all genus open-closed Gromov-Witten invariants from another viewpoint on the B-model, the Eynard-Orantin's topological recursion [9]. This prediction from the topological recursion is called *the remodeling conjecture*. This conjecture is proved later in [10, 13, 14].

More precisely, there is a certain type of Lagrangian submanifolds, the *Aganagic-Vafa branes* in $\mathcal{X}$. In case such branes are not gerby, they are all homeomorphic to $S^1 \times \mathbb{R}^2$. We fix such a Lagrangian $\mathcal{L} \subset \mathcal{X}$, and consider the open Gromov-Witten potential

$$F^{\mathcal{X},\mathcal{L}}_{g,n}(\boldsymbol{\tau}; \tilde{X}_1, \dots, \tilde{X}_n),$$

which is a generating function parametrizing the number of holomorphic maps from a genus g bordered Riemann surface with n

boundary components to $\mathcal{X}$ while the boundary lands on $\mathcal{L}$. The Kähler parameter $\boldsymbol{\tau}$ records the extended Kähler class of $\mathcal{X}$. In case that $\mathcal{X}$ is a smooth manifold, by the divisor equation the power of $e^{\boldsymbol{\tau}}$ records the homology class of the image of this map, while the power of $\tilde{X}_i$ records the winding number of each boundary component into $\mathcal{L} \cong S^1 \times \mathbb{R}^2$.

The mirror curve of $\mathcal{X}$ is an affine curve $C_q = \{H_q(X,Y) = 0\}$ in $(\mathbb{C}^*)^2$. Here $H_q(X,Y) = 0$ is the equation for C_q. The conjecture of Aganagic-Klemm-Vafa [2,3] predicts

$$F_{0,1}^{\mathcal{X},\mathcal{L}}(\boldsymbol{\tau};\tilde{X}) = \int_X \log Y \frac{dX}{X},$$

under certain explicit open-closed mirror map

$$\boldsymbol{\tau} = \log q + O(q), \quad \tilde{X} = X(1 + O(q)).$$

One should understand this integral as anti-derivative and $\log Y$ is a function of X near a particular point on $\bar{C}_q$ with $X = 0$, where $\bar{C}_q$ is a compactification of C_q.

The Eynard-Orantin's topological recursion starts from a choice of Lagrangian subspace of $H^1(\bar{C}_q;\mathbb{C})$ where the symplectic pairing is the cohomology pairing $(\alpha,\beta) \to \int_{\bar{C}_q} \alpha \cup \beta$. Then one can recursively and uniquely constructs a meromorphic and symmetric n-form $\omega_{g,n}$ on $(\bar{C}_q)^n$. Then the BKMP remodeling conjecture says under the same open-closed mirror map

$$F_{g,n}^{\mathcal{X},\mathcal{L}}(\boldsymbol{\tau};\tilde{X}_1,\dots,\tilde{X}_n) = \int_{X_1} \cdots \int_{X_n} \omega_{g,n}.$$

1.2. **String Kähler moduli and global mirror symmetry.** The topological recursion on the mirror curve as the B-model automatically carries many interesting properties. For example, the modularity of the recursion algorithm was already addressed in [9] when such algorithm was proposed.

There are many "phases" of A-model theories. If $\mathcal{X}$ is a smooth manifold, then the Gromov-Witten theory of $\mathcal{X}$ is a *theory at a large radius limit*. In general there are many limit points on the stringy Kähler moduli space $\mathcal{M}_K$ of $\mathcal{X}$. When $\mathcal{X}$ is a toric Calabi-Yau 3-fold, $\mathcal{M}_K$ can be identified with its secondary toric variety. Around each torus fixed point $\mathfrak{s}_i$ of $\mathcal{M}_K$, we can associate a toric Calabi-Yau 3-orbifold $\mathcal{X}_i$ depending on the GIT stability condition. To one of these torus fixed point $\mathfrak{s}_0$, $\mathcal{X}_0 = \mathcal{X}$ itself, while at other points they are McKay equivalent toric Calabi-Yau 3-orbifolds, related by being a partial crepant resolution pairs of a same singular toric variety.

The A-model theory around each torus fixed point $\mathfrak{s}_i$ in $\mathcal{M}_K$ is the orbifold Gromov-Witten theory of the toric Calabi-Yau 3-fold $\mathcal{X}_i$. *A priori* there is no reason these theories about $\mathcal{X}_i$ could patch together globally over $\mathcal{M}_K$. However this desired global behavior is more accessible from the B-model.

The mirror B-model considered in this paper is an affine curve C_q together with its compactification $\bar{C}_q$, where q is the complex parameter. We will see that $q \in \mathcal{M}_K$ and construct a *family of mirror curves* $\mathfrak{C}$, inside a family of toric surfaces $\mathcal{S}$, over $\mathcal{M}_K$. At each q the fiber of this family is indeed $\bar{C}_q$ while $C_q = \bar{C}_q \setminus (\partial \mathcal{S}_q)$.

The global mirror curve $\mathfrak{C}$ implies the existence of a global B-model over the stringy Kähler moduli space $\mathcal{M}_K$. The modularity of the B-model generating function is automatic given such a global mirror curve $\mathfrak{C}$. The B-model theory near each limit point $\mathfrak{s}_i$ are related by analytic continuation, since at every point in $\mathcal{M}_K$ the B-model theory is well-defined. Translated back into the A-model Gromov-Witten theory, one obtains the modularity of the Gromov-Witten theory and the crepant resolution conjecture.

1.3. **The structure of this paper.** We will illustrate the construction of $\mathfrak{C}$ and discuss its implication by a main example $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$. In Section 2 we state the BKMP remodeling conjecture for both $\mathcal{X}$ and its orbifold phase $\mathcal{X}' = \mathbb{C}^3/\mathbb{Z}_3$. Then we explain how to construct a global mirror curve $\mathfrak{C}$ for this example in Section 3, and we will also explain the crepant resolution conjecture and the modularity of the Gromov-Witten theory from mirror symmetry.

1.4. **Acknowledgement.** The author would like to thank Chiu-Chu Melissa Liu and Zhengyu Zong for the wondrous collaboration in [13, 14] and several current ongoing projects – this paper's goal is to explain some of which. The author would also like to thank Bai-Ling Wang for a fantastic workshop at Kioloa in Jan 2016, without which this paper would not be possible.

2. THE REMODELING CONJECTURE FOR $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$

2.1. **A toric Calabi-Yau 3-fold $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$.** A toric Calabi-Yau 3-fold is given by a triangulated defining polytope. Let $N = \mathbb{Z}^3$ and

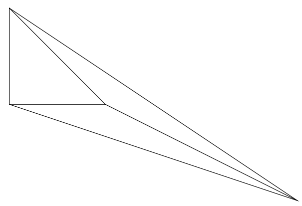

FIGURE 1. The triangulated defining polytope of $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$

$M = N^\vee$. We also choose $e_3^\vee = (0,0,1) \in M$ and let $M' = M/\langle e_3^\vee \rangle \cong \mathbb{Z}^2$ and $N' = \ker(e_3^\vee) \subset N$.

For $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$, let

$$P = \mathrm{Conv}((0,0),(3,-1),(0,1)) \subset N',$$

and its triangulation is given in the Figure 1.

The 1-(resp. 2-, 3-)cones in the *fan* data of $\mathcal{X}$ are cones from the origin in $N_{\mathbb{R}}$ over vertices (resp. edges, faces) of the triangulated $P \times \{1\} \subset N'_{\mathbb{R}} \times \{1\} \subset N_{\mathbb{R}}$. We write down the generators of 1-cones here:

$$b_1 = (0,0,1),\ b_2 = (1,0,1),\ b_3 = (0,1,1),\ b_4 = (3,-1,1).$$

By toric geometry, the fan data prescribes a torus action $G \cong \mathbb{C}^*$ on $\mathbb{C}^4$:

$$t \cdot (Z_1, Z_2, Z_3, Z_4) = (tZ_1, tZ_2, tZ_3, t^{-3}Z_4).$$

The smooth variety $\mathcal{X}$ is defined as the following quotient

$$\mathcal{X} = (\mathbb{C}^4 \setminus ((0,0,0) \times \mathbb{C}))/G.$$

The moment map $\tilde{\mu}$ for action of the maximal compact subgroup $G_{\mathbb{R}}$ is

$$(Z_1, Z_2, Z_3, Z_4) \mapsto |Z_1|^2 + |Z_2|^2 + |Z_3|^2 - 3|Z_4|^2.$$

Then $\mathcal{X}$ is also obtained as a symplectic quotient

$$\mathcal{X} = \tilde{\mu}^{-1}(r)/G_{\mathbb{R}},\ r > 0.$$

The parameter r is the Kähler parameter, which is the symplectic area of the base $\mathbb{P}^2$.

We denote the 3-dimensional torus $\mathbb{T} = N \otimes_{\mathbb{Z}} \mathbb{C}^*$, which acts on and is also open and dense in $\mathcal{X}$. Let $\mathbb{T}'$ be the 2-dimensional subtorus which acts trivially on its canonical bundle. Let $\mathbb{T}'_{\mathbb{R}}$ the maximal compact subgroup of $\mathbb{T}'$ one may consider its moment map $\mu' : \mathcal{X} \to M'_{\mathbb{R}} \cong \mathbb{R}^2$. The one-dimensional $\mathbb{T}'$-invariant subvariety $\mathcal{X}^1$ of $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$ is the union of three $\mathbb{P}^1$ in $\mathbb{P}^2$ and the fibers over three

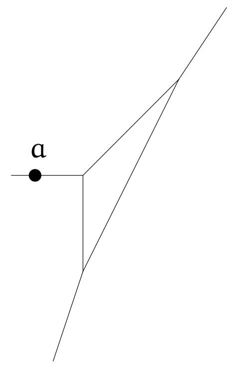

FIGURE 2. The toric graph of $\mathcal{X}$ and the image of an outer Aganagic-Vafa brane $\mathcal{L}$.

torus fixed point in $\mathbb{P}^2$. The image of $\mathcal{X}^1$ under μ' is the *toric graph*, shown in Figure 2.

An *Aganagic-Vafa* brane is an Lagrangian submanifold in the preimage of a non-vertex point in the toric graph. Precisely, we define an Aganagic-Vafa brane $\mathcal{L}$ below.

$$|Z_1|^2 - |Z_2|^2 = |Z_1|^2 - |Z_3|^2 = c > 0, \ \mathrm{Arg}(Z_1 \dots Z_4) = \mathrm{const}.$$

This Lagrangian brane $\mathcal{L}$ is homeomorphic to $\mathbb{R}^2 \times S^1$. It is *outer* since its image under μ' is the point a on a non-compact leg of the toric graph, which is also illustrated in Figure 2. We label the unique $\mathbb{T}$-fixed point on the 1-dimensional $\mathbb{T}$-invariant subvariety that $\mathcal{L}$ intersects by p_0. Let $\iota_0 : p_0 \hookrightarrow \mathcal{X}$ be the embedding.

2.2. **The Gromov-Witten theory of $\mathcal{X}$.** We define the closed Gromov-Witten primary correlators where $\gamma_1, \dots, \gamma_n \in H^*(\mathcal{X}; \mathbb{C})$

$$\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X}}_{g,n,\beta} = \int_{[\bar{\mathcal{M}}_{g,n}(\mathcal{X};\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^* \gamma_1 \cup \dots \cup \mathrm{ev}_n^* \gamma_n.$$

Here $\bar{\mathcal{M}}_{g,n}(\mathcal{X}; \beta)$ is the moduli space of stable maps from genus g, n-marked points to $\mathcal{X}$ in homology class $\beta \in H_2(\mathcal{X}; \mathbb{Z}) \cong \mathbb{Z}$, $[\bar{\mathcal{M}}_{g,n}(\mathcal{X}; \beta)]^{\mathrm{vir}}$ is its virtual fundamental class, and ev_i is the i-th evaluation map. Similarly the notion $\langle \gamma_1, \dots, \gamma_n \rangle^{\mathcal{X},\mathbb{T}}_{g,n,\beta}$ is for the equivariant Gromov-Witten theory where $\gamma_i \in H^*_{\mathbb{T}}(\mathcal{X}; \mathbb{C})$. Replacing $\mathbb{T}$ by other groups acting on $\mathcal{X}$ is self-evident. In the rest of this section, we only fix notations in the non-equivariant setting while the equivariant invariants are completely parallel. The descendant correlators are

$$\langle \tau_{a_1}(\gamma_1), \dots, \tau_{a_n}(\gamma_n) \rangle^{\mathcal{X}}_{g,n,\beta} = \langle \gamma_1 \psi_1^{a_1}, \dots, \gamma_n \psi_n^{a_n} \rangle^{\mathcal{X}}_{g,n,\beta}$$
$$= \int_{[\bar{\mathcal{M}}_{g,n}(\mathcal{X};\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^* \gamma_1 \cup \psi_1^{a_1} \cup \dots \cup \mathrm{ev}_n^* \gamma_n \cup \psi_n^{a_n}.$$

The psi-class ψ_i is the first Chern-class of the i-th tautological line bundle on $\bar{\mathcal{M}}_{g,n}(\mathcal{X};\beta)$.

The double brackets are

$$\langle\langle \gamma_1 \psi_1^{a_1}, \dots, \gamma_n \psi_n^{a_n} \rangle\rangle^{\mathcal{X}}_{g,n} = \sum_{\beta \geq 0, \ell \geq 0} \frac{1}{\ell !} \langle \gamma_1 \psi_1^{a_1}, \dots, \gamma_n \psi_n^{a_n}, \boldsymbol{\tau}, \dots, \boldsymbol{\tau} \rangle^{\mathcal{X}}_{g,n+\ell,\beta}.$$

So whenever double brackets appear they are functions of $\boldsymbol{\tau} \in H^2(\mathcal{X};\mathbb{C})$. In the equivariant setting the notion $\langle\langle \dots \rangle\rangle^{\mathcal{X},\mathbb{T}}_{g,n}$ is a function of $\boldsymbol{\tau} \in H^2_{\mathbb{T}}(\mathcal{X};\mathbb{C})$. Here we do not need to introduce Novikov variables to deal with the issue of convergence (see [14, Remark 3.2]).

We define the genus g free energy of $\mathcal{X}$ as

$$F^{\mathcal{X}}_g(Q) = \langle\langle \rangle\rangle^{\mathcal{X}}_{g,0}.$$

This is a power series in $Q = e^{\tau}$.

Open GW invariants for $(\mathcal{X}, \mathcal{L})$ count holomorphic maps

$$u : (\Sigma, x_1, \dots, x_\ell, \partial\Sigma = \coprod_{j=1}^{n} R_j) \to (\mathcal{X}, \mathcal{L})$$

where Σ is a bordered Riemann surface with interior marked points x_i and $R_j \cong S^1$ are connected components of $\partial\Sigma$. These invariants depend on the following data:

- the topological type (g, n) of the coarse moduli of the domain, where g is the genus of Σ and n is the number of connected components of $\partial\Sigma$,
- the degree $\beta' = u_*[\Sigma] \in H_2(\mathcal{X}, \mathcal{L}; \mathbb{Z})$,
- the winding numbers $\mu_1, \dots, \mu_n \in H_1(\mathcal{L}; \mathbb{Z}) \cong \mathbb{Z}$,
- the framing $f \in \mathbb{Z}$ of $\mathcal{L}$.

We call the pair $(\mathcal{L}, f)$ a framed Aganagic-Vafa Lagrangian brane. We write $\vec{\mu} = (\mu_1, \dots, \mu_n)$. Let $\bar{\mathcal{M}}_{(g,n),\ell}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu})$ be the compactified moduli space parametrizing stble maps described above. Evaluation at the i-th marked point x_i gives a map $\mathrm{ev}_i : \bar{\mathcal{M}}_{(g,n),\ell}(\mathcal{X}, \mathcal{L} \mid \beta', \vec{\mu}) \to X$.

The framing f specifies a subtorus $\mathbb{T}'_f = \ker((0,1) - f(1,0))$, where $(0,1),(1,0) \in M'$ are characters for $\mathbb{T}'$. For $\gamma_1,\dots,\gamma_n \in H^*_{\mathbb{T}'_f}(\mathcal{X};\mathbb{C})$, we define by localization

$$\langle \gamma_1,\dots,\gamma_\ell\rangle^{\mathcal{X},(\mathcal{L},f)}_{g,\beta,\vec{\mu}} := \int_{[\bar{\mathcal{M}}_{(g,n),\ell}(\mathcal{X},\mathcal{L}|\beta',\vec{\mu})^{\mathbb{T}'_{\mathbb{R}}}]^{\mathrm{vir}}} \frac{\prod_{i=1}^{\ell} \mathrm{ev}_i^*\gamma_i}{e_{\mathbb{T}'_{\mathbb{R}}}(N^{\mathrm{vir}})}\Bigg|_{(\mathbb{T}'_f)_{\mathbb{R}}}$$
$$\in \mathbb{C}v^{\sum_{i=1}^{\ell}(\frac{\deg \gamma_i}{2}-1)}$$

where $\mathbb{T}'_{\mathbb{R}}$ and $(\mathbb{T}_f)_{\mathbb{R}}$ are the corresponding real sub-torus of $\mathbb{T}'$ and $\mathbb{T}_f$ that preserves the Lagrangian $\mathcal{L}$, $H^*_{\mathbb{T}'_f}(\mathrm{pt}) = \mathbb{C}[v]$, $\beta \in H_2(\mathcal{X};\mathbb{Z}) \cong \mathbb{Z}$ and $\beta' = \beta + \sum \mu_i \in H_2(\mathcal{X},\mathcal{L};\mathbb{Z})$.

Then we define the open Gromov-Witten potential by

$$F^{\mathcal{X},(\mathcal{L},f)}_{g,n}(\boldsymbol{\tau};\hat{X}_1,\dots,\hat{X}_n) = \sum_{\vec{\mu}=(\mu_1,\dots,\mu_n),\mu_i>0}\ \sum_{\beta\ge 0,\ell\ge 0} \frac{\langle \boldsymbol{\tau}^{\ell}\rangle^{\mathcal{X},(\mathcal{L},f)}_{g,\beta,\vec{\mu}}}{\ell!}\hat{X}_1^{\mu_1}\dots\hat{X}_n^{\mu_n}.$$

This potential does depend on the choice of f, and is in degree 0 of v. Mirror symmetry predicts these $F^{\mathcal{X},(\mathcal{L},f)}_{g,n}$ from the mirror curve of $\mathcal{X}$. The free energy $F^{\mathcal{X}}_g$ is the special case for $n = 0$, involves only closed invariants and does not depend on f.

2.3. **Mirror curve as the B-model.** The mirror curve of $\mathcal{X}$ is the following

$$\{U_1U_2U_3U_4^{-3} = q,\ U_1 + U_2 + U_3 + U_4 = 0\}/\mathbb{C}^*.$$

Here q is the complex parameter. The overall $\mathbb{C}^*$ action rescales $U_1,\dots,U_4$ simultaneously.

One can rewrite the mirror curve in an equation

$$C_q = \{H_q(X,Y) = X + Y + 1 + qX^3Y^{-1} = 0\} \subset (\mathbb{C}^*)^2. \quad (1)$$

We will see that these specific choice of coordinates are related to the phase (location) of $\mathcal{L}$. Each term of H_q corresponds to an integer point in the defining polytope P.

When $|q|$ is small, the curve C_q is a genus 1 curve with three punctures (see Figure 3). The affine curve C_q allows a natural compactification $\bar{C}_q$ in $\mathbb{S}_P$, the toric surface associated to the defining polytope. In this particular example $\mathbb{S}_P \cong \mathbb{P}^2/\mathbb{Z}_3$, which is a singular toric Fano surface. The curve $\bar{C}_q$ is a compact Riemann surface of genus 1. There are three puncture points in C_q, which are the intersection $\bar{C}_q \cap \mathbb{S}_P$. When $q = 0$, $\bar{C}_q$ degenerates into a compact nodal curve $\bar{C}_0$, while the curve C_q also denegerates into a nodal curve $C_0 \subset \bar{C}_0$. We

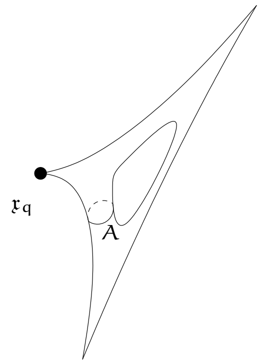

FIGURE 3. The mirror curve C_q of $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$.

denote a very small ball $\mathcal{B}$ containing 0 such that when $q \in \mathcal{B} \setminus \{0\}$ the curve C_q and $\bar{C}_q$ are smooth.

One of the punctures in C_q for $q \in \mathcal{B}$ has coordinates $(X, Y) = (0, -1)$. We label this point as a *open large radius limit* $\mathfrak{x}_q$, which corresponds to the large radius in the open parameter.

The choice of framing in the A-model is interpreted of changing coordinates in the B-model. Let

$$X = \hat{X}\hat{Y}^{-f}, \ Y = \hat{Y}.,$$

The mirror curve equation becomes

$$H_q = \hat{X}\hat{Y}^{-f} + \hat{Y} + 1 + q\hat{X}^3\hat{Y}^{-1-3f}.$$

We introduce the Seiberg-Witten form

$$\Phi = \log \hat{Y} \frac{d\hat{X}}{\hat{X}}.$$

This form is multi-valued – it is well-defined on the universal cover of C_q. The B-model genus 0 disk potential is defined as

$$\check{F}^{\mathcal{X}}_{0,1}(q; X) \text{“}=\text{”} \int_{\hat{X}} \log \hat{Y} \frac{d\hat{X}}{\hat{X}}.$$

Let $\mathcal{U}_q$ be a small neighborhood of $\mathfrak{x}_q$ in $\bar{C}_q$. We should understand this integral as an anti-derivative, where $\log \hat{Y}$ is expanded in terms of $\hat{X}$ in $\mathcal{U}_q$. One should discard the anti-derivative from the degree 0 term in the expansion of $\log \hat{Y}$, and obtain a power series in $\hat{X}$ with no degree 0 term. There is a constant ambiguity while taking $\log$ – however it is in the discarded part and does not contribute. We use the notation “ = ” to denote that the degree-0 term in $\hat{X}$ is discarded.

The higher genus B-model theory is defined via the Eynard-Orantin topological recursion. It starts from a spectral curve, which contains the following data

- an affine curve C_q and its compactification $\bar{C}_q$;
- two holomorphic Morse functions $\hat{X}$ and $\hat{Y}$ on C_q and meromorphic on $\bar{C}_q$ – their critical points do not coincide;
- a fundamental bidifferential form $\omega_{0,2}$ (Bergman kernel), which is a meromorphic symmetric form on $\bar{C}_q^2$.

We explain $\omega_{0,2}$ a little bit here. It is uniquely determined by a Lagrangian subspace $\mathcal{A} \subset H_1(\bar{C}_q;\mathbb{C})$. The symplectic pairing on this space is the cohomology pairing (PD is Poincarè pairing)

$$(a,b) = \int_{\bar{C}_q} \mathrm{PD}(a) \cup \mathrm{PD}(b).$$

The fundamental form $\omega_{0,2}$ is uniquely characterized by $\mathcal{A}$ and its pole behavior:

- For any cycle $A \in \mathcal{A}$,
$$\int_{z_2 \in A} \omega_{0,2}(z_1, z_2) = 0.$$
- The only pole of $\omega_{0,2}$ is the double pole at the diagonal and normalized at
$$\omega_{0,2} = \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{holomorphic part}.$$

We let $X = e^{-x}, Y = e^{-y}, \hat{X} = e^{-\hat{x}}, \hat{Y} = e^{-\hat{y}}$. Near each ramification point of $\hat{x}$, we denote $\bar{p}$ to be the point such that $\hat{x}(\bar{p}) = \hat{x}(p)$ and $\bar{p} \neq p$.

The Eynard-Orantin's topological recursion produces a meromorphic symmetry n-form on $\bar{C}_q^n$ recursively as below.

(2)

$$\omega_{g,n}(p_1,\dots,p_n) = \sum_{d\hat{x}|_{p'}=0} \mathrm{Res}_{p=p'} \frac{\int_{\xi=p}^{\bar{p}} B(p_n,\xi)}{2(\Phi(p)-\Phi(\bar{p}))} \Big(\omega_{g-1,n+1}(p,\bar{p},p_1,\dots,p_{n-1}) + \sum'_{g_1+g_2=g,\ I\sqcup J=\{1,\dots,n-1\}} \omega_{g_1,|I|+1}(p,p_I)\omega_{g_2,|J|+1}(\bar{p},p_J)\Big).$$

Here the sum symbol $\sum'$ excludes the case $(g_1,|I|) = (0,1),(0,n-1),(g,1)$ or $(g,n-1)$.

The resulting form $\omega_{g,n}$ (for $2g-2+n>0$) is smooth away from the ramification point $d\hat{x}=0$. In particular they are holomorphic in $\mathcal{U}_q^n$, where $\mathcal{U}_q$ is the small open neighborhood around $\mathfrak{r}_q$ in each copy of $\bar{C}_q$.

We still need to specify $\mathcal{A}$ as a Lagrangian subspace of $H_1(\bar{C}_q;\mathbb{C})$ to completely write down the mirror curve as a spectral curve and produce higher genus invariants.

2.4. **The remodeling conjecture.** The BKMP remodeling conjecture [6,7,17] predicts $F_{g,n}^{\mathcal{X},(\mathcal{L},f)}$ by $\omega_{g,n}$. Both sides are related by a change of variables called the *mirror map*.

We explicitly write down the mirror map here for $\mathcal{X}$. The cohomology $H^*_{\mathbb{T}'_f}(\mathcal{X};\mathbb{C})$ is a 2-dimensional $\mathbb{C}$-vector space – the equivariant parameter v *and* any lift of the hyperplane class form a basis. We let H be the equivariant lift of the hyperplane class such that $\iota_0^* H = 0$. (Recall that $\iota_0 : p_0 \to \mathcal{X}$ is the embedding of the $\mathbb{T}$-fixed point "closest" to $\mathcal{L}$)

$$\boldsymbol{\tau} = (\log q - 3\sum_{d>0} \frac{(-1)^{d-1}(3d-1)!}{(d!)^3} q^d)H, \tag{3}$$
$$\log \check{X} = \log \hat{X} + \sum_{d>0} \frac{(-1)^{d-1}(3d-1)!}{(d!)^3} q^d.$$

These mirror maps have geometric interpretation. There exists an cycle $A \in H_1(C_q;\mathbb{Z})$ such that

$$\boldsymbol{\tau} = \tau H = (\frac{1}{2\pi\sqrt{-1}} \int_A \Phi)H.$$

We denote the image of this cycle in $H_1(\bar{C}_q;\mathbb{Z})$ by $\bar{A}$. It spans a Lagrangian subspace of $H_1(\bar{C}_q;\mathbb{C})$. Therefore the mirror curve is then equipped with a spectral curve structure.

We define B-model open potentials

$$\check{F}^{\mathcal{X}}_{0,2}(q;\hat{X}_1,\hat{X}_2) = \int_{\hat{X}_1}\int_{\hat{X}_2}\left(\omega_{0,2} - \frac{d\hat{X}_1 d\hat{X}_2}{(\hat{X}_1-\hat{X}_2)^2}\right),$$
$$\check{F}^{\mathcal{X}}_{g,n}(q;\hat{X}_1,\dots,\hat{X}_n) = \int_{\hat{X}_1}\cdots\int_{\hat{X}_n}\omega_{g,n},\ 2g-2+n>0.$$

Similarly to $\check{F}^{\mathcal{X}}_{0,1}$, we consider the expansion of $\omega_{g,n}$ in $\mathcal{U}_q^n$, and the resulting integrals (anti-derivatives) are power series in $\hat{X}_1,\dots,\hat{X}_n$ with no degree 0 term. Notice $\omega_{0,2}$ has diagonal pole so we need to subtract the principal part first.

Theorem 2.1. *We have the following mirror symmetry statements, where* $q \in \mathcal{B}$ *and* $\hat{X} \in \mathcal{U}_q$*:*

- *Disk mirror theorem* [2,3], *proved in* [15]*:*
$$F_{0,1}^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau};\tilde{X}) = \check{F}_{0,1}^{\mathcal{X}}(q;\hat{X}).$$
- *Higher genus mirror symmetry, a.k.a. the BKMP remodeling conjecture* [6,7,17], *proved in* [10,14]*:*
$$F_{g,n}^{\mathcal{X},(\mathcal{L},f)}(\boldsymbol{\tau};\tilde{X}_1,\ldots,\tilde{X}_n) = \check{F}_{g,n}^{\mathcal{X}}(q;\hat{X}_1,\ldots,\hat{X}_n).$$
- *For* $g>1$*, the free energy*
$$F_g^{\mathcal{X}}(\boldsymbol{\tau}) = \frac{1}{2-2g}\sum_{d\hat{x}(p_0)=0}\mathrm{Res}_{p=p_0}\omega_{g,1}(p)\int\Phi(p).$$
Here $\int\Phi$ *is the anti-derivative of* Φ*, which we regard as a local function around each ramification point (the ambiguity does not affect the residue).*

2.5. **The remodeling conjecture for $\mathbb{C}^3/\mathbb{Z}_3$.** We let $\mathcal{X}' = \mathbb{C}^3/\mathbb{Z}_3$, the quotient stack. The orbifold $\mathcal{X}'$ is obtained by the same polytope P in Section 2.1 while there is no further triangulation inside the polytope. It is given by the GIT quotient at a different stability condition.
$$\mathcal{X}' = (\mathbb{C}^3\times\mathbb{C}^*)/G,$$
where the torus $G\cong\mathbb{C}^*$ acts by
$$t\cdot(Z_1,\ldots,Z_4) = (tZ_1,tZ_2,tZ_3,t^{-3}Z_4).$$
It is also a symplectic quotient
$$\mathcal{X}' = \tilde{\mu}^{-1}(r)/G_{\mathbb{R}},$$
where $r<0$. The Aganagic-Vafa brane $\mathcal{L}'$ is in the pre-image of the point a' for the moment polytope $\mu'_{\mathbb{R}}$ in the toric graph as in Figure 4. We should consider the Chern-Ruan orbifold cohomology $H^*_{\mathrm{CR}}(\mathcal{X}';\mathbb{C})$ for the extended Kähler classes. In particular, $H^2_{\mathrm{CR}}(\mathcal{X}';\mathbb{C})$ is generated by an age 1 element. The open-closed Gromov-Witten potentials are defined as
$$F_{g,n}^{\mathcal{X}',(\mathcal{L}',f)}(\boldsymbol{\tau}';\hat{X}'_1,\ldots,\hat{X}'_n) = \sum_{\vec{\mu}=(\mu_1,\ldots,\mu_n),\mu_i>0}\sum_{\ell\geq 0}\frac{\langle\boldsymbol{\tau}'^{\ell}\rangle_{g,\vec{\mu}}^{\mathcal{X}',(\mathcal{L}',f)}}{\ell!}\hat{X}_1'^{\mu_1}\ldots\hat{X}_n'^{\mu_n}\in\mathbb{Q}$$
for $\boldsymbol{\tau}'\in H^2_{\mathrm{CR},\mathbb{T}'_f}(\mathcal{X}';\mathbb{C})$. When $n=0$, this is usually written as $F_g^{\mathcal{X}}$, and it involves only closed Gromov-Witten invariant which do not depend on f.

The mirror curve is also explicitly given by
$$H'_q(X',Y') = 1 + X'^3Y'^{-1} + Y' + q'X'; \tag{4}$$

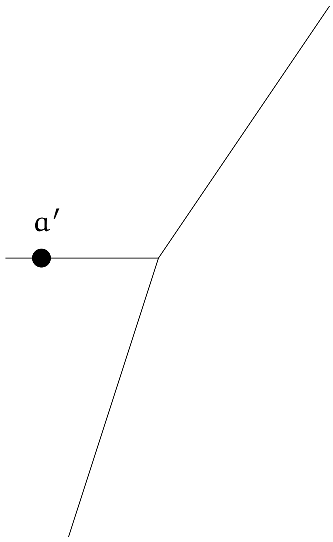

FIGURE 4. The toric graph of $\mathcal{X}' = \mathbb{C}^3/\mathbb{Z}_3$.

while the framed mirror curve equation is also given by a simple change of variables as below.

$$X' = \hat{X}'\hat{Y}'^{-f}, \ Y' = \hat{Y}',$$
$$H'_q = 1 + \hat{X}'^3\hat{Y}'^{-1-3f} + \hat{Y}' + q'\hat{X}'\hat{Y}'^{-f};$$

We denote this mirror curve by $C'_{q'}$ and its compactification by $\bar{C}'_{q'}$. When $|q'|$ is very small, $C'_{q'}$ is also a 3-punctured curve of genus 1, while $\bar{C}'_{q'}$ is a compact Riemann surface of genus 1. When $q' = 0$, $\bar{C}'_0$ is *not* singular, unlike the mirror curve of $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$. We denote a small neighborhood $\mathcal{B}'$ of 0 such that when $q' \in \mathcal{B}'$, $C'_{q'}$ and $\bar{C}'_{q'}$ are smooth.

There is also a distinguished point $\mathfrak{s}'_{q'}$ in $C'_{q'} \subset \bar{C}'_{q'}$ given by $X' = 0$ and $Y' = -1$ for $q' \in \mathcal{B}'$. We use $\mathcal{U}'_q$ to denote an small open neighborhood of $\mathfrak{x}'_{q'}$ in $\bar{C}'_{q'}$. Then we define open B-model disk potential as below

$$\check{F}_{0,1}^{\mathcal{X}'} \text{ “ = ” } \int_{\hat{X}'} \Phi',$$

where $\Phi' = \log \hat{Y}' \frac{d\hat{X}'}{\hat{X}'}$. We consider this integral as an anti-derivative of Φ' expanded in $\mathcal{U}_{q'}$, and define $\check{F}_{0,1}^{\mathcal{X}'}$ by discarding degree-0 terms in $\hat{X}'$.

To construct higher genus B-model open potential, one also runs the Eynard-Orantin topological recursion. The cohomology $H^2_{\mathbb{T}'_f}(\mathcal{X};\mathbb{C})$

is a 2-dimensional $\mathbb{C}$-vector space. Let $\mathbf{1}_1$ be the generator of the age 1 elements. We also have a cycle $A' \in H_1(C'_q;\mathbb{C})$ such that the integral

$$\boldsymbol{\tau}' = (\frac{1}{2\pi\sqrt{-1}}\int_{A'}\Phi')\mathbf{1}_1 = q'(\sum_{k\geq 0}\frac{\Gamma(2/3)^3}{\Gamma(\frac{2}{3}-k)^3(3k)!}3q'^{3k})\mathbf{1}_1.$$

The open mirror map is trivial for $\mathcal{X}' = \mathbb{C}^3/\mathbb{Z}_3$:

$$\tilde{X}' = \hat{X}'.$$

The Lagrangian subspace $\mathcal{A}'$ spanned by $\bar{A}'$, the image of A' in $H_1(\bar{C}'_{q'};\mathbb{C})$, is the last piece of information to make $C'_{q'}$ and $\bar{C}'_{q'}$ into a spectral curve. Then the Eynard-Orantin topological recursion produces $\omega'_{g,n}$. We define

$$\check{F}^{\mathcal{X}'}_{0,2}(q';\hat{X}'_1,\hat{X}'_2) = \int_{\hat{X}'_1}\int_{\hat{X}'_2}\left(\omega'_{0,2} - \frac{d\hat{X}'_1 d\hat{X}_2}{(\hat{X}'_1-\hat{X}'_2)^2}\right),$$

$$\check{F}^{\mathcal{X}'}_{g,n}(q;\hat{X}'_1,\dots,\hat{X}'_n) = \int_{\hat{X}'_1}\cdots\int_{\hat{X}'_n}\omega'_{g,n},\ 2g-2+n>0.$$

These integrals are understood as anti-derivatives for the relevant differential forms in $(\mathcal{U}'_{q'})^n \subset \bar{C}'^n_{q'}$.

Theorem 2.2. *We have the following mirror symmetry statements under the open-closed mirror map, where $q' \in \mathcal{B}'$ and $\hat{X}' \in \mathcal{U}'_{q'}$:*

- *Disk mirror theorem, proved in* [12]*:*

$$F^{\mathcal{X}',(\mathcal{L}',f)}_{0,1}(\boldsymbol{\tau}';\tilde{X}') = \check{F}^{\mathcal{X}_1}_{0,1}(q';\hat{X}').$$

- *Higher genus mirror symmetry, a.k.a. the BKMP remodeling conjecture* [7]*, proved in* [13]*:*

$$F^{\mathcal{X}',(\mathcal{L}',f)}_{g,n}(\boldsymbol{\tau};\tilde{X}'_1,\dots,\tilde{X}'_n) = \check{F}^{\mathcal{X}_1}_{g,n}(q';\hat{X}'_1,\dots,\hat{X}'_n).$$

- *When $g>1$, the free energy*

$$F^{\mathcal{X}'}_g(\boldsymbol{\tau}) = \frac{1}{2-2g}\sum_{d\hat{x}'(p_0)=0}\mathrm{Res}_{p=p_0}\omega'_{g,1}(p)\int\Phi'(p).$$

3. THE CONSTRUCTION OF THE GLOBAL MIRROR CURVE

3.1. **Family of mirror curves.** The mirror curve equations (1) and (4) are the same after a simple change of variables:

$$q = q'^{-3},\ X = X'q',\ Y = Y'.$$

So C_q and $C'_{q'}$ should form a family of affine curves. Here we give a toric construction such that $\bar{C}_q$ and $\bar{C}_{q'}$ form a family of compact curves over the weighted projective line $\mathbb{P}(1,3)$.

Recall that $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$. Its fan is the cone over the defining polytope P, as shown in Figure 1.

Its secondary stacky fan $\mathfrak{S}$ is a complete fan in $\mathbb{R}$. The generators of is 1-cones are

$$\mathfrak{b}_1 = 1, \quad \mathfrak{b}_2 = 1, \quad \mathfrak{b}_3 = 1, \quad \mathfrak{b}_4 = -3.$$

The toric orbifold $\mathcal{M}_K \cong \mathbb{P}(1,3)$ defined by $\mathfrak{S}$ is the moduli space of the B-model, or conjecturally, is the stringly Kähler moduli space of the mirror A-model on $\mathcal{X}$. Denote the stacky torus fixed point by $\mathfrak{s}_{\mathrm{orb}}$ and the non-stacky smooth torus fixed point by $\mathfrak{s}_{\mathrm{LRL}}$.

FIGURE 5. The secondary fan of $\mathcal{X} = \mathcal{O}_{\mathbb{P}^2}(-3)$.

We now define the following extended secondary fan $\tilde{\mathfrak{S}}$ as a complete fan in $N_K \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^3$, where $N_K = \mathbb{Z}^3$. The generators of its 1-cones in N_K are

$$\tilde{\mathfrak{b}}_1 = (0,0,1), \quad \tilde{\mathfrak{b}}_2 = (-1,0,1), \quad \tilde{\mathfrak{b}}_3 = (0,-1,1), \quad \tilde{\mathfrak{b}}_4 = (-1,-1,-3),$$
$$\tilde{\mathfrak{b}}_5 = (1,1,0), \quad \tilde{\mathfrak{b}}_6 = (-2,1,0), \quad \tilde{\mathfrak{b}}_7 = (1,-2,0).$$

The top dimensional cones are spanned by $\tilde{\mathfrak{b}}_i$ where i ranges from the following index sets

$$\{4,5,6\},\ \{4,6,7\},\ \{4,5,7\},\ \{5,1,2\},\ \{5,1,3\},$$
$$\{6,1,2\},\ \{6,2,3\},\ \{7,2,3\},\ \{7,1,3\},\ \{1,2,3\}.$$

The 2-cones are faces of 3-cones. We denote the toric orbifold associated to the fan $\tilde{\mathfrak{S}}$ by $\tilde{\mathcal{M}}_K$.

There is an obvious fan map $\pi' : \mathbb{R}^3 \to \mathbb{R}^2$ that maps $\tilde{\mathfrak{S}}$ to $\mathfrak{S}$ which forgets the first two entries. It induces a toric map $\pi : \tilde{\mathcal{M}}_K \to \mathcal{M}_K$. The fiber $\pi^{-1}(\mathfrak{s})$ for $\mathfrak{s} \neq \mathfrak{s}_{LRL}$ is a toric orbifold defined by the stacky fan given by $\tilde{\mathfrak{b}}_5, \tilde{\mathfrak{b}}_6, \tilde{\mathfrak{b}}_7$ (on $\mathbb{R}^2$). It is isomorphic to $\mathbb{P}^2/\mathbb{Z}_3$. Over the smooth torus fixed point, the fiber $\pi^{-1}(\mathfrak{s}_{\mathrm{LRL}})$ is three $\mathbb{P}^2$ intersecting along three $\mathbb{P}^1$ with normal crossing singularities. If one intersects the fan $\tilde{\mathfrak{S}}$ by a vertical plane, at different horizontal position, we get the fan of each fiber toric surface. See Figure 7.

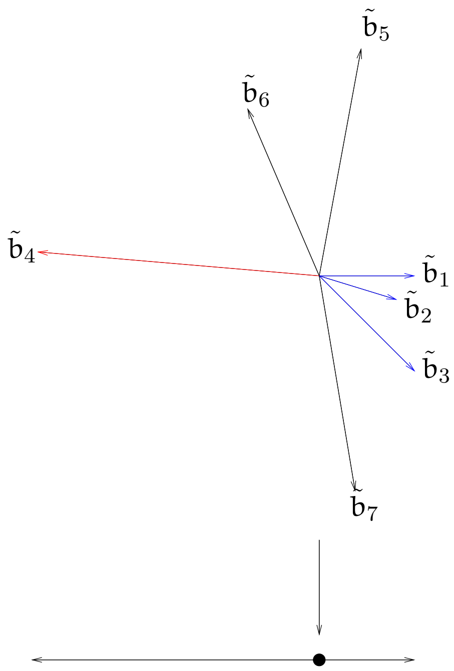

FIGURE 6. The extended secondary fan of $\mathcal{O}_{\mathbb{P}^2}(-3)$. The third coordinates of the generators $\tilde{b}_i$ are the same if they are in the same color. The rays $\tilde{b}_5, \tilde{b}_6, \tilde{b}_7$ form the toric graph of $\mathbb{C}^3/\mathbb{Z}_3$. There is an obvious fan map $\tilde{\mathfrak{S}} \to \mathfrak{S}$.

One can also understand $\tilde{\mathfrak{S}}$ in the following way. The fan $\mathfrak{S}$ lives inside $G_{\mathbb{R}}^\vee \cong \mathbb{R}$, and $\mathcal{X}_r = \tilde{\mu}^{-1}(r)/G_{\mathbb{R}}$ (here $\mathcal{X} \cong \mathcal{X}_r,\ r > 0$ and $\mathcal{X}' \cong \mathcal{X}_r,\ r < 0$). The intersection $\pi^{-1}(r) \cap (\tilde{\mathfrak{S}}(2) \cup \tilde{\mathfrak{S}}(1) \cup \tilde{\mathfrak{S}}(0))$ is precisely the toric graph of $\mathcal{X}_r$.

We understand X, Y, q as characters in $\mathrm{Hom}(\mathbb{T}_K, \mathbb{C}^*) = N_K^\vee$, where $\mathbb{T}_K$ is the open dense 3-torus in $\tilde{\mathcal{M}}_K$, and $N_K \cong \mathbb{Z}^3$ is the lattice that $\tilde{b}_i$ belong to. Then X, Y, q corresponds to $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ in $N_K^\vee$ respectively. They are sections of a line bundle $\mathbb{L} = \mathcal{O}_{\tilde{\mathcal{M}}_K}(\sum_{i=1}^6 D_i)$ (here each D_i is the toric divisor corresponding to each $\tilde{b}_i$). We define a section $H \in H^0(\mathbb{L})$

$$H = X + Y + 1 + qX^3Y^{-1}.$$

We define the compactified global mirror curve $\mathfrak{C} = H^{-1}(0) \subset \tilde{\mathcal{M}}_K$. It is parametrized over $\mathcal{M}_K$ by $\pi_{\mathfrak{C}} = \pi|_{\mathfrak{C}} : \mathfrak{C} \to \mathcal{M}_K$. For any $\mathfrak{s} \in \mathcal{M}_K$, the fiber $\pi_{\mathfrak{C}}^{-1}(\mathfrak{s})$ is a compact (possibly singular) curve. Let $\mathcal{M}_{K,0}$ be the part of $\mathcal{M}_K$ where $\pi_{\mathfrak{C}}^{-1}(\mathcal{M}_{K,0})$ is smooth. As shown in Figure

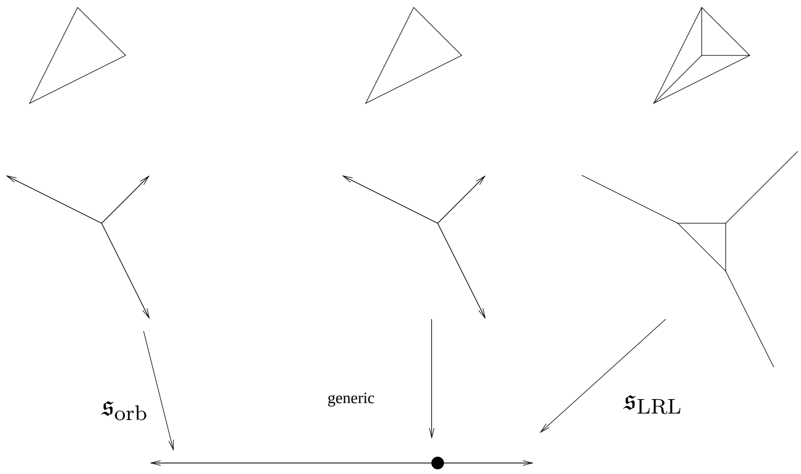

FIGURE 7. Over $\mathcal{M}_K$, we have a family of toric surfaces given by π. When $\mathfrak{s} \neq \mathfrak{s}_{\mathrm{LRL}}$, the fiber $\pi^{-1}(\mathfrak{s}) \cong \mathbb{P}^2/\mathbb{Z}_3$, given by the stacky fan spanned by $\tilde{b}_5, \tilde{b}_6, \tilde{b}_7$. Over $\mathfrak{s}_{\mathrm{LRL}}$, the toric surface degenerates to a normal crossing of three $\mathbb{P}^2$, as shown by the "fan" and the polytope. The first rows are polytopes and the second rows are fans for fiber toric surfaces at different points in $\mathcal{M}_K$.

8, $\mathfrak{s}_{\mathrm{LRL}} \notin \mathcal{M}_{K,0}$ since the fiber is a nodal curve (three $\mathbb{P}^1$ with nodal singularities), while $\mathfrak{s}_{\mathrm{orb}} \notin \mathcal{M}_{K,0}$ since itself is a stacky point. There is another point other than $\mathfrak{s}_{\mathrm{LRL}}$ not in $\mathcal{M}_{K,0}$, where the fiber has one nodal singularity. This point is called the conifold point $\mathfrak{s}_{\mathrm{con}}$. Thus $\mathcal{M}_{K,0} = \mathcal{M}_K \setminus \{\mathfrak{s}_{\mathrm{LRL}}, \mathfrak{s}_{\mathrm{orb}}, \mathfrak{s}_{\mathrm{con}}\}$.

By our notation, $\bar{C}_q$ and $\bar{C}'_{q'}$ are identified with $\mathfrak{C}_{\mathfrak{s}}$ when $q = q'^{-3}$, where $\mathfrak{C}_{\mathfrak{s}} = \pi_{\mathfrak{C}}^{-1}(\mathfrak{s})$, $q(\mathfrak{s}) = q$, and $q'(\mathfrak{s}) = q'$.

3.2. **Open crepant resolution conjecture for disk potentials.** The crepant resolution conjecture (CRC) for disk potentials is a direct consequence of the global mirror curve $\mathfrak{C}$. A CRC result should relate Gromov-Witten invariants around the large radius point to orbifold Gromov-Witten invariants around the orbifold points. The CRC for disk potentials, by its name, should relate $F_{0,1}^{\mathcal{X},(\mathcal{L},f)}$ and $F_{0,1}^{\mathcal{X}',(\mathcal{L}',f)}$.

We pick a path $\gamma : [0,1] \to \mathcal{M}_{K,0}$ such that $\gamma(0) = \mathfrak{s}_{\mathrm{LRL}}$ and $\gamma(1) = \mathfrak{s}_{\mathrm{orb}}$. We also pick a lift of this γ to $\tilde{\gamma} : [0,1] \to \mathfrak{C}$ such that $\tilde{\gamma}(0) = \mathfrak{x}_0$, $\tilde{\gamma}(1) = \mathfrak{x}'_0$, and $\pi \circ \tilde{\gamma} = \gamma$.

The function $\log \hat{Y} = \log \hat{Y}'$ is a well-defined analytic function from a small tubular neighborhood of $\tilde{\gamma}([0,1])$ in $\mathfrak{C}$ to $\mathbb{C}/\langle 2\pi\sqrt{-1}\rangle$. By the

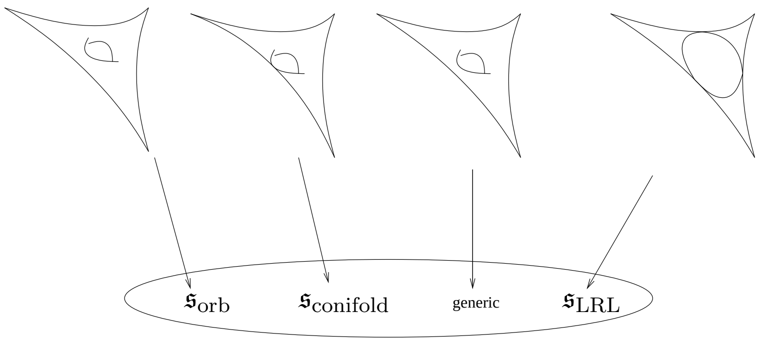

FIGURE 8. Over $\mathcal{M}_K$, we have a family of compactified mirror curves $\mathfrak{C}$. At $\mathfrak{s}_{\mathrm{con}}$ and $\mathfrak{s}_{\mathrm{LRL}}$ the mirror curves are singular. As before, the sharp ends in the mirror curve picture are the punctures on the mirror curve. After compactification, they become compact curves in $\pi^{-1}(\mathfrak{s})$. All puncture points are smooth.

disk mirror theorem

$$\hat{X}\frac{d}{d\hat{X}}F_{0,1}^{\mathcal{X},(\mathcal{L},f)} = \log\hat{Y}, \quad \hat{X}'\frac{d}{d\hat{X}'}F_{0,1}^{\mathcal{X}',(\mathcal{L}',f)} = \log\hat{Y}'$$

up to degree-0 terms in $\hat{X}$ or $\hat{X}'$. Since we know the degree-0 term of $\log\hat{Y}$'s expansion in terms of $\hat{X}$ is $\log(-1)$ (the deg-0 term of the expansion $\log\hat{Y}'$ in $\hat{X}'$ is also $\log(-1)$), one can analytically continuate $\hat{X}\frac{d}{d\hat{X}}F_{0,1}^{\mathcal{X},(\mathcal{L},f)}$, considered as a function near $\mathfrak{x}_0$, along $\tilde{\gamma}$. The resulting holomorphic function near $\mathfrak{x}_0'$ differs with $\hat{X}'\frac{d}{d\hat{X}'}F_{0,1}^{\mathcal{X}',(\mathcal{L}',f)}$ by an integral multiple of $2\pi\sqrt{-1}$.

3.3. **Modular invariance of fundamental normalized differentials of the second kind.** The mirror curve C_q (and its compactification $\bar{C}_q$) is a spectral curve. The genus of the compactified mirror curve $\bar{C}_q$ is 1. We fix two sets of Torelli markings $(\bar{A},\bar{B}),(\bar{A}',\bar{B}')$ on $\bar{C}_q$, such that

$$(\bar{A},\bar{B}) = (\bar{A}',\bar{B}') = 1,\ (\bar{A},\bar{A}) = (\bar{B},\bar{B}) = (\bar{A}',\bar{A}') = (\bar{B}',\bar{B}') = 0.$$

They differ by an $SL(2;\mathbb{Z})$ transformation

$$\begin{pmatrix}\bar{A}\\ \bar{B}\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\bar{A}'\\ \bar{B}'\end{pmatrix}$$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z})$. Let ω be the non-trivial holomorphic form on $\bar{C}_q$ given by the Torelli marking $(\bar{A},\bar{B})$, i.e.

$$\int_{\bar{A}} \omega = 1.$$

The period θ is given by

$$\theta = \int_{\bar{B}} \omega.$$

We know $\mathrm{Im}\theta > 0$. It depends on the choice of cycles A, B and the parameter q. Similarly,

$$\int_{\bar{A}'} \omega' = 1, \quad \theta' = \int_{\bar{B}'} \omega'.$$

We have

$$\theta' = \frac{\theta a - c}{d - \theta b}, \quad \omega' = J\omega,$$

where $J = (d - \theta b)^{-1}$.

Define the modified cycles

$$\bar{A}(\theta) = \bar{A} - \kappa \bar{B}(\theta), \quad \bar{B}(\theta) = \bar{B} - \theta \bar{A},$$
$$\bar{A}'(\theta') = \bar{A}' - \kappa \bar{B}(\theta'), \quad \bar{B}'(\theta') = \bar{B}' - \theta' \bar{A}.$$

Here

$$\kappa(\theta, \bar{\theta}) = \frac{1}{\bar{\theta} - \theta}$$

is a function of θ (not holomorphic). As a convention, we denote the fundamental differential associated to the A-cycle $\bar{A}$ by $\omega_{0,2}$, and the fundamental differential associated to the modified A-cycles $\bar{A}(\theta)$ by $\tilde{\omega}_{0,2}$. We also denote the fundamental differential associated to $\bar{A}'$ by $\eta_{0,2}$, while the fundamential differential associated to $\bar{A}'(\theta')$ by $\tilde{\eta}_{0,2}$.

By direct calculation, Eynard-Orantin show that in [9]

$$\tilde{\omega}_{0,2} = \omega_{0,2} + 2\pi\sqrt{-1}\theta\kappa(\theta, \bar{\theta})\theta.$$

They also show that

$$\eta_{0,2} = \omega_{0,2} + 2\pi\sqrt{-1}\theta\hat{\kappa}(\theta)\theta,$$

where $\hat{\kappa} = bJ$.

The fact that

$$J\kappa(\theta', \bar{\theta}')J + \hat{\kappa}(\theta) = \frac{1}{\bar{\theta} - \theta}$$

implies

$$\tilde{\eta}_{0,2} = \tilde{\omega}_{0,2}.$$

Proposition 3.1 (Eynard-Orantin)**.** *Given any Torelli marking* $(\bar{A}, \bar{B})$*, the modified fundamental differential* $\tilde{\omega}_{0,2}$ *given by the modified Torelli marking* $(\bar{A}(\theta), \bar{B}(\theta))$ *is independent of the choice of* $(\bar{A}, \bar{B})$*.*

This property implies that given a fixed spectral curve, we have a preferred choice of the fundamental differential $\tilde{\omega}_{0,2}$ independent of the choice of the A-cycles. Moreover, under the limit $\mathrm{Im}\theta \to \infty$, $\tilde{\omega}_{0,2} \to \omega_{0,2}$. Notice the parameter θ and $\omega_{0,2}$ depends on the choice of the A-cycle.

From the explicit expression of the Eynard-Orantin recursion (Equation (2)), for any spectral curve, we can define its modified B-model invariants $\tilde{\omega}_{g,n}$ based on this modified fundamental differential $\tilde{\omega}_{0,2}$, with

$$\lim_{\mathrm{Im}\theta \to \infty} \tilde{\omega}_{g,n} = \omega_{g,n}.$$

3.4. **Modularity.** The monodromies of the Gauss-Manin connection on the local system $H^1(\mathfrak{C}_{\mathfrak{s}}; \mathbb{C}) \cong H_1(\mathfrak{C}_{\mathfrak{s}}; \mathbb{C})$ over $\mathcal{M}_{K,0}$ (as computed in [1]) gives the *modular group* Γ of this local system. It is a normal subgroup of the symplectic group $SL(2; \mathbb{Z})$ of index 3.

Over $\mathcal{M}_{K,0}$, we have a smooth family of mirror curves, and the coordinates X, Y are well defined. So X, Y are invariant under the action of the modular group Γ. If we use the modified fundamental differential $\tilde{\omega}_{0,2}$ to define the higher genus B-model invariants $\tilde{\omega}_{g,n}$, then they are all well-defined global invariants on $\mathfrak{C}|_{\mathcal{M}_{K,0}}$. In other words, if one uses Torelli-marking-sensitive coordinate θ to express these $\tilde{\omega}_{g,n}$, they are invariant under the action of the modular group Γ.

Using the mirror map (3) we define the open potential in the holomorphic polarization under A-model flat coordinates when $2g - 2 + n > 0$.

$$\tilde{F}^{\mathcal{X},(\mathcal{L},f)}_{g,n}(\tilde{X}_1, \ldots, \tilde{X}_n, \tau) = \int_{\hat{X}_1} \cdots \int_{\hat{X}_n} \tilde{\omega}_{g,n}.$$

The A-model coordinate $Q = e^{\tau}$ is well-defined around the LRL point, and is related to B-model coordiante q around the LRL point under the closed mirror map. The open potential $\tilde{F}^{\mathcal{X},(\mathcal{L},f)}_{g,n}$ has non-holomorphic dependence on $\mathfrak{s}$ (q or θ), in contrast to the name "holomorphic polarization". Under the holomorphic limit

$$\lim_{\mathrm{Im}\theta \to \infty} \tilde{\omega}_{g,n} = \omega_{g,n}.$$

With the BKMP remodeling conjecture (Theorem 2.1), for $2g-2+n>0$ and $n\geq 1$

$$\lim_{\mathrm{Im}\theta\to\infty}\tilde{F}^{\mathcal{X},(\mathcal{L},f)}_{g,n}=F^{\mathcal{X},(\mathcal{L},f)}_{g,n}. \tag{5}$$

If one defines

$$\tilde{F}^{\mathcal{X}}_g=\frac{1}{2-2g}\sum_{d\hat{x}(p_0)=0}\mathrm{Res}_{p=p_0}\tilde{\omega}_{g,1}(p)\int\Phi(p),$$

then for $g\geq 2$

$$\lim_{\mathrm{Im}\theta\to\infty}\tilde{F}^{\mathcal{X}}_g=F^{\mathcal{X}}_g.$$

The potential $\tilde{F}^{\mathcal{X},(\mathcal{L},f)}_{g,n}$ and $\tilde{F}^{\mathcal{X}}_g$ are globally defined over $\mathcal{M}_K$, although their expansions in $Q=e^\tau$ are only defined around $\mathfrak{s}_{\mathrm{LRL}}$ since Q is a flat coordinate around $\mathfrak{s}_{\mathrm{LRL}}$. Their dependence on $\mathfrak{s}\in\mathcal{M}_K$ is not holomorphic.

Theorem 3.2. *The Gromov-Witten potential $F^{\mathcal{X}}_g$ can be completed into an analytic function $\tilde{F}^{\mathcal{X}}_g$, which under the mirror map* (3) *is globally defined on $\mathcal{M}_K$. When $\mathcal{X}=\mathcal{O}_{\mathbb{P}^2}(-3)$, $\mathcal{M}_K$ is a modular curve, the function $\tilde{F}^{\mathcal{X}}_g$ is a function of θ and modular invariant.*

Remark 3.3. *In the unstable cases $(g,n)=(0,0),(0,1),(0,2),(1,0)$, the theorem also holds but we need to treat these cases separately. We did not very clearly spell out what this "anti-holomorphic completion" is, as it should be stronger than* (5). *Indeed, $\tilde{F}^{\mathcal{X}}_g$ can be written as a polynomial in $\frac{1}{\mathrm{Im}\theta}$ with holomorphic coefficients* [9, 11]. *The lowest order of $\mathrm{Im}\theta$ is $2-2g$, and each coefficient in non-holomorphic terms are given by combinations of $F^{\mathcal{X}}_{g'}$, $g'<g$ and their derivatives in a graph sum formula.*

Remark 3.4. *One could use the modularity property to compute higher genus Gromov-Witten invariants for certain toric Calabi-Yau 3-(orbi)folds, thanks to the complete structure theorem of almost holomorphic modular forms. See* [1, 4, 18] *for numerical calculations and closed formulae for some $\tilde{F}^{\mathcal{X}}_g$ and $F^{\mathcal{X}}_g$.*

REFERENCES

[1] M. Aganagic, V. Bouchard, and A. Klemm, *Topological strings and (almost) modular forms*, Comm. Math. Phys. **277** (2008), no. 3, 771–819.

[2] M. Aganagic, A. Klemm, and C. Vafa, *Disk instantons, mirror symmetry and the duality web*, Z. Naturforsch. A **57** (2002), no. 1-2, 1–28.

[3] M. Aganagic and C. Vafa, *Mirror symmetry, D-branes and counting holomorphic discs*. arXiv:hep-th/0012041.

[4] M. Alim, E. Scheidegger, S.-T. Yau, and J. Zhou, *Special polynomial rings, quasi modular forms and duality of topological strings*, Adv. Theor. Math. Phys. **18** (2014), no. 2, 401–467.
[5] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Comm. Math. Phys. **165** (1994), no. 2, 311–427.
[6] V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti, *Remodeling the B-model*, Comm. Math. Phys. **287** (2009), no. 1, 117–178.
[7] V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti, *Topological open strings on orbifolds*, Comm. Math. Phys. **296** (2010), no. 3, 589–623.
[8] T.-M. Chiang, A. Klemm, S.-T. Yau, and E. Zaslow, *Local mirror symmetry: calculations and interpretations*, Adv. Theor. Math. Phys. **3** (1999), no. 3, 495–565.
[9] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory Phys. **1** (2007), no. 2, 347–452.
[10] B. Eynard and N. Orantin, *Computation of open Gromov-Witten invariants for toric Calabi-Yau 3-folds by topological recursion, a proof of the BKMP conjecture*, Comm. Math. Phys. **337** (2015), no. 2, 483–567.
[11] B. Eynard, N. Orantin, and M. Mariño, *Holomorphic anomaly and matrix models*, J. High Energy Phys. **6** (2007), 058, 20.
[12] B. Fang, C.-C. M. Liu, and H.-H. Tseng, *Open-closed Gromov-Witten invariants of* 3*-dimensional Calabi-Yau smooth toric DM stacks*. arXiv:1212.6073.
[13] B. Fang, C.-C. M. Liu, and Z. Zong, *All genus open-closed mirror symmetry for affine toric Calabi-Yau* 3*-orbifolds*. arXiv:1310.4818.
[14] ———, *On the remodeling conjecture for toric Calabi-Yau 3-orbifolds*. arXiv:1604.07123.
[15] T. Graber and E. Zaslow, *Open-string Gromov-Witten invariants: calculations and a mirror "theorem"*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 107–121.
[16] A. Klemm and E. Zaslow, *Local mirror symmetry at higher genus*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), AMS/IP Stud. Adv. Math., vol. 23, Amer. Math. Soc., Providence, RI, 2001, pp. 183–207.
[17] M. Mariño, *Open string amplitudes and large order behavior in topological string theory*, J. High Energy Phys. **3** (2008), 060, 34.
[18] J. Zhou, *Arithmetic Properties of Moduli Spaces and Topological String Partition Functions of Some Calabi-Yau Threefolds*, ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–Harvard University.

BOHAN FANG, BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, 5 YIHEYUAN ROAD, BEIJING 100871, CHINA
E-mail address: bohanfang@gmail.com