GROMOV-WITTEN INVARIANTS AND RATIONALLY CONNECTEDNESS

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1. Introduction

One ultimate goals of algebraic geometry is to classify all the projective varieties $X \subset \mathbb{P}^n$ over $\mathbb{C}$ up to isomophism. Rational curves play a very important role in the classification theory of algebraic varieties.Rational curves are, by definition, images of the projective line $\mathbb{P}^1$ by non-constant morphisms, thus very special objects in algebraic geometry. In fact, a "very general algebraic variety", called non-uniruled varieties, will not carry any rational curve on it, while many varieties, called uniruled varieties, carry sufficiently many rational curves. For those algebraic varieties which do not carry much rational curves, one wishes to construct a "minimal model" by a sequence of contractions analogous to the blow-downs. The study of rational curves on algebraic varieties includes the following topics: techniques to find rational curves on certain class of varieties; characterization of uniruled varieties; generic semipositivity of the cotangent bundle of non-uniruled varieties; decomposition of a given variety into the "non-uniruled part" and the "rationally connected part"; and their applications.

In 80's, based on the understanding of the role of rational cuves on algebraic varieties, Mori proposed his Minimal Model Program (MMP) for birational classification of higher-dimensional algebraic varieties. Mori program divides algebraic varieties into two categories: uniruled varieties and non-uniruled varieties. The minimal model program was carried out to a large extent in higher dimensions. See [BCHM, D, K1, KM, Mat1, Siu] for more details. Roughly speaking, if we take MMP as a classification machine, then we will obtain either a minimal model or a Mori fiber space when we put an algebraic varieties into the machine, i.e.,

X
↓
MMP
↙ ↘
X_{min}:a minimal model X_{mori}: a Mori fiber space

Among uniruled varieties, there is a special class consisting of rationally connected varieties. Roughly speaking, a variety X is said to be uniruled if

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X contains a rational curve through every point. If two arbitrary points on X can be joined by a rational curve on X, X is called rationally connected. Even for uniruled algebraic manifolds, the classification is also difficult. Uniruled (or rationally connected) curves are nothing but rational curves. In dimension two, a uniruled (resp. rationally connected) surface is a ruled (resp. rational) surface.

In symplectic geometry category, symplectic geometers [Ltj1, Ltj2, M1, M2, M3]was looking for a suitable classification theory for symplecitic manifolds long since. In the early 90's, Ruan [R1, R2] first speculated that Mori's MMP could be extend to symplectic geometry via the newly created Gromov-Witten theory, this means that there should be a birational symplectic geometric program. In particular, Ruan [R2] extended Mori's extremal rays to the symplectic category and used it to study the symplectomorphism group. Similar to the role of rational curves in the classification theory of algebraic varieties, one needs to find its corresponding in symplectic category. Gromov [Gr] introduced the peudo-holomorphic curves into the symplectic category. Ruan-Tian [R1, RT1, RT2] first established Gromov-Witten theory for semi-positive symplectic manifolds. Later, semi-positivity condition has been removed by many authors [B, FO, LT1, LT2, R3, S]. The flexibility of symplectic geometry should give a better understanding of birational algebraic geometry in the same way that the Gromov-Witten theory gave a much better understanding of the role rational curves in Mori theory. Kollár [K2] and Ruan [R3] used Gromov-Witten invariants to characterize the uniruledness of projective manifolds, and proved that a smooth projective uniruled manifold carries a nonzero genus zero Gromov-Witten invariant with a point insertion. This result convinced symplectic geometers of the existence of birational symplectic geometry.

To classify symplectic manifolds in birational way, one first needs to generalize the notion of birational equivalence in algebraic geometry to symplectic geometry. Simple birational operations such as blow-up/blow-down were known in symplectic geometry for a long time [GS, MS]. But there was no direct generalizaiton of a general birational map in the flexible symplectic category. Recall that two algebraic varieties are birational equivalent if and only if there is an isomorphism between Zariski open sets, i. e., there is a birational map between them, but not necessarily defined everywhere. If a birational map is defined everywhere, it is called as a contraction. Since a contraction changes a lower dimensional uniruled subvariety only, this enables us to view it as a topological surgery. On the other hand, Abramovich-Karu-Matsuki-Wlodarczyk [AKMW, Mat2] showed the weak factorization theorem that any birational map between projective manifolds can be decomposed as a sequence of blow-ups and blow-downs. This result matches perfectly the picture of the wall crossing of symplectic reductions analyzed by Guillemin and Sternberg in the 80's [GS]. Therefore, together Tian-Jun Li and Yongbin Ruan, the author [HLR] proposed to use their notion of cobordism in [GS] as the symplectic analogue of the birational equivalence, called symplectic birational cobordism, see subsection 2.1. Based on the theorem of Kollár-Ruans [K2, R3], we call a symplectic manifold (X, ω) (symplectically or numerically) uniruled if there is a nonzero genus zero Gromov-Witten invariant with a point insertion. Furthermore, we also showed that symplectic uniruledness is a birational invariant. This result took the decisive step toward the classification of symplectic manifolds, i. e., birational symplectic geometry.

In Mori's theory (dimension ≥ 3), if the fundamental birational invariant – the Kodaira dimension $\kappa \geq 0$, then in a birational equivalence class MMP may produce many minimal models which are isomorphic in codimension one. But when we look at the other end of MMP, i. e., the Mori fiber space which is uniruled, they may not even be isomorphic in codimension one even in a fixed birational equivalence class. Therefore it is necessary to further analyse the structure of uniruled varieties. Among uniruled varieties, rationally connected variety is a class of varieties with stronger property, on which a general pair of points can be connected by a rational curve. If we hope to characterize the geometry and topology of rationally connected varieties similar to the uniruled case as we do in [HLR], then we propose to define symplectically (or numerically) rationally connectedness via a nonzero genus zero Gromov-Witten invariant with two point insertions. If we choose this definition, then we need to ask the whether a rationally connected projective variety is symplectically rationally connected or not, and whether symplectically rationally connectedness is invariant under symplectic birational cobordism. This expository paper will survey the development along this direction.

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2. Preliminaries

2.1. **Birational Cobordism.** In this subsection, we will introduce symplectic biratonal cobordism. Readers may find all the details in [GS, HLR]. We start with the definition which is essentially contained in [GS].

Definition 2.1. *Two symplectic manifolds (X, ω) and (X', ω') are birational cobordant if there are a finite number of symplectic manifolds (X_i, ω_i), $0 \leq i \leq k$, with $(X_0, \omega_0) = (X, \omega)$ and $(X_k, \omega_k) = (X', \omega')$, and for each i, (X_i, ω_i) and (X_{i+1}, ω_{i+1}) are symplectic reductions of a semi-free Hamiltonian S^1-symplectic manifold W_i (of 2 more dimension).*

Here an S^1-action is called semi-free if it is free away from the fixed point set. Before state the generalization of weak factorization theorem to symplectic geometry, we first need to introduce $\mathbb{Z}$-linear deformation of symplectic structures.

Definition 2.2. *A $\mathbb{Z}$-linear deformation is a path of symplectic form $\omega + t\kappa$, $t \in I$, where κ is a closed 2-form representing an integral class and I is an interval. Two symplectic forms are $\mathbb{Z}$-linear deformation equivalent if they are joined by a finite number of $\mathbb{Z}$-linear deformations.*

According to [GS], we have the following basic factorization result:

Theorem 2.3. *A birational cobordism can be decomposed as a sequence of elementary ones, which are modeled on blow-up, blow-down and $\mathbb{Z}$-linear deformation of symplectic structure.*

Comparing with the weak factorization theorem, we have

Theorem 2.4. *Two birational projective manifolds with any polarizations are birational as symplectic manifolds.*

2.2. **GW-invariants.** As mentioned in the introduction, in this note, we want to use Gromov-Witten invariant to characterize the property of rationally connectedness of symplectic manifolds. So in this subsection, we introduce the Gromov-Witten invariant and its degeneration formula.

Suppose that (X, ω) is a compact symplectic manifold and J is a tamed almost complex structure.

Definition 2.5. *A stable J-holomorphic map is an equivalence class of pairs (Σ, f). Here Σ is a connected nodal marked Riemann surface with arithmetic genus g, k smooth marked points $x_1, ..., x_k$, and $f : \Sigma \longrightarrow X$ is a continuous map whose restriction to each component of Σ (called a component of f in short) is J-holomorphic. Furthermore, it satisfies the stability condition: if $f|_{S^2}$ is constant (called a ghost bubble) for some S^2-component, then the S^2-component has at least three special points (marked points or nodal points). (Σ, f), (Σ', f') are equivalent, or $(\Sigma, f) \sim (\Sigma', f')$, if there is a biholomorphic map $h : \Sigma' \longrightarrow \Sigma$ such that $f' = f \circ h$.*

An essential feature of Definition 2.5 is that, for a stable J-holomorphic map (Σ, f), the automorphism group

$$\mathrm{Aut}(\Sigma, f) = \{h \mid h \circ (\Sigma, f) = (\Sigma, f)\}$$

is finite. We define the moduli space $\overline{\mathcal{M}}_A^X(g, k, J)$ to be the set of equivalence classes of stable J-holomorphic maps such that $[f] = f_*[\Sigma] = A \in H_2(X, \mathbb{Z})$. The virtual dimension of $\overline{\mathcal{M}}_A^X(g, k, J)$ is computed by index theory,

$$\mathrm{virdim}_{\mathbb{R}} \overline{\mathcal{M}}_A^X(g, k, J) = 2c_1(A) + 2(n-3)(1-g) + 2k,$$

where n is the complex dimension of X.

Unfortunately, $\overline{\mathcal{M}}_A^X(g, k, J)$ is highly singular and may have larger dimension than the virtual dimension. Fortunately, $\overline{\mathcal{M}}_A^X(g, k, J)$ may carry a virtual fundamental class $[\overline{\mathcal{M}}_A^X(g, k, J)]^{vir}$, and we still can extract the Gromov-Witten invariant via the virtual integration against its virtual fundamental class, see [CLW, FO, R3, LT1, LT2, S] for the details.

There are evaluation maps

$$ev_i : \overline{\mathcal{M}}_A(g, k, J) \longrightarrow X, \quad (\Sigma, f) \to f(x_i),$$

for $1 \leq i \leq k$.

Definition 2.6. *The (primary) Gromov-Witten invariant is defined as*

$$\langle \alpha_1, \cdots, \alpha_k \rangle^X_{g,A} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \wedge_i (ev_i^* \alpha_i),$$

where $\alpha_i \in H^*(X; \mathbb{R})$. *For the genus zero case, we also write* $\langle \alpha_1, \cdots, \alpha_k \rangle^X_A =$ $\langle \alpha_1, \cdots, \alpha_k \rangle^X_{0,A}$.

Definition 2.7. *For each marked point* x_i, *we define an orbifold complex line bundle* $\mathcal{L}_i$ *over* $\overline{\mathcal{M}}^X_A(g, k, J)$ *whose fiber is* $T^*_{x_i}\Sigma$ *at* (Σ, f). *Denote* $c_1(\mathcal{L}_i)$, *the first Chern class of* $\mathcal{L}_i$, *by* ψ_i.

Definition 2.8. *The descendent Gromov-Witten invariant is defined as*

$$\langle \tau_{d_1} \alpha_1, \cdots, \tau_{d_k} \alpha_k \rangle^X_{g,A} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \wedge_i (\psi_i^{d_i} \wedge ev_i^* \alpha_i),$$

where $\alpha_i \in H^*(X; \mathbb{R})$.

Remark 2.9. *In the stable range* $2g + k \geq 3$, *one can also define non-primary GW invariants (See e.g.* [R1]*). Recall that there is a map* $\pi : \overline{\mathcal{M}}^X_A(g, k, J) \to \overline{\mathcal{M}}_{g,k}$ *contracting the unstable components of the source Riemann surface. We can introduce a class* κ *from the Deligne-Mumford space via* π *to define the ancestor GW invariants*

$$\langle \kappa \mid \Pi_i \alpha_i \rangle^X_{g,A} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \pi^* \kappa \wedge_i (ev_i^* \alpha_i).$$

The primary Gromov-Witten invariants are the special invariants with the point class in $\overline{\mathcal{M}}_{0,k}$.

Remark 2.10. *For each* $\langle \tau_{d_1} \alpha_1, \cdots, \tau_{d_k} \alpha_k \rangle^X_{g,A}$, *we can conveniently associate a simple graph* Γ *of one vertex decorated by* (g, A) *and a tail for each marked point. We then further decorate each tail by* (d_i, α_i) *and call the resulting graph* $\Gamma(\{(d_i, \alpha_i)\})$ a weighted graph. *Using the weighted graph notation, we denote the above invariant by* $\langle \Gamma(\{(d_i, \alpha_i)\}) \rangle^X$. *We can also consider the disjoint union* $\Gamma^\bullet$ *of several such graphs and use* $A_{\Gamma^\bullet}, g_{\Gamma^\bullet}$ *to denote the total homology class and total arithmetic genus. Here the total arithmetic genus is* $1 + \sum (g_i - 1)$. *Then, we define* $\langle \Gamma^\bullet(\{(d_i, \alpha_i)\}) \rangle^X$ *as the product of Gromov-Witten invariants of the connected components.*

2.3. **Relative GW-invariants.** The degeneration formula [LR, IP, Li2] provides a rigorous formulation about the change of Gromov-Witten invariants under semi-stable degeneration, or symplectic cutting. The formula relates the absolute Gromov-Witten invariant of X to the relative Gromov-Witten invariants of two smooth pairs. In this subsection, we will review the relative GW-invariants. The readers can find more details in the reference [LR].

Let $Z \subset X$ be a real codimension 2 symplectic submanifold. Suppose that J is an ω-tamed almost complex structure on X preserving TZ, i.e. making Z an almost complex submanifold. The relative Gromov-Witten invariants are defined by counting the number of stable J-holomorphic maps intersecting Z at finitely many points with prescribed tangency. More precisely, fix a k-tuple $T_k = (t_1, \cdots, t_k)$ of positive integers, consider a marked pre-stable curve

$$(C, x_1, \cdots, x_m, y_1, \cdots, y_k)$$

and stable J-holomorphic map $f : C \longrightarrow X$ such that the divisor f^*Z is

$$f^*Z = \sum_i t_i y_i.$$

To form a compact moduli space of such maps we thus must allow the target X to degenerate as well (compare with [Li1]). For any non-negative integer m, construct Y_m by gluing together m copies of $Y = \mathbb{P}(N_{Z|X} \oplus \mathbb{C})$, where the infinity section of the i^{th} component is glued to the zero section of the $(i+1)^{st}$ component for $1 \leq i \leq m$. Denote the zero section of the i^{th} component by Z_{i-1}, and the infinity section by Z_i, so Sing $Y_m = \cup_{i=1}^{m-1} Z_i$. We will also denote Z_m by Z_∞ if there is no possible confusion. Define X_m by gluing X along Z to Y_m along Z_0. Thus the singular part of X_m is $S(X_m) = \cup_{i=0}^{m-1} Z_i$, and $X_0 = X$. $X_0 = X$ will be referred to as the root component and the other irreducible components will be called the bubble components. Let $\mathrm{Aut}_Z Y_m$ be the group of automorphisms of Y_m preserving Z_0, Z_m, and the morphism to Z. And let $\mathrm{Aut}_Z X_m$ be the group of automorphisms of X_m preserving X (and Z) and with restriction to Y_m being contained in $\mathrm{Aut}_Z Y_m$ (so $\mathrm{Aut}_Z X_m = \mathrm{Aut}_Z Y_m \cong (\mathbb{C}^*)^m$, where each factor of $(\mathbb{C}^*)^m$ dilates the fibers of the $\mathbb{P}^1$-bundle $Y_i \longrightarrow Z_i$). Denote by $\pi[m] : X_m \longrightarrow X$ the map which is the identity on the root component X_0 and contracts all the bubble components to Z_0 via the fiber bundle projections.

Now consider a nodal curve C mapped into X_m by $f : C \longrightarrow X_m$ with specified tangency to Z. There are two types of marked points:

(i) absolute marked points whose images under f lie outside Z, labeled by x_i,

(ii) relative marked points which are mapped into Z by f, labeled by y_j.

A relative J-holomorphic map $f : C \longrightarrow X_m$ is said to be pre-deformable if $f^{-1}(Z_i)$ consists of a union of nodes such that for each node $p \in f^{-1}(Z_i), i =$

$1, 2, \cdots, m$, the two branches at the node are mapped to different irreducible components of X_m and the orders of contact to Z_i are equal.

An isomorphism of two such J-holomorphic maps f and f' to X_m consists of a diagram

$$\begin{array}{ccc} (C, x_1, \cdots, x_l, y_1, \cdots, y_k) & \xrightarrow{f} & X_m \\ h\downarrow & & \downarrow t \\ (C', x'_1, \cdots, x'_l, y'_1, \cdots, y'_k) & \xrightarrow{f'} & X_m \end{array}$$

where h is an isomorphism of marked curves and $t \in \mathrm{Aut}_Z(X_m)$. With the preceding understood, a relative J-holomorphic map to X_m is said to be stable if it has only finitely many automorphisms.

We introduced the notion of a weighted graph for an absolute stable map in Remark 2.10. We need to refine it for relative stable maps to (X, Z). A (connected) relative graph Γ consists of the following data:

(1) a vertex decorated by $A \in H_2(X; \mathbb{Z})$ and genus g,

(2) a tail for each absolute marked point,

(3) a relative tail for each relative marked point.

Definition 2.11. *Let Γ be a relative graph with k (ordered) relative tails and $T_k = (t_1, \cdots, t_k)$, a k-tuple of positive integers forming a partition of $A \cdot Z$. A relative J-holomorphic map to (X, Z) with type (Γ, T_k) consists of a marked curve $(C, x_1, \cdots, x_l, y_1, \cdots, y_k)$ and a map $f : C \longrightarrow X_m$ for some non-negative integer m such that*

(i) C is a connected curve (possibly reducible) of arithmetic genus g,

(ii) the map

$$\pi_m \circ f : C \longrightarrow X_m \longrightarrow X$$

satisfies $(\pi_m \circ f)_*[C] = A$,

(iii) the $x_i, 1 \le i \le l$, are the absolute marked points,

(iv) the $y_i, 1 \le i \le k$, are the relative marked points,

(v) $f^* Z_m = \sum_{i=1}^k t_i y_i$.

Let $\overline{\mathcal{M}}_{\Gamma,T_k}(X, Z, J)$ be the moduli space of pre-deformable relative stable J-holomorphic maps with type (Γ, T_k). Notice that for an element $f : C \to X_m$ in $\overline{\mathcal{M}}_{\Gamma,T_k}(X, Z, J)$ the intersection pattern with $Z_0, ..., Z_{m-1}$ is only constrained by the genus condition and the pre-deformability condition.

Similar to the absolute case, the moduli space $\overline{\mathcal{M}}_{\Gamma,T_k}(X, Z, J)$ also carries a virtual fundamental class $[\overline{\mathcal{M}}_{\Gamma,T_k}(X, Z, J)]^{vir}$, see [LR] for the detail.

In addition to the evaluation maps on $\overline{\mathcal{M}}_{\Gamma,T_k}(X, Z, J)$,

$$\begin{array}{rccl} ev_i^X : & \overline{\mathcal{M}}_{\Gamma,T_k}(X, Z, J) & \longrightarrow & X, \qquad 1 \le i \le l, \\ & (\Sigma, x_1, \cdots, x_l, y_1, \cdots, y_k, f) & \mapsto & f(x_i), \end{array}$$

there are also the evaluations maps

$$\begin{array}{cccc} ev_j^Z: & \overline{\mathcal{M}}_{\Gamma,T_k}(X,Z,J) & \longrightarrow & Z, \qquad 1\le j\le k, \\ & (\Sigma, x_1,\cdots,x_l,y_1,\cdots,y_k,f) & \mapsto & f(y_j), \end{array}$$

where $Z=Z_m$ if the target of f is X_m.

Definition 2.12. *Let $\alpha_i\in H^*(X;\mathbb{R})$,$1\le i\le l$, $\beta_j\in H^*(Z;\mathbb{R})$, $1\le j\le k$. Define the relative Gromov-Witten invariant*

$$\langle \Pi_i\tau_{d_i}\alpha_i \mid \Pi_j\beta_j\rangle^{X,Z}_{\Gamma,T_k} = \frac{1}{|Aut(T_k)|}\int^{vir} \Pi_i\psi_i^{d_i}\wedge (ev_i^X)^*\alpha_i\wedge \Pi_j(ev_j^Z)^*\beta_j,$$

where the virtual integral is against $[\overline{\mathcal{M}}_{\Gamma,T_k}(X,Z,J)]^{vir}$, and $Aut(T_k)$ is the symmetry group of the partition T_k. Denote by $\mathcal{T}_k=\{(t_j,\beta_j)\mid j=1,\cdots,k\}$ the weighted partition of $A\cdot Z$. If the vertex of Γ is decorated by (g,A), we will sometimes write

$$\langle \Pi_i\tau_{d_i}\alpha_i \mid \mathcal{T}_k\rangle^{X,Z}_{g,A}$$

for $\langle \Pi_i\tau_{d_i}\alpha_i \mid \Pi_j\beta_j\rangle^{X,Z}_{\Gamma,T_k}$.

Remark 2.13. *In* [LR] *only invariants without descendent classes were considered. But it is straightforward to extend the definition of* [LR] *to include descendent classes.*

We can decorate the tail of a relative graph Γ by (d_i,α_i) as in the absolute case. We can further decorate the relative tails by the weighted partition $\mathcal{T}_k$. Denote the resulting weighted relative graph by $\Gamma\{(d_i,\alpha_i)\}|\mathcal{T}_k$. In [LR] the source curve is required to be connected. We will also need to use a disconnected version. For a disjoint union $\Gamma^\bullet$ of weighted relative graphs and a corresponding disjoint union of partitions, still denoted by T_k, we use $\langle\Gamma^\bullet\{(d_i,\alpha_i)\}|\mathcal{T}_k\rangle^{X,Z}$ to denote the corresponding relative invariants with a disconnected domain, which is simply the product of the connected relative invariants. Notice that although we use $\bullet$ in our notation following [MP], our disconnected invariants are different. The disconnected invariants there depend only on the genus, while ours depend on the finer graph data.

2.4. **Partial orderings on relative GW invariants.** In [MP], the authors first introduced a partial order on the set of relative Gromov-Witten invariants of a $\mathbb{P}^1$-bundle. The authors, [HLR], refined their partial order on the set of relative Gromov-Witten invariants of a Blow-up manifold relative to the exceptional divisor, and used this partial order to obtain a Blow-up correspondence of absolute/relative Gromov-Witten theory. In this subsection, we will review the partial order on the set of relative Gromov-Witten invariants.

First of all, all Gromov-Witten invariants vanish if $A\in H_2(X,\mathbb{Z})$ is not an effective curve class. We define a partial ordering on $H_2(X,\mathbb{Z})$ as follows:

$$A' < A$$

if $A - A'$ is a nonzero effective curve class.

The set of pairs (m, δ) where $m \in \mathbb{Z}_{>0}$ and $\delta \in H^*(Z, \mathbb{Q})$ is partially ordered by the following size relation

$$(m, \delta) > (m', \delta') \tag{1}$$

if $m > m'$ or if $m = m'$ and $\deg(\delta) > \deg(\delta')$.

Let μ be a partition weighted by the cohomology of Z, i.e.,

$$\mu = \{(\mu_1, \delta_{r_1}), \cdots, (\mu_{\ell(\mu)}, \delta_{r_{\ell(\mu)}})\}.$$

We may place the pairs of μ in decreasing order by size (1). We define

$$\deg(\mu) = \sum \deg(\delta_{r_i}).$$

A lexicographic ordering on weighted partitions is defined as follows:

$$\mu \overset{l}{>} \mu'$$

if, after placing the pairs in μ and μ' in decreasing order by size, the first pair for which μ and μ' differ in size is larger for μ.

For the nondescendent relative Gromov-Witten invariant

$$\langle \varpi \mid \mu \rangle^{X,Z}_{g,A},$$

denote by $\|\varpi\|$ the number of absolute insertions.

Definition 2.14. *A partial ordering $\overset{\circ}{<}$ on the set of nondescendent relative Gromov-Witten invariants is defined as follows:*

$$\langle \varpi' \mid \mu' \rangle^{X,Z}_{g',A'} \overset{\circ}{<} \langle \varpi \mid \mu \rangle^{X,Z}_{g,A}$$

if one of the conditions below holds

(a) $A' < A$,
(b) *equality in (a) and* $g' < g$,
(c) *equality in (a)-(b) and* $\|\varpi'\| < \|\varpi\|$,
(d) *equality in (a)-(c) and* $\deg(\mu') > \deg(\mu)$,
(e) *equality in (a)-(d) and* $\mu' \overset{l}{>} \mu$.

2.5. **Degeneration formula.** Now we describe the degeneration formula of GW-invariants under symplectic cutting.

As an operation on topological spaces, the symplectic cut is essentially collapsing the circle orbits in the hypersurface $H^{-1}(0)$ to points in Z.

Suppose that $X_0 \subset X$ is an open codimension zero submanifold with a Hamiltonian S^1-action. Let $H : X_0 \to \mathbb{R}$ be a Hamiltonian function with 0 as a regular value. If $H^{-1}(0)$ is a separating hypersurface of X_0, then we obtain two connected manifolds $X_0^\pm$ with boundary $\partial X_0^\pm = H^{-1}(0)$. Suppose further that S^1 acts freely on $H^{-1}(0)$. Then the symplectic reduction $Z = H^{-1}(0)/S^1$ is canonically a symplectic manifold of dimension 2 less. Collapsing the S^1-action on $\partial X^\pm = H^{-1}(0)$, we obtain closed smooth manifolds $\overline{X_0}^\pm$ containing respectively real codimension 2 submanifolds $Z^\pm = Z$

with opposite normal bundles. Furthermore $\overline{X_0}^{\pm}$ admits a symplectic structure $\overline{\omega}^{\pm}$ which agrees with the restriction of ω away from Z, and whose restriction to $Z^{\pm}$ agrees with the canonical symplectic structure ω_Z on Z from symplectic reduction.

This is neatly shown by considering $X_0 \times \mathbb{C}$ equipped with appropriate product symplectic structures and the product S^1-action on $X_0 \times \mathbb{C}$, where S^1 acts on $\mathbb{C}$ by complex multiplication. The extended action is Hamiltonian if we use the standard symplectic structure $\sqrt{-1}dw \wedge d\bar{w}$ or its negative on the $\mathbb{C}$ factor.Then the moment map is

$$\mu_+(u, w) = H(u) + |w|^2 : X_0 \times \mathbb{C} \to \mathbb{R},$$

and $\mu_+^{-1}(0)$ is the disjoint union of S^1-invariant sets

$$\{(u, w)|H(u) = -|w|^2 < 0\} \quad \text{and} \quad \{(u, 0)|H(z) = 0\}.$$

We define $\overline{X_0}^+$ to be the symplectic reduction $\mu_+^{-1}(0)/S^1$. Then $\overline{X}_0^+$ is the disjoint union of an open symplectic submanifold and a closed codimension 2 symplectic submanifold identified with (Z, ω_Z). The open piece can be identified symplectically with

$$X_0^+ = \{u \in X_0|H(u) < 0\} \subset X_0$$

by the map $u \to (u, \sqrt{-H(u)})$.

Similarly, if we use $-\mathbf{i}dw \wedge d\bar{w}$, then the moment map is

$$\mu_-(u, w) = H(u) - |w|^2 : X_0 \times \mathbb{C} \to \mathbb{R}$$

and the corresponding symplectic reduction $\mu_-^{-1}(0)/S^1$, denoted by $\overline{X}_0^-$, is the disjoint union of an open piece identified symplectically with

$$X_0^- = \{u \in X_0|H(u) > 0\}$$

by the map $\phi_0^- : u \to (u, \sqrt{H(u)})$, and a closed codimension 2 symplectic submanifold identified with (Z, ω_Z).

We finally define $\overline{X}^+$ and $\overline{X}^-$. $\overline{X}^+$ is simply $\overline{X_0}^+$, while $\overline{X}^-$ is obtained from gluing symplectically X^- and $\overline{X}_0^-$ along X_0 via ϕ_0^-. Notice that $\overline{X}^- = (X^- - X_0) \cup \overline{X_0}^-$ as a set.

The two symplectic manifolds $(\overline{X}^{\pm}, \overline{\omega}^{\pm})$ are called the symplectic cuts of X along $H^{-1}(0)$.

Thus we have a continuous map

$$\pi : X \to \overline{X}^+ \cup_Z \overline{X}^-.$$

As for the symplectic forms, we have $\omega^+|_Z = \omega^-|_Z$. Hence, the pair (ω^+, ω^-) defines a cohomology class of $\overline{X}^+ \cup_Z \overline{X}^-$, denoted by $[\omega^+ \cup_Z \omega^-]$. It is easy to observe that

$$\pi^*([\omega^+ \cup_Z \omega^-]) = [\omega]. \tag{2}$$

Let $B \in H_2(X; \mathbb{Z})$ be in the kernel of

$$\pi_* : H_2(X; \mathbb{Z}) \longrightarrow H_2(\overline{X}^+ \cup_Z \overline{X}^-; \mathbb{Z}).$$

By (2) we have $\omega(B) = 0$. Such a class is called a vanishing cycle. For $A \in H_2(X;\mathbb{Z})$ define $[A] = A + \mathrm{Ker}(\pi_*)$ and

$$\langle \Pi_i \tau_{d_i} \alpha_i \rangle^X_{g,[A]} = \sum_{B \in [A]} \langle \Pi_i \tau_{d_i} \alpha_i \rangle^X_{g,B}. \tag{3}$$

Notice that ω has constant pairing with any element in $[A]$. It follows from the Gromov compactness theorem that there are only finitely many such elements in $[A]$ represented by J-holomorphic stable maps. Therefore, the summation in (3) is finite.

The degeneration formula expresses $\langle \Pi_i \tau_{d_i} \alpha_i \rangle^X_{g,[A]}$ in terms of relative invariants of $(\overline{X}^+, Z)$ and $(\overline{X}^-, Z)$ possibly with disconnected domains.

To begin with, we need to assume that the cohomology class α_i is of the form

$$\alpha_i = \pi^*(\alpha^+ \cup_Z \alpha^-).$$

Here $\alpha_i^\pm \in H^*(\overline{X}^\pm;\mathbb{R})$ are classes with $\alpha_i^+|_Z = \alpha_i^-|_Z$ so that they give rise to a class $\alpha_i^+ \cup_Z \alpha_i^- \in H^*(\overline{X}^+ \cup_Z \overline{X}^-;\mathbb{R})$.

Next, we proceed to write down the degeneration formula. We first specify the relevant topological type of a marked Riemann surface mapped into $\overline{X}^+ \cup_Z \overline{X}^-$ with the following properties:

- (i) Each connected component is mapped either into $\overline{X}^+$ or $\overline{X}^-$ and carries a respective degree 2 homology class;
- (ii) The images of two distinct connected components only intersect each other along Z;
- (iii) No two connected components which are both mapped into $\overline{X}^+$ or $\overline{X}^-$ intersect each other;
- (iv) The marked points are not mapped to Z;
- (v) Each point in the domain mapped to Z carries a positive integer (representing the order of tangency).

By abuse of language we call the above data a $(\overline{X}^+, \overline{X}^-)-$graph. Such a graph gives rise to two relative graphs of $(\overline{X}^+, Z)$ and $\overline{X}^-, Z)$ from (i-iv), each possibly being disconnected. We denote them by $\Gamma^\bullet_+$ and $\Gamma^\bullet_-$ respectively. From (v) we also get two partitions T_+ and T_-. We call a $(\overline{X}^+, \overline{X}^-)-$graph a degenerate $(g, A, l)-$graph if the resulting pairs $(\Gamma^\bullet_+, T_+)$ and $(\Gamma^\bullet_-, T_-)$ satisfy the following constraints: the total number of marked points is l, the relative tails are the same, i.e. $T_+ = T_-$, and the identification of relative tails produces a connected graph of X with total homology class $\pi_*[A]$ and arithmetic genus g.

Let $\{\beta_a\}$ be a self-dual basis of $H^*(Z;\mathbb{R})$ and $\eta^{ab} = \int_Z \beta_a \cup \beta_b$. Given g, A and l, consider a degenerate $(g, A, l)-$graph. Let $T_k = T_+ = T_-$ and $\mathcal{T}_k$ be a weighted partition $\{t_j, \beta_{a_j}\}$. Let $\mathcal{T}_k' = \{t_j, \beta_{a_{j'}}\}$ be the dual weighted partition.

The degeneration formula for $\langle \Pi_i \tau_{d_i} \alpha_i \rangle^X_{g,[A]}$ then reads as follows,

$$(4) \qquad \begin{aligned} & \langle \Pi_i \tau_{d_i} \alpha_i \rangle^X_{g,[A]} \\ = & \sum \langle \Gamma^\bullet \{(d_i, \alpha_i^+)\} | \mathcal{T}_k \rangle^{\overline{X}^+, Z} \Delta(\mathcal{T}_k) \langle \Gamma^\bullet \{(d_i, \alpha_i^-)\} | \mathcal{T}_k' \rangle^{\overline{X}^-, Z}, \end{aligned}$$

where the summation is taken over all degenerate $(g, A, l)-$graphs, and

$$\Delta(\mathcal{T}_k) = \prod_j t_j |\mathrm{Aut}\,(T_k)|.$$

The advantage of the degeneration formula is to establish the connections between the absolute and relative Gromov-Witten invariants, called the comparison theorem. In [HR], we obtained such a comparison theorem.

Let X be a compact symplectic manifold and $Z \subset X$ be a smooth symplectic submanifold of codimension 2. $\iota : Z \longrightarrow X$ is the inclusion map. The cohomological push-forward

$$\iota^! : H^*(Z, \mathbb{R}) \longrightarrow H^*(X, \mathbb{R})$$

is determined by the pullback ι^* and Poincaré duality.

Definition 2.15. *A symplectic divisor Z is said to be* positive *if for some tamed almost complex structure J, $C_1(N_Z)(A) > 0$ for any A represented by a non-trivial J-sphere in Z.*

This is a generalization of ample divisor from algebraic geometry. Define

$$(5) \quad V := \min\{C_1(N_{Z|X})(A) > 0 \mid A \in H_2(Z, \mathbb{Z}) \text{ is a stably effective class}\}.$$

Then we have

Theorem 2.16 (HR, Corollary 4.4)**.** *Suppose that Z is a positive divisor and $V \geq l$. Then for $A \in H_2(X, \mathbb{Z})$, $\alpha_i \in H^*(X, \mathbb{R})$, $1 \leq i \leq \mu$, and $\beta_j \in H^*(Z, \mathbb{R})$, $1 \leq j \leq l$, we have*

$$\begin{aligned} & \langle \alpha_1, \cdots, \alpha_\mu, \iota^!(\beta_1), \cdots, \iota^!(\beta_l) \rangle^X_A \\ = & \sum_{\mathcal{T}} \langle \alpha_1, \cdots, \alpha_\mu \mid \mathcal{T} \rangle^{X,Z}_A, \end{aligned}$$

where the summation runs over all possible weighted partitions $\mathcal{T} = \{(1, \gamma_1), \cdots, (1, \gamma_q), (1, [Z]), \cdots, (1, [Z])\}$ where γ_i's are the products of some β_j classes.

3. Rationally connected manifolds

3.1. **Rationally connectedness in algebraic geometry.**

The basic reference for this subsection is [A]. We refer to [C, D, K1, KMM1, KMM2, V] for more details.

Let us recall the notion of rationally connectedness in algebraic geometry.

Definition 3.1. *Let X be a smooth complex projective variety of positive dimension. We say that X is rationally connected if one of the following equivalent conditions holds.*

(1) *Any two points of X can be connected by a rational curve (called as rationally connected).*
(2) *Two general points of X can be connected by a chain of rational curves (called as rationally chain-connected).*
(3) *Any finite set of points in X can be connected by a rational curve.*
(4) *Two general points of X can be connected by a very free rational curve. Here we say that a rational curve $C \subset X$ is a very free curve if there is a surjective morphism $f : \mathbb{P}^1 \longrightarrow C$ such that*

$$f^*TX \cong \oplus_{i=1}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i), \quad \text{with all } a_i \geq 1.$$

Next let us look at some properties of rationally connected varieties.

Proposition 3.2. *The following properties of rationally connected manifolds hold:*

(1) *Rationally connectedness is a birational invariant.*
(2) *Rationally connectedness is invariant under smooth deformation.*
(3) *If X is rationally connected, then $H^0(X, (\Omega_X^1)^{\otimes m}) = 0$ for every $m \geq 1$.*
(4) *Fano varieties (i.e., smooth complex projective varieties X for which $-K_X$ is ample) are rationally connected. In particular, smooth hypersurfaces of degree d in $\mathbb{P}^n$ are rationally connected for $d \leq n$.*
(5) *Rationally connected varieties are well behaved under fibration, i.e., let X be a smooth complex projective variety. Assume that there exists a surjective morphism $f : X \longrightarrow Y$ with Y and the general fiber of f rationally connected. Then X is rationally connected.*

An important theorem connecting birational geometry to Gromov-Witten theory is the result of Kollár and Ruan [K2], [R1]: a uniruled projective manifold has a nonzero genus zero GW-invariant with a point insertion. Similarly, we also hope to use the Gromov-Witten invariant to characterize the rationally connected variety. Z. Tian [T1, T2] showed the following result.

Proposition 3.3. (i) *If a rationally connected 3-fold X is Fano or has Picard number 2, then there is a nonzero Gromov-Witten invariant with two point insertions.*
(ii) *If X is a smooth projective Fano 4-fold of pseudo-index at least 2, then there is a nonzero Gromov-Witten invariant with two point insertions.*

These results give us some evidence for the following conjecture:

Conjecture 3.4. *Any rationally connected projective manifolds carry a nonzero genus zero Gromov-Witten invariant with two insertions.*

3.2. **Symplectically rationally connected manifolds.** Inspired by Conjecture 3.4, we first introduce the notion of k-point rational connectedness.

Definition 3.5. *Let $A \in H_2(X,\mathbb{Z})$ be a nonzero class. A is said to be a k-point rationally connected class if there is a nonzero Gromov-Witten invariant*

$$\langle \tau_{d_1}[pt], \cdots, \tau_{d_k}[pt], \tau_{d_{k+1}}\alpha_{k+1}, \cdots, \tau_{d_l}\alpha_l\rangle_A^X, \tag{6}$$

where $\alpha_i \in H^*(X,\mathbb{R})$ and $d_1, \cdots, d_l$ are non-negative integers. We call a class $A \in H_2(X,\mathbb{Z})$ a k-point strongly rationally connected class if $d_i = 0$, $1 \le i \le l$, in (6).

Definition 3.6. *X is said to be (symplectic) k-point (strongly) rationally connected if there is a k-point (strongly) rationally connected class. We simply call a 2-point (strongly) rationally connected symplectic manifold as (strongly) rationally connected symplectic manifold.*

Remark 3.7. *From the definition of uniruledness of* [HLR]*, a 1-point rationally connected symplectic manifold is equivalent to a uniruled symplectic manifold. From the definitions, we know that a k-point (strongly) rational connected symplectic manifold must be uniruled.*

Remark 3.8. *It is possible that k-point rational connectedness is equivalent to k-point strongly rational connectedness. We do not know how to prove this.*

Example 3.9. *It is well-known that for any positive integer k, the projective space $\mathbb{P}^n$ is k-point strongly rationally connected.*

Example 3.10. *Let $G(k,n)$ be the Grassmannian manifold of k-planes in $\mathbb{C}^n$. It is well-known that the classical cohomology of $Gr(k,n)$ has a basis of Schubert classes σ_λ, as λ varies over partitions whose Young diagram fits in a k by $n-k$ rectangle. The (complex) codimension of σ_λ is $|\lambda| = \sum \lambda_i$, the number of boxes in the Young diagram. The quantum cohomology of the Grassmannian is a free module over the polynomial ring $\mathbb{Z}[q]$, with a basis of Schubert classes; the variable q has (complex) degree n. The quantum product $\sigma_\lambda \star \sigma_\mu$ is a finite sum of terms $q^d\sigma_\nu$, the sum over $d \ge 0$ and $|\nu| = |\lambda| + |\mu| - dn$, each occurring with a nonnegative coefficient (a Gromov-Witten invariant). Denote by $\rho = \sigma_{((n-k)^k)}$ the class of a point. In* [BCF]*, the authors proved that $\sigma_\rho \star \sigma_\rho = q^k\sigma_{((n-2k)^k)}$ if $k \le n-k$, and $\sigma_\rho \star \sigma_\rho = q^{n-k}\sigma_{((n-k)^{2k-n})}$ if $n-k \le k$. This means that the Grassmannian $Gr(k,n)$ is symplectic rationally connected.*

Example 3.11. *For any integer $d \ge 0$, Consider the Grassmannian $G(d,2d)$. Buch-Kresch-Tamvakis* [BKT] *proved that for three points $U, V, W \in G(d,2d)$ pairwise in general position, there is a unique morphism $f : \mathbb{P}^1 \longrightarrow G(d,2d)$ of degree d such that $f(0) = U$, $f(1) = V$ and $f(\infty) = W$. This implies that the Gromov-Witten invariant $\langle [pt],[pt],[pt]\rangle_d^{G(d,2d)} = 1$. Therefore, $G(d,2d)$ are 3-point strongly rationally connected.*

Example 3.12. *Let $\mathbb{H} = Hilb^2(\mathbb{P}^1 \times \mathbb{P}^1)$ the Hilbert scheme of points on $\mathbb{P}^1 \times \mathbb{P}^1$. In* [P]*, the author gave a $\mathbb{Q}$-basis for $A * (\mathbb{H})$ as follows: $T_0 = [\mathbb{H}]$,*

T_1, T_2, T_3, T_4, $T_5 = T_1T_2$, $T_6 = T_1^2$, $T_7 = T_2^2$, $T_8 = T_1T_3$, $T_9 = T_2T_3$, $T_{10} = C_2 + F$, $T_{11} = C_1 + F$, $T_{12} = C_1 + C_2 + F$ *and* T_{13} *the class of a point. The author also computed the quantum product* $T_4 \star T_4 = T_{13} + 2q_1q_2q_3^2T_0$ *and* $T_4 \star T_4 \star T_4 = 2q_1q_2q_3^2T_4$. *So we have* $T_{13} \star T_{13} = -2q_1q_2q_3^2T_{13} \neq 0$. *This implies that* $\mathbb{H}$ *is symplectic rationally connected. From the computation of* [G], *it is easy to know that* $Hilb^2(\mathbb{P}^2)$ *also is symplectic rationally connected.*

Since Gromov-Witten invariants are invariant under smooth symplectic deformations, k-point rational connectedness is invariant under smooth symplectic deformations. Conjecture 3.4 means that it is not yet known whether a projective rationally connected manifold is symplectic rationally connected. Moreover, so far, we could not show that this notion is invariant under symplectic birational cobordisms defined in subsection 2.1 or [HLR]. However, we would like to mention some partial results along this direction. From the blowup formula of Gromov-Witten invariants in [H1, H2, H3, HZ, La], we have

Proposition 3.13. *Suppose that* X *is a* k*-point strongly rationally connected symplectic manifold. Let* $\tilde{X}$ *be the blowup of* X *along a finite number of points or some special submanifolds with convex normal bundles. Then* $\tilde{X}$ *is* k*-point strongly rationally connected.*

As we saw in Proposition 3.2, rationally connectedness is a birationally invariant property of projective manifolds. Proposition 3.13 gives some evidence for the following conjecture:

Conjecture 3.14. *Symplectic rationally connectedness is invariant under symplectic birational cobordism.*

If two complex projective or compact Kähler manifolds X and Y belong to the same class for the equivalence relation generated by Kähler deformations and symplectomorphism, then they are said to be symplectically equivalent. Along this direction, Voisin [V] proved

Theorem 3.15 (V, Theorem 0.10). *Let* X, Y *be compact Kähler* 3*-folds. Suppose that* X *and* Y *are symplectically equivalent. Then*

(i) *If* X *is Fano, then* Y *is rationally connected.*
(ii) *If* X *is rationally connected and* $b_2(X) \leq 2$, *then* Y *is rationally connected.*

Z. Tian [T1, T2] also studied the symplectic geometry of rationally connected 3-folds and 4-folds. He proved

Theorem 3.16 (T1, Theorem 1.6). *Rational connectedness is a symplectic deformation invariant in complex dimension* 3.

4. Rationally connected symplectic divisors

In this section, we want to apply degeneration technique to study the k-point rationally connectedness properties. We will show that a symplectic manifold is k-point strongly rationally connected if it contains a k-point strongly rationally connected symplecitc divisor with sufficiently positive normal bundle.

4.1. **Relative Gromov-Witten invariants of $\mathbb{P}^1$-bundle.** When we apply the degeneration formula, we often need to compute some special terms where the degeneration graph completely lies on the side of the $\mathbb{P}^1$-bundle. Before we prove our main result of this section, we first recall some results of [HR] about the relative Gromov-Witten invariants of $\mathbb{P}^1$-bundles.

Suppose that L is a line bundle over Z and $Y = \mathbb{P}(L \oplus \mathbb{C})$. Let D, Z be the infinity section and zero section of Y respectively. Let $\beta_1, \cdots, \beta_{m_Z}$ be a self-dual basis of $H^*(Z,\mathbb{Q})$ containing the identity element. We will often denote the identity by β_{id}. The degree of β_i is the real grading in $H^*(Z,\mathbb{Q})$. We view β_i as an element of $H^*(Y,\mathbb{Q})$ via the pull-back by $\pi : Y = \mathbb{P}(L\oplus\mathbb{C}) \longrightarrow Z$. Let $[Z]$, $[D] \in H^2(Y,\mathbb{Q})$ denote the cohomology classes associated to the divisors. Denote by F the homology class of the fiber of Y. Then we have

Proposition 4.1 (HR, Proposition 3.5). *Let $s > 0$, $\alpha_i \in H^*(Z,\mathbb{R})$, $1 \le i \le q$.*

(i) *Let $\mathcal{T}_k = \{(t_i, \beta_i)\}$ be a cohomology weighted partition of s. Then*

$$\langle \pi^*\alpha_1, \cdots, \pi^*\alpha_q, \beta_1\cdot[Z], \cdots, \beta_l\cdot[Z] \mid \mathcal{T}_k\rangle^{Y,D}_{sF} = 0$$

except for $s = k = 1$ and $q = 0$.

(ii) *For $s > 0$, we have the two-point relative invariant*

$$\langle \tau_{d-1}(\beta_0\cdot[Z]) \mid (s,\beta_\infty)\rangle^{Y,D}_{sF} = \begin{cases} \frac{1}{s!}\int_Z \beta_0\wedge\beta_\infty, & d = s \\ 0, & d \neq s \end{cases},$$

where $\beta_0 \in H^*(Z,\mathbb{Q})$ and $\beta_\infty \in H^*(D,\mathbb{Q})$.

(iii) *For $s = k = 1$, we have*

$$\langle \iota^!(\beta_1), \cdots, \iota^!(\beta_l) \mid (1,\gamma)\rangle^{Y,D}_F = \int_Z \beta_1\wedge\cdots\wedge\beta_l\wedge\gamma.$$

Let Γ_0 be a relative graph with the following data:

(i) a vertex decorated by a homology class $A \in H_2(Y,\mathbb{Q})$ and genus zero,

(ii) $l+q$ tails associated to $l+q$ absolute marked points,

(iii) k relative tails associated to k relative marked points.

Denote by A the homology class of the relative stable map (Σ, f) to (Y,D). Let $T_k = \{t_1, \cdots, t_k\}$ be a partition of $D\cdot A$ and d_i, $1 \le i \le l$, be positive integers. Denote $d = \sum_{i=1}^l d_i$. Denote by $\iota : Z \longrightarrow Y$ the inclusion of Z into Y via the zero section of Y. Then for any $\beta \in H^*(Z,\mathbb{R})$, the inclusion map ι pushes forward the class β to a cohomology class $\iota^!(\beta) \in H^*(Y,\mathbb{Q})$, determined by the pull-back map ι^* and Poincaré duality.

Proposition 4.2 (HR, Proposition 3.1)**.** *Suppose $A \neq sF$ or $k+l+q \geq 3$. Assume that $Z^*(A) \geq \sum d_i$ and $c_1(L)(C) \geq 0$ for any J-holomorphic curve C into Z. Then for any $\beta_i \in H^*(Z, \mathbb{Q})$, $1 \leq i \leq l$, and any weighted partition $\mathcal{T}_k = \{(t_i, \delta_i)\}$ of $D \cdot A$, we have*

$$\langle \varpi, \tau_{d_1-1}\iota^!(\beta_1), \cdots, \tau_{d_l-1}\iota^!(\beta_l) \mid \mathcal{T}_k \rangle^{Y,D}_{\Gamma_0, T_k} = 0,$$

where ϖ consists of insertions of the form $\pi^*\alpha_1, \cdots, \pi^*\alpha_q$, and $\alpha_i \in H^*(Z, \mathbb{R})$, $1 \leq i \leq q$.

Next, we will consider the case where the relative invariants of (Y, D) with empty relative insertion on D are no longer zero and the invariant of Z will contribute in a nontrivial way.

Suppose that $A \in H_2(Z, \mathbb{Z})$. Denote by $\iota : Z \longrightarrow Y$ the embedding of Z into Y as the zero section. Consider the relative invariant of (Y, D)

$$\langle \tau_{i_1}(\beta_1[Z]), \tau_{i_2}(\beta_2[Z]), \cdots, \tau_{i_k}(\beta_k[Z]), \varpi \mid \emptyset \rangle^{Y,D}_{0,A},$$

where ϖ consists of insertions of the form $\pi^*\alpha_1, \cdots, \pi^*\alpha_q$. The dimension condition is

$$2\sum(i_t+1) + \sum \deg \beta_t + \deg \varpi = 2(C_1^Y(A) + n - 3 + k + q).$$

The dimension condition of the divisor invariant $\langle \tau_{i_1}(\beta_1), \cdots, \tau_{i_k}(\beta_k), i^*\varpi \rangle^Z_{0,A}$ is

$$2(C_1^Z(A) + n - 1 - 3 + k + q) = 2\sum i_t + \sum \deg \beta_t + \deg \varpi.$$

Since $C_1^Y(A) = C_1^Z(A) + Z \cdot A$, so both invariants are well-defined only when $k = Z \cdot A + 1$.

Theorem 4.3. *Let $A \in H_2(Z, \mathbb{Z})$. Suppose that $k = Z \cdot A + 1$ and $C_1(L)(C) \geq 0$ for any holomorphic curve C into Z. Then*

$$\begin{aligned} & \langle \varpi, \beta_1 \cdot [Z], \beta_2 \cdot [Z], \cdots, \beta_k \cdot [Z] \mid \emptyset \rangle^{Y,D}_{0,A} \\ = \quad & \langle \iota^*\varpi, \beta_1, \cdots, \beta_k \rangle^Z_{0,A}, \end{aligned}$$

where ϖ consists of insertions of the form $\pi^*\alpha_1, \cdots, \pi^*\alpha_l$ and $\alpha_i \in H^*(Z, \mathbb{R})$, $1 \leq i \leq q$.

Proof. Choose a Hermitian metric and a unitary connection on L such that they induce a splitting

$$0 \longrightarrow V \longrightarrow TY \longrightarrow \pi^*TZ \longrightarrow 0,$$

where V is the vertical tangent bundle. We choose a metric of TY as the direct sum of a metric on V and π^*TZ, where the second one is induced from a metric on Z. The Levi-Civita connection is a direct sum. Therefore, we may choose almost complex structures J_Z on TZ and J_V on V such that we may choose the direct sum $J_Z \oplus J_V$ as an almost complex structure J_Y on TY. It is easy to see that $\bar{\partial}$ commutes with π.

From Lemma 3.2 in [HR], we know that the projection $\pi : Y \longrightarrow Z$ induces a map $\pi_{\mathcal{S}_e}$ between the virtual fundamental classes and an isomorphism between obstruction bundles. Therefore, by the definition of Gromov-Witten invariants, we have

$$\begin{aligned} & \langle \varpi, \beta_1 \cdot [Z], \beta_2 \cdot [Z], \cdots, \beta_k \cdot [Z] \mid \emptyset \rangle^{Y,D}_{0,A} \\ = \; & \deg(\pi_{\mathcal{S}_e}) \langle i^* \varpi, \beta_1, \cdots, \beta_k \rangle^Z_{0,A}. \end{aligned} \tag{7}$$

We claim that $\deg(\pi_{\mathcal{S}_e}) = 1$.

In fact, by the construction of virtual neighborhoods, we know that for every generic element $(\mathbb{P}^1, x_1, \cdots, x_{k+q}, \tilde{f})$ in the top strata of the moduli space $\mathcal{M}^Z_A(0, J_Z)$, there is a section ν of the obstruction bundle E_Z such that $\bar{\partial}_{J_Z} \tilde{f} = \nu$.

Suppose that a generic element $(\mathbb{P}^1, x_1, \cdots, x_{k+q}, f)$ in the top strata of the moduli space $\mathcal{M}^{Y,D}_A(0, J_Y)$ is a preimage of $(\mathbb{P}^1, x_1, \cdots, x_{k+q}, \tilde{f})$ under $\pi_{\mathcal{S}_e}$. That is, f is a lifting of $\tilde{f}$ to Y vanishing at the marked points $x_1, \cdots, x_k$. Therefore, from the fact that $\bar{\partial}$ commutes with π, we have that $(\mathbb{P}^1, x_1, \cdots, x_{k+q}, f)$ satisfies

$$\bar{\partial}_{J_Y} f = \pi^*_{\mathcal{S}_e} \nu. \tag{8}$$

If we choose a local coordinate (z, s) on Y, where s is the Euclidean coordinate on the fiber $\mathbb{P}^1$, then locally we may write $f = (\tilde{f}, f^V)$. Therefore (8) locally can be written as

$$\begin{cases} \bar{\partial}_{J_Z} \tilde{f} & = \nu, \\ \bar{\partial}_{J_V} f^V & = 0 \end{cases} \tag{9}$$

Since $\bar{\partial}^2 = 0$ always holds on $\mathbb{P}^1$, it follows from a well-known fact of complex geometry that f^*L is a holomorphic line bundle over $\mathbb{P}^1$. Moreover, (9) shows that f^V gives rise to a holomorphic section of the bundle f^*L, up to $\mathbb{C}^*$, which vanishes at the marked points $x_1, \cdots, x_k$. Since $\deg(f^*L) = Z \cdot A$, therefore, from our assumption that $k = Z \cdot A + 1$, we know that $f^*L \otimes (-x_1 - \cdots - x_k)$ has no nonzero holomorphic sections. Therefore, $f^V \equiv 0$. This says that the only preimage of a generic element $(\mathbb{P}^1, x_1, \cdots, x_{k+q}, \tilde{f})$ in the top strata of the moduli space $\mathcal{M}^Z_A(0, J_Z)$ is itself. This implies $\deg(\pi_{\mathcal{S}_e}) = 1$. This proves the theorem.

□

4.2. **From divisor to ambient space.** In 1991, McDuff [M3] first observed that a semi-positive symplectic 4-manifold, which contains a submanifold P symplectomorphic to $\mathbb{P}^1$ whose normal Chern number is non-negative, must be uniruled. In [LtjR], the authors generalize McDuff's result to more general situations. More importantly, they gave a rather general *from divisor to ambient space* inductive construction of uniruled symplectic manifolds. In this subsection, we will generalize their inductive construction to the case of rationally connected symplectic manifolds.

Suppose that X is a compact symplectic manifold and $Z \subset X$ is a symplectic submanifold of codimension 2. Denote by $N_{Z|X}$ the normal bundle of Z in X. Denote by $\iota : Z \subset X$ the inclusion of Z into X. Let V be the minimal normal Chern number defined in (5). We call a class $A \in H_2(Z, \mathbb{R})$ a **minimal class** if $Z \cdot A = V$.

Theorem 4.4. *Suppose that X is a compact symplectic manifold and $Z \subset X$ is a symplectic submanifold of codimension 2. If Z is k-point strongly rationally connected and $A \in H_2(Z, \mathbb{Z})$ is a minimal class such that*

$$\langle \iota^*\alpha_1, \cdots, \iota^*\alpha_l, [pt], \cdots, [pt], \beta_{k+1}, \cdots, \beta_r \rangle^Z_A \neq 0 \tag{10}$$

for some $r \leq V + 1$, $\beta_i \in H^*(Z, \mathbb{R})$ and $\alpha_j \in H^*(X, \mathbb{R})$, then X is k-point strongly rationally connected. In particular, if $\iota : Z \to X$ is homologically injective, then X is k-point strongly symplectic rational connected if $k \leq V + 1$.

Proof. Since we can always increase the number of Z-insertions by adding divisor insertions in (10), therefore, without loss of generality, we may assume that $r = Z \cdot A + 1$. Consider the following Gromov-Witten invariant of X:

$$\langle \alpha_1, \cdots, \alpha_l, [pt], \cdots, [pt], \beta_{k+1} \cdot [Z], \cdots, \beta_r \cdot [Z] \rangle^X_A. \tag{11}$$

If the invariant (11) is nonzero, then we are done. So in the following we assume that the invariant (11) equals zero.

To find a nonzero Gromov-Witten invariant of X with at least k point insertions, we first apply the degeneration formula to the invariant (11) to obtain a nonzero relative Gromov-Witten invariant of (X, Z) with at least k point insertions, then use our comparison theorem to obtain a nonzero Gromov-Witten invariant of X.

We perform the symplectic cutting along the boundary of a tubular neighborhood of Z. Then we have $\overline{X}^- = X$, $\overline{X}^+ = Y = \mathbb{P}(N_{Z|X} \oplus \mathbb{C})$. Since $\beta_i \in H^*(Z, \mathbb{R})$, we choose the support of $[pt]$, $\beta_i \cdot Z$ near Z. Then,$([pt])^- = 0$, $(\beta_i \cdot Z)^- = 0$, $([pt])^+ = [pt]$, $(\beta_i \cdot Z)^+ = \iota^!(\beta_i)$. Here ι in the second term is understood as the inclusion map of Z via the zero section into $Y = \mathbb{P}(N_{Z|X} \oplus \mathbb{C})$. Up to a rational multiple, each α_i is Poincaré dual to an immersed submanifold W_i. We can perturb W_i to be transverse to Z. In a neighborhood of Z, W_i is $\pi^{-1}(W_i \cap Z)$, where $\pi : N_{Z|X} \longrightarrow Z$ is the projection. Clearly, π induces the projection $\mathbb{P}(N_{Z|X} \oplus \mathbb{C}) \longrightarrow Z$, still denoted by π. The symplectic cutting naturally decomposes W_i into $W_i^- = W_i$, $W_i^+ = h^*(\alpha_i|_Z) = h^*\tilde{\alpha}_i$. In other words, we can choose $\alpha_i^- = \alpha_i$, $\alpha_i^+ = h^*(\alpha_i|_Z)$.

Now we apply the degeneration formula for the invariant

$$\langle \alpha_1, \cdots, \alpha_l, [pt], \cdots, [pt], \beta_{k+1} \cdot [Z], \cdots, \beta_r \cdot [Z] \rangle^X_A.$$

and express it as a summation of products of relative invariants of (X, Z) and (Y, D). Moreover, from the degeneration formula, each summand Ψ_C may consist of a product of relative Gromov-Witten invariants with disconnected domain curves of both (X, Z) and (Y, D).

On the side of Y, there may be several disjoint components. We claim that every component on the side of Y is a multiple of the fiber class. Suppose that there is a component on the side of Y which has a homology class $A' + \mu F$ where $A' \in H_2(Z, \mathbb{Z})$ and F is the fiber class. Then we have $Z \cdot (A' + \mu F) = Z \cdot A' + \mu$. Since every component must intersect the infinity section of Y, $\mu > 0$. Therefore, we have $Z \cdot (A' + \mu F) \geq V + 1$. From Proposition 4.2, we know that the contribution of this component to the corresponding relative Gromov-Witten invariants of (Y, D) must be zero. So the corresponding summand in the degeneration formula must be zero. From Proposition 4.1 and the same argument as in the proof of Theorem 2.16 , we have

$$\begin{aligned} 0 &= \langle \alpha_1, \cdots, \alpha_l, [pt], \cdots, [pt], \beta_{k+1} \cdot [Z], \cdots, \beta_r \cdot [Z] \rangle_A^X \\ &= \sum_\mu C_\mu \langle \alpha_1, \cdots, \alpha_l \mid \mu \rangle_A^{X,Z} \\ &\quad + \langle \iota^* \alpha_1, \cdots, \iota^* \alpha_l, [pt], \cdots, [pt], \beta_{k+1} \cdot [Z], \cdots, \beta_r \cdot [Z] \mid \emptyset \rangle_A^{Y,D} \\ &= \sum_\mu C_\mu \langle \alpha_1, \cdots, \alpha_l \mid \mu \rangle_A^{X,Z} \\ &\quad + \langle \iota^* \alpha_1, \cdots, \iota^* \alpha_l, [pt], \cdots, [pt], \beta_{k+1}, \cdots, \beta_r \rangle_A^Z, \end{aligned}$$

where we used Theorem 4.3 in the last equality and the summation runs over the possible partitions $\mu = \{(1, [pt]), \cdots, (1, [pt]), (1, \gamma_1), \cdots, (1, \gamma_q), (1, [Z]), \cdots, (1, [Z])\}$ where γ_i are the product of some β_i classes.. From our assumption (10), we have

$$\sum_\mu C_\mu \langle \alpha_1, \cdots, \alpha_l \mid \mu \rangle_A^{X,Z} \neq 0. \tag{12}$$

Denote by $\langle \alpha_1, \cdots, \alpha_l \mid \mu_0 \rangle_A^{X,Z}$ the minimal nonzero relative invariant in the summand (12)in the sense of Definition 2.14. Write $\mu_0 = \{(1, [pt]), \cdots, (1, [pt]), (1, \gamma_1), \cdots, (1, \gamma_q), (1, [Z]), \cdots, (1, [Z])\}$. Then it is easy know that the product of any two γ_i and γ_j vanishes.

Now we consider the following absolute Gromov-Witten invariant

$$\langle \alpha_1, \cdots, \alpha_l, [pt], \cdots, [pt], \gamma_1 \cdot [Z], \cdots, \gamma_l \cdot [Z], [Z], \cdots, [Z] \rangle_A^X.$$

Applying the degeneration formula and always distribute the point insertions to the side of (Y, D). Therefore, from Theorem 2.16, we have

$$\begin{aligned} &\langle \alpha_1, \cdots, \alpha_l, [pt], \cdots, [pt], \gamma_1 \cdot [Z], \cdots, \gamma_q \cdot [Z] \rangle_A^X \\ = &\langle \alpha_1, \cdots, \alpha_l \mid \mu_0 \rangle_A^{X,Z} \neq 0. \end{aligned}$$

This implies that X is k-point strongly rationally connected. This proves our theorem.

□

It is well-known that $\mathbb{P}^{n-1}$ is strongly rationally connected. Therefore, from Theorem 4.4, we have

Corollary 4.5. *Let (X, ω) be a compact $2n$-dimensional symplectic manifold which contains a submanifold P symplectomorphic to $\mathbb{P}^{n-1}$ whose normal Chern number $m \geq 2$. Then X is strongly rationally connected.*

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