THE GROUP OF INVERTIBLE ELEMENTS OF CERTAIN BANACH ALGEBRAS OF OPERATORS ON HILBERT SPACE

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0. INTRODUCTION

If E denotes a real or complex Hilbert space, J a complex structure on E and $\mathfrak{S}$ a separable symmetrically normed ideal in the algebra $\mathcal{B}(E)$ of bounded operators on E then one may define a subalgebra $\mathcal{B}_{\mathfrak{S}}(E)$ of $\mathcal{B}(E)$ by

$$\mathcal{B}_{\mathfrak{S}}(E) = \{A \in \mathcal{B}(E) | AJ - JA \in \mathfrak{S}\} .$$

Then $\mathcal{B}_{\mathfrak{S}}(E)$ may be normed to become a Banach algebra. The homotopy type of the group $G_{\mathfrak{S}}(E)$ of invertible elements of $\mathcal{B}_{\mathfrak{S}}(E)$ may be determined (it is the same as that of a classifying space for a certain functor of K-theory). When $\mathfrak{S} = \mathfrak{S}_2$ (the Hilbert-Schmidt ideal) the orthogonal or unitary retracts of $G_{\mathfrak{S}}(E)$ have a physical interpretation in terms of automorphisms of the infinite dimensional Clifford algebra. Moreover the first K-group, for E real, $K_1\left(\mathcal{B}_{\mathfrak{S}_2}(E)\right) \cong \mathbb{Z}_2$, relates to the existence of two distinct phases in the Ising model below the critical temperature and for E complex, $K_1\left(\mathcal{B}_{\mathfrak{S}_2}(E)\right) \cong \mathbb{Z}$, may be interpreted in terms of the electric charge in the second quantised Dirac theory of the electron.

1. ALGEBRAS OF OPERATORS

Let E denote a real separable Hilbert space, H a complex separable Hilbert space, J a fixed complex structure on E or H (i.e. $J^2 = -1$, $J^* = -J$) and $\mathfrak{S}$ a separable symmetrically normed ideal in the algebra of bounded operators on E or H (see [4] or [7] for a definition). Specifically $\mathfrak{S}$ is an ideal of compact operators which is a Banach space in its norm: $\| \|_{\mathfrak{S}}$, contains the finite rank operators and satisfies:

$$\|ASB\|_{\mathfrak{S}} \leq \|A\|_\infty \|S\|_{\mathfrak{S}} \|B\|_\infty , \quad S \in \mathfrak{S}, \quad A, B \text{ bounded.}$$

($\| \|_\infty$ = uniform norm).

Write $\mathcal{B}_{\mathfrak{S}}(E)$ (resp $\mathcal{B}_{\mathfrak{S}}(H)$) for the subalgebra of the bounded operators on E (resp H) consisting of those A which almost commute with J in the sense that $AJ - JA \in \mathfrak{S}$. It is not difficult to see that $\mathcal{B}_{\mathfrak{S}}(E)$ and $\mathcal{B}_{\mathfrak{S}}(H)$ are Banach algebras in the norm

$$\|A\| = \|A\|_\infty + \|AJ - JA\|_{\mathfrak{S}} \tag{1.1}$$

Denote by $G_{\mathfrak{S}}(E)$ and $G_{\mathfrak{S}}(H)$ the respective groups of invertible elements of these algebras.

Let $\mathcal{O}_{\mathfrak{S}}$ (resp $\mathcal{U}_{\mathfrak{S}}$) denote the orthogonal (resp unitary) subgroup of $G_{\mathfrak{S}}(E)$ (resp $G_{\mathfrak{S}}(H)$). Then $\mathcal{O}_{\mathfrak{S}}$ and $\mathcal{U}_{\mathfrak{S}}$ are deformation retracts of the respective groups of invertible elements and hence have the same homotopy type as those groups.

To determine this homotopy type one argues as follows (the case of $\mathcal{O}_{\mathfrak{S}}$ only is sketched, $\mathcal{U}_{\mathfrak{S}}$ is similar).

$\mathcal{O}_{\mathfrak{S}}$ acts transitively by conjugation on the space X consisting of all complex structures differing from J by an element of $\mathfrak{S}$. Moreover this action is jointly continuous when X is equipped with the metric topology: $\|J_1 - J_2\|_{\mathfrak{S}}$. As the isotropy subgroup of $J \in X$ is clearly the group $\mathcal{U}(E)$ of unitary operators on E (equipped with complex structure J) we have $\mathcal{O}_{\mathfrak{S}}/\mathcal{U}(E)$ homeomorphic to X. Furthermore the quotient map $\mathcal{O}_{\mathfrak{S}} \to X$ is a locally trivial principal fibration (using the fact that $\mathcal{O}_{\mathfrak{S}}$ is a Banach Lie group) and a standard argument exploiting contractibility of the fibre $\mathcal{U}(E)$ shows that $\mathcal{O}_{\mathfrak{S}}$ and X have the same homotopy type. On the other hand the homotopy type of X is known from the proof of Bott periodicity (see for example Milnor [6]) to be that of $O(\infty)/U(\infty) = \varinjlim O(n)/U(n)$ (where the RHS of this equality means the inductive limit of these homogeneous spaces as $n \to \infty$ and $O(\infty)$ and $U(\infty)$ denote the stable orthogonal and unitary groups respectively).

In particular $\pi_0(\mathcal{O}_{\mathfrak{S}}) \cong \mathbb{Z}_2$ and X is just the classifying space for the functor K^{-2} of real topological K-theory [5]. For $\mathcal{U}_{\mathfrak{S}}$ one obtains $\pi_0(\mathcal{U}\ ) \cong \mathbb{Z}$ (provided J satisfies (1.4) below) and $\mathcal{U}_{\mathfrak{S}}$ has the homotopy type of the space of Fredholm operators on H (a classifying space for K^0 of complex K-theory). This latter remark indicates the existence of an index map for these groups. By this I mean roughly an analytic method of determining in which connected component of $\mathcal{O}_{\mathfrak{S}}$ or $\mathcal{U}_{\mathfrak{S}}$ a given operator lies.

Thus define $i_{\mathcal{O}} : \mathcal{O}_{\mathfrak{S}} \to \mathbb{Z}_2$ by

(1.2) $\quad i_{\mathcal{O}}(R) = \dim_{\mathbb{C}} \ker(JR+RJ) \pmod 2 \ , \quad R \in \mathcal{O}_{\mathfrak{S}}$

and $i_{\mathcal{U}} : \mathcal{U}_{\mathfrak{S}} \to \mathbb{Z}$ by

(1.3) $$i_{\mathcal{U}}(U) = \dim \ker\left(P_+UP_+\right) - \dim \ker\left(P_+U^*P_+\right)$$

$$= \text{Fredholm index of } P_+UP_+$$

where $J = i\left(P_+-P_-\right)$ is the spectral decomposition of J and for surjectivity of $i_{\mathcal{U}}$ we need:

(1.4) $\quad P_{\pm}$ both have infinite dimensional range.

Both $i_{\mathcal{O}}$ and $i_{\mathcal{U}}$ are well defined as $JR + RJ$ and P_+UP_+ are Fredholm operators for all $R \in \mathcal{O}_{\mathfrak{S}}$ and $U \in \mathcal{U}_{\mathfrak{S}}$ respectively.

These index maps have another interpretation. By definition $K_1\left(\mathcal{B}_{\mathfrak{S}}(E)\right)$ is the quotient of the group of invertible elements of $(\mathcal{B}_{\mathfrak{S}}(E) \otimes K(E))^+$ by its connected component of the identity $\left(K(E)\right.$ denotes the compact operators$\left.\right)^*$. To determine $K_1\left(\mathcal{B}_{\mathfrak{S}}(E)\right)$ it is sufficient to find π_0 of the unitary group of $\mathcal{B}_{\mathfrak{S}}(E) \otimes M_n$ [8] for each n, where M_n denotes the $n \times n$ matrices (over $\mathbb{R}$). This latter problem is easily solved as

$$\mathcal{B}_{\mathfrak{S}}(E) \otimes M_n \cong \mathcal{B}_{\mathfrak{S}}\left(E \otimes \mathbb{R}^n\right)$$

* $(\quad)^+$ denotes unit adjoined

where $E \otimes \mathbb{R}^n$ is equipped with the complex structure $J \otimes I_n$ ($I_n = n \times n$ identity matrix). So π_0 of the unitary group of $B_{\mathfrak{S}}(E) \otimes M_n$ is $\mathbb{Z}_2$ independently of n. Thus $K_1(B_{\mathfrak{S}}(E)) \cong \mathbb{Z}_2$. Similarly $K_1(B_{\mathfrak{S}}(H)) \cong \mathbb{Z}$.

2. APPLICATIONS

The infinite dimensional Clifford algebra $C(E)$ over E is generated by $\{c(u) | u \in E\}$ with $c(u)c(v) + c(v)c(u) = 2\langle u, v\rangle . I$; $u, v \in E$. Let $\mathfrak{S}_2$ denote the Hilbert-Schmidt operators. Then J defines a representation π_J of $C(E)$ and $\mathcal{O}_{\mathfrak{S}_2}$ consists of those orthogonal operators R on E for which there is a unitary operator $\Gamma_J(R)$ on the Hilbert space of π_J such that

$$\Gamma_J(R)\pi_J(c(u))\Gamma_J(R)^{-1} = \pi_J(c(Ru)) . \tag{2.1}$$

As $\Gamma_J(R)$ is determined only up to multiplication by a complex number of modulus one by (2.1), the map $R \to \Gamma_J(R)$ is a homomorphism from $\mathcal{O}_J$ into the projective unitary group of the Hilbert space of π_J.

Similarly, regarding H as E equipped with a fixed complex structure J_0 (commuting with J) one has $\mathcal{U}_{\mathfrak{S}_2}$ consisting of unitary R on E for which (2.1) holds (in general $\mathcal{U}_{\mathfrak{S}}$ is a subgroup of $\mathcal{O}_{\mathfrak{S}}$). Thus, through the representation theory of the Clifford algebra, $\mathcal{U}_{\mathfrak{S}_2}$ and $\mathcal{O}_{\mathfrak{S}_2}$ enter into quantum field theory and statistical mechanics.

The physical interpretation of K_1 arises through the index maps (1.2) and (1.3) which define the isomorphisms

$$K_1\big(\mathcal{B}_{\mathfrak{S}}(E)\big) \cong \mathbb{Z}_2\ , \qquad K_1\big(\mathcal{B}_{\mathfrak{S}}(H)\big) \cong \mathbb{Z}\ .$$

Notice that $i_{\mathcal{O}}$ factors through X via the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{S}} & \cdots\cdots\!> & X \\ & {\scriptstyle i_{\mathcal{O}}} \searrow \quad \swarrow {\scriptstyle j_{\mathcal{O}}} & \\ & \mathbb{Z}_2 & \end{array}$$

where the horizontal arrow is the quotient map.

To give a simple expression for $j_{\mathcal{O}}$ we introduce

$$\tilde{E} = E \oplus E\ ; \qquad \Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \qquad \Gamma\ :\ E \to E\ ;$$

and regard $\tilde{E}$ as a complex Hilbert space with complex structure $J \oplus -J$. Then one may define a map $R \to \begin{pmatrix} T_1 & T_2 \\ T_2 & T_1 \end{pmatrix}$, where $T_1 = \tfrac{1}{2}(R - JRJ)$ $T_2 = \tfrac{1}{2}(R + JRJ)$ which is an isomorphism of $\mathcal{O}_{\mathfrak{S}}$ with the group $\mathcal{U}_\Gamma$ of unitary operators U on $\tilde{E}$ commuting with Γ and such that $P_+UP_- + P_-UP_+ \in \mathfrak{S}_2$ where $P_- = 1 - P_+$ and $P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Then X is homeomorphic to the space of all projections P on $\tilde{E}$ satisfying $\Gamma P \Gamma = 1 - P$ and $P - P_+ \in \mathfrak{S}_2$ and

$$j_0(P) = \text{rank}(P_+ \wedge P) \quad (\text{mod } 2)$$

where $P_+ \wedge P$ denotes the projection onto the eigenspace of $P_+ - P$ corresponding to the eigenvalue one. In this form it was shown by Araki and Evans [1] that the mod 2 index j_0 is related to the existence of two phases in the Ising model below the critical temperature.

(Briefly one observes that below the critical temperature one has 2 states on the algebra of observables for the Ising model and these may be defined in terms of two projections P_+ and $P \in X$. The observable properties of the Ising model are all determined by the even subalgebra of $C(E)$ and the representations π_J, and π_{J_1} (with $J \Leftrightarrow P_+$, $J_1 \Leftrightarrow P$) restricted to the even subalgebra are equivalent if and only if $j_0(P) = 0$. As this latter condition is not satisfied one deduces that the Ising model exhibits two distinct phases below the critical temperature.)

In the case of the Dirac theory there is a cyclic vector Ω (the vacuum) in the Hilbert space of π_J and a charge operator Q (the generator of the one parameter group $\theta \to \Gamma(e^{i\theta}I)$, $\theta \in \mathbb{R}$) such that for each $U \in \mathcal{U}_{\mathfrak{G}_2}$

$$Q\Gamma(U)\Omega = i_{\mathcal{U}}(U)\Gamma(U)\Omega$$

(that is the index determines the charge of the state $\Gamma(U)\Omega$.) In the case of 2-dimensional quantum field theory one chooses $H = L^2(\mathbb{R}, \mathbb{C}^2)$ and introduces the group of local gauge transformations consisting of those functions $\phi : \mathbb{R} \to U(2)$ which lie in $\mathcal{U}_{\mathfrak{S}}$ (as multiplication operators on H) . In this way the index $i_{\mathcal{U}}$ is expressed in terms of topological data of the symbol ϕ of the matrix valued Wiener-Hopf operator $P_+\phi P_+$.

More details on the results sketched here may be found in [2]. Other applications are discussed in [3].

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