NPM planar algebras and the Guionnet-Jones-Shlyakhtenko construction

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1 Introduction

In [Jon83] Jones initiated the study of modern subfactor theory by defining the index of a subfactor. A stronger combinatorial invariant, called the standard invariant, was later developed and axiomitized by Ocneanu’s paragroups [Ocn88], Popa’s $\lambda$-lattices [Pop95], and Jones’ planar algebras [Jon].

Given $\lambda$-lattice, Popa constructed a $\text{II}_1$-subfactor whose standard invariant is exactly the given $\lambda$-lattice [Pop95]. Later, in work with Shlyakhtenko, it was shown that the factors in Popa’s construction can be made to be isomorphic to the free group factor on infinitely many generators, $L(\mathbb{F}\infty)$ [PS03]. Guionnet, Jones, and Shlyakhtenko gave a diagrammatic (planar algebraic) proof of Popa’s result [GJS10]. In the finite depth case, they showed that the factors involved in the construction are interpolated free group factors [GJS11]. The author later showed that in the infinite depth case, the factors involved in the construction are all isomorphic to $L(\mathbb{F}\infty)$ [Har13].

This article initially appeared as the final chapter of the author’s graduate thesis, and is based on a problem posed by Vaughan Jones. The problem is as follows: Given a subfactor planar algebra, $Q$, one can consider the algebras $\text{Gr}^+_{k}(Q)$ as defined in [GJS10], and place the following “toy potential” on $Q$:

$$\text{tr}(x) = \begin{bmatrix} \sum V \\ x \end{bmatrix}$$

where $V$ is a rotationally invariant set of elements in $Q$. If one is fortunate, $\text{tr}$ is positive definite on $Q$ and left multiplication is bounded on $L^2(\text{Gr}(Q))$. To this end, it is an interesting problem to study the von Neumann algebras, $\mathcal{N}^{\pm}_k(\text{Gr}^\pm_k(Q)^\prime\prime)$ associated to $Q$ and $V$.

The case that will be considered here is the case where $Q$ is the standard invariant for a subfactor $N \subset M$ that contains an intermediate subfactor, $P$. As such, it follows that $Q$ contains the Fuss Catalan planar algebra as a sub planar algebra [BJ97]. Therefore, we can consider the following potential on $\text{Gr}^+_0(Q)$:

$$\text{tr}(x) = \begin{bmatrix} \sum FC \\ x \end{bmatrix}$$
where $\sum FC$ represents the sum of all Fuss Catalan diagrams. Recall that Fuss Catalan diagrams are planar two-colored string diagrams satisfying the following condition: If the colors of the strings are $a$ and $b$, then the colors of strings intersecting the boundary form the pattern, $aabbaabbaabb$...

To attack this problem, it is natural to introduce a new planar algebra, $P$ which will be called an $N - P - M$ planar algebra. A pleasing feature of this planar algebra is that the boundary conditions of the input disks can be taken to be any word in $aa$ and $bb$. The planar algebra $Q$ will be realized as a subalgebra of $P$ by replacing each strand with an $a$ strand next to a $b$ strand so that the pattern formed by the input disks in the $Q-$tangles is of the form $aabbaabbaabb$.... Whenever $Q$ is itself the Fuss Catalan planar algebra, we can take $P$ to be generated the set of all string diagrams having boundary conditions any word in $aa$ and $bb$.

The purpose of the $N - P - M$ planar algebra is that it gives one a natural way to decouple the strings $a$ and $b$ and treat them as free generators in an appropriate sense (see Section 5 for how this is done). We form algebras $M_\alpha$ for $\alpha$ a suitable word in $a$ and $b$ which are the $N - P - M$ analogues of the algebras $M_{\pm k}$ from [GJS10], and will use a semifinite algebra as in [GJS11] to find the isomorphism class of the algebras $M_\alpha$. More precisely, we will prove the following theorem:

**Theorem A.** Let $\alpha$ be a word in $a$ and $b$ where $a$ appears $n$ times, and $b$ appears $m$ times. Define $\delta_\alpha = \delta_a^n \cdot \delta_b^m$. $M_\alpha$ is a II$_1$ factor and is isomorphic to $L(\mathbb{F}(1 + 2I\delta_a^{-2}(\delta_a + \delta_b - 2)))$ for $P$ finite depth. Here, and $I = \sum_{v \in \Gamma_N} \mu(v)^2$ with $\mu$ the Perron Frobenius weighting on the principal graphs on $P$.

This formula has some interest, because it contains information about the inclusions $N \subset P$ and $P \subset M$ ($\delta_a$ and $\delta_b$ respectively) as well as the larger inclusion $N \subset M$ (the global index $I$). Just as in the case for the GJS algebras, we will also prove the following theorem:

**Theorem B.** $M_\alpha \cong L(\mathbb{F}_\infty)$ when $P$ is infinite depth.

By taking a tensor product of planar algebras, one obtains the following diagrammatic corollary

**Corollary ([PS03]).** Given any standard invariant, $Q$, there exist II$_1$-factors $N \subset M$ having standard invariant $Q$ and both isomorphic to $L(\mathbb{F}_\infty)$.

In addition to understanding the algebraic structure of the $N_k$, we will make use of the semifinite algebra to show that the law of $\cup \in N^+_0$ has a nice expression in terms of known laws. Unfortunately, the author has not yet been able to identify the isomorphism classes of the algebras $N_k^\pm$, however the semifinite algebra will show that there is evidence that the algebras $N_k^\pm$ are free group factors:

**Theorem C.** The von Neumann algebras, $N_k^\pm$, are each contained in a free group factor and contain a free group factor.

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2 \( N - P - M \) planar algebras

The object that is at the heart of all of these computations is an \( N - P - M \) planar algebra which can be thought of an augmentation of such a \( Q \) as above. Given parameters \( a \) and \( b \), we denote \( W \) as the set of finite words on \( aa \) and \( bb \). We define what it means for a planar tangle to be an \( N - P - M \) tangle:

**Definition 2.1.** A planar tangle is said to be an \( N - P - M \) tangle if its regions are shaded by three colors, \( N, P, \) and \( M \) such that the following conditions are met:

- A region colored \( N \) only borders regions colored \( P \)
- A region colored \( M \) only borders regions colored \( P \)
- A region colored \( P \) can border regions colored \( N \) or \( M \) but not \( P \).

We will denote the set of \( N - P - M \) tangles as the \( N - P - M \) planar operad. All tangles will be drawn so that the internal and external disks are rectangles. In order for smooth isotopy to make sense, the rectangles have their corners smoothed out.

Any string serving as the boundary string of a region colored \( N \) will be called an \( a \) string and any string bordering a region colored \( M \) will be called a \( b \) string. We note that the conditions on the regions show that if \( \alpha \) is a word in \( a \) and \( b \), then \( \alpha \) serves as a word of strings intersecting a disk if and only if \( \alpha \in \Delta \) where \( \Delta \) is the set

\[
\Delta = \{awa : w \in W\} \cup \{bwb : w \in W\} \cup W
\]

The following notation will be useful:

**Notation 2.2.** We denote \( \Delta^Q \) to be the set of all words in \( \Delta \) where the shading before the first letter is \( Q \) (for \( Q = N, P, \) or \( M \)). If \( \alpha \in \Delta \), we define \( s(\alpha) \) to be the first letter of the word \( \alpha \). Whenever a word is mentioned, part of the data is its initial region (hence the choice of region between every pair of letters), not just its letters. If a tangle, \( T \), has boundary condition \( \alpha \) on its outer disk, we will call \( T \) a planar \( \alpha \) tangle. Any internal rectangle with boundary condition \( \alpha \) will be called an \( \alpha \) rectangle.

We remark that just as for shaded planar algebras, there is a natural gluing operation. Namely, if we have two planar tangles \( S, T \) satisfying the following boundary condition:

- Some internal rectangle \( D_S \) of \( S \) has boundary data which agrees with \( T \), i.e. the shadings along the boundaries of \( T \) and \( D_S \) agree when counting clockwise from the marked point.

then we may compose \( S \) and \( T \) to get the planar tangle \( S \circ_{D_S} T \) by taking \( S \) union the interior of \( T \), removing the boundary of \( D_S \), smoothing the strings, and removing any closed loops.

Given \( \alpha \in \Delta \), we let \( \overline{\alpha} \) be \( \alpha \) read in the opposite order. When a string appears with a label \( \alpha \), then the string is meant to be a band of strings having colors ordered by the word \( \alpha \). The strings are read in the order of top to bottom and left to right. Also, unless otherwise marked, all marked regions of rectangles will be assumed to be on the top-left corner of the box. In addition, whenever there is a box written without a tangle, it is assumed that the box is placed in a larger tangle whose boundary data agrees with the boundary data for the box and whose marked region is the same as the marked region of the box.

We now define an \( N - P - M \) planar algebra:
**Definition 2.3.** An \( N - P - M \) planar algebra consists of the following data:

- Given parameters \( a \) and \( b \) as above, there is a finite dimensional complex vector space \( \mathcal{P}_\alpha \) for every nonempty word, \( \alpha \in \Delta \). There are three vector spaces \( \mathcal{P}_\emptyset^N, \mathcal{P}_\emptyset^P, \) and \( \mathcal{P}_\emptyset^M \) in the case when \( \alpha \) is empty. These are one dimensional complex vector spaces.

- An action of planar tangles by multilinear maps, i.e., for each planar \( \alpha \) tangle \( T \), whose rectangles \( D_i(T) \) are \( \alpha_i \) rectangles, there is a multilinear map

\[
Z_T : \prod_{i \in I} P_{\alpha_i} \rightarrow P_{\alpha}
\]

satisfying the following axioms:

**Isotopy:** If \( \theta \) is an orientation preserving diffeomorphism of \( \mathbb{R}^2 \), then \( Z_{\theta(T)} = Z_T \). That is, let \( T^0 \) be the interior of \( T \), and let \( f \in \prod_{D \subset T^0} P_{a_D} \). Then

\[
Z_{\theta(T)}(f_\theta) = Z_T(f)
\]

where \( f_\theta(\theta(D)) = f(D) \).

**Naturality:** For \( S, T \) composable tangles, \( Z(S \circ_D T) = Z(S) \circ_D Z(T) \), where the composition on the right hand side is the composition of multilinear maps.

- \( \mathcal{P} \) is unital \([\text{Jon11}]\): Let \( S \) be an \( N - P - M \) tangle with no input disks and boundary condition \( \alpha \in \Delta \). Then, there is an element \( Z(S) \in P_\alpha \) so that the following holds:

Let \( S \) be a tangle a nonempty set of internal disks such that \( S \) can be glued into the internal disk \( D^S \) of \( T \). Then

\[
Z(T \circ S) = Z(T) \circ Z_S.
\]

Here \( (Z(T) \circ Z_S)(f) = \tilde{f} \) where

\[
\tilde{f}(D) = \begin{cases} f(D) & \text{if } D \neq D^S \\ Z(S) & \text{if } D = D^S \end{cases}
\]

This condition allows isotopy classes of such an \( S \) to be elements of \( P_\alpha \). This action allows us to identify the empty diagrams (shaded \( N, P, \) and \( M \)) with the scalar \( 1 \in \mathbb{C} \). We make this assumption in the rest of this chapter. The naturality axiom, combined with this identification, forces closed strings with parameters \( a \) and \( b \) to be replaced by scalars \( \delta_a \) and \( \delta_b \) respectively.

- There is a conjugate linear involution, \( * : \mathcal{P}_\alpha \rightarrow \mathcal{P}_{\bar{\alpha}} \). It is compatible with reflection of tangles i.e., if \( T \) is a tangle which is produced by an orientation reversing diffeomorphism, \( \varphi \), of \( T \), then we have

\[
(Z_T(f))^* = Z_{\bar{T}}(\bar{f})
\]

where \( \bar{f}(\varphi(D)) = f(D)^* \).

- Each \( \mathcal{P}_\alpha \) comes equipped with the positive definite sesquilinear form:

\[
\langle x, y \rangle = \begin{vmatrix} x \end{vmatrix}_\alpha y^*
\]

where \( x, y \) are elements of \( \mathcal{P}_\alpha \).
• \( P \) is spherical, i.e. for all \( \alpha \in \Delta \) and all \( x \in P_{\alpha\overline{\alpha}} \), we have

\[
\text{tr}(x) = \begin{pmatrix} \alpha \\ x \end{pmatrix} \pi = \begin{pmatrix} x \end{pmatrix} \alpha.
\]

This says that we can think of our planar tangles as living in a sphere instead of a plane.

**Remark 2.4.** In viewing the action of a tangle, the letter \( Z \) will often be omitted.

A-priori, it is not clear that an \( N - P - M \) planar algebra should exist. The following example shows that this is the case. For the rest of this paper, an \( a \) string will be colored blue and a \( b \) string will be colored red.

**Example 2.5.** Let \( \delta_a, \delta_b \in \{ 2 \cos(\pi/n) : n = 3, 4, 5, \ldots \} \cup [2, \infty) \) and define \( P_\alpha \) by

\[
P_\alpha = \text{span}\{\text{planar string diagrams with with boundary condition } \alpha\}
\]
i.e., \( P_\alpha \) is the \( \mathbb{C} \)-linear span of isotopy classes of \( \alpha \) tangles with no input disks and no loops. For example,

\[
P_{a b b a} = \text{span}\left\{\begin{array}{c} \includegraphics[width=2cm]{blue-string-diagram} \\ \includegraphics[width=2cm]{red-string-diagram} \end{array}\right\}.
\]

The action of \( N - P - M \) tangles is as follows: All string diagrams are inserted into the necessary input disks. The result of this operation is a new string diagram with except with some loops. These loops are replaced with a parameter \( \delta_a \) or \( \delta_b \), depending on the color of a loop. The adjoint operation is the conjugate linear extension of reflection of diagrams.

It is straightforward to check that \( P \) satisfies all of the axioms of an \( N - P - M \) planar algebra except positive definiteness. Given \( \alpha \in \Delta \), we form the word \( \alpha' \) which is a word of colors that can appear in a Fuss Catalan diagram, and is obtained from \( \alpha \) by inserting the minimal numbers of \( a a \)'s or \( b b \)'s between letters in \( \alpha \). For example, if \( \alpha = aaabbaabbbba \), then \( \alpha' = a(bb)aabbaaab(aa)bba \). We then define a map \( \phi : P_\alpha \to P_{\alpha'} \) which is given by inserting a cup of the appropriate color whenever that color has been inserted into \( \alpha \), and then dividing by \( \delta_a^m \cdot \delta_b^n \). Here, \( aa \) was inserted \( m \) times and \( bb \) was inserted \( n \) times. For example,

\[
\phi \left( \begin{array}{c} \includegraphics[width=2cm]{blue-string-diagram} \\ x \end{array}\right) = \frac{1}{\delta_a^2 \delta_b} \cdot \begin{pmatrix} \alpha \\ x \end{pmatrix}.
\]

This map is easily seen to be preserve the desired sesquilinear form, and we know that this form is positive semidefinite on the Fuss Catalan algebras, with positive definiteness in the case \( \delta_a, \delta_b \geq 2 \), from \( [BJ97] \). Therefore, after taking a quotient in the case that \( \delta_a \) or \( \delta_b \) is less than 2, this example produces an \( N - P - M \) planar algebra.

By unitality, this planar algebra is represented in every \( N - P - M \) planar algebra.
2.1 Principal graphs of $N - P - M$ planar algebras

We first remark that if $\alpha \in \Delta$, then the axioms of an $N - P - M$ planar algebra show that $P_{\alpha \bar{\alpha}}$ is a finite dimensional $C^*$ algebra with multiplication given by

$$x \cdot y = \begin{array}{c|c}
\alpha & x \\
\hline
\gamma & y
\end{array}.$$  

Let $p \in P_{\alpha \bar{\alpha}}$ and $q \in P_{\gamma \gamma}$ be projections. Then we say $p$ is equivalent to $q$ if there is a $u \in P_{\alpha \gamma}$ so that

$$\begin{array}{c|c}
\alpha & u \\
\hline
\gamma & p
\end{array} \quad \text{and} \quad \begin{array}{c|c}
\alpha & u \\
\hline
\gamma & q
\end{array} = q.$$  

To an $N - P - M$ planar algebra $P$, there are three principal graphs associated to $P$, $\Gamma_N$, $\Gamma_P$, and $\Gamma_M$. We will call $\Gamma_Q$ the $Q$-principal graph of $P$. Each $\Gamma_Q$ has three sets of vertices, $\Gamma_Q^N$, $\Gamma_Q^P$, and $\Gamma_Q^M$. They are described by the following procedure:

The vertices $v \in \Gamma_Q^{Q_2}$ correspond to equivalence classes of minimal projections $p_v$ in the finite-dimensional $C^*$ algebra $P_{\alpha \bar{\alpha}}$ for some $\alpha$ depending on $v$ where $\alpha \bar{\alpha} \in \Delta^{Q_1}$ and $\bar{\alpha} \bar{\alpha} \in \Delta^{Q_2}$. There are $a-$colored edges connecting the vertices $\Gamma_Q^{N}$ to the vertices $\Gamma_Q^{P}$ as well as $b-$colored edges connecting the vertices $\Gamma_Q^{P}$ to $\Gamma_Q^{M}$. The $a-$colored edges are defined as follows:

Suppose $v \in \Gamma_Q^{N}$ and $w \in \Gamma_Q^{P}$, let $p \in P_{\beta a}$ be equivalent to $p_v$. It follows that the element

$$i_a(p) = \begin{array}{c}
\beta \\
\hline
p
\end{array}$$

is a projection in $P_{\beta a a \bar{a}}$. We draw $n$ $a-$colored edges between $v$ and $w$ if $n$ is the maximal number such that there exist orthogonal projections $q_1, \ldots, q_n \in P_{\beta a a \bar{a}}$ which are each equivalent to $p_w$ and satisfy $\sum_{i=1}^{n} q_i \leq i_a(p)$. We can also get edges from $w$ to $v$ in a similar manner. In principle, the construction of the $a-$edges leads to oriented edges, however, the presence of the Jones basic construction shows that the edges can be unoriented. More precisely, consider the projection

$$e = \frac{1}{\delta_a} \begin{array}{c}
\beta \\
\hline
\delta_a
\end{array} \in P_{\beta a a a \bar{a}},$$

and $e$ be its central support. We note that $P_{\beta \bar{a}}$ unitally includes into $P_{\beta a a a \bar{a}}$ by applying the map $i_a$ twice. It is also a straightforward check to see that the mapping $P_{\beta \bar{a}} \rightarrow P_{\beta a a a \bar{a}}$ given by $x \mapsto i_a(i_a(x)) e$ is an isometry, and $ei_a(y)e = i_a(E_{P_{\beta \bar{a}}} (y)) e$ for $y \in P_{\beta a a a \bar{a}}$. Therefore, from [JS97] it follows $zP_{\beta a a a \bar{a}}z$ is isomorphic to the basic construction of $P_{\beta \bar{a}}$ in $P_{\beta a a a \bar{a}}$. If $A, B,$ and $C$ are finite dimensional $C^*$ algebras with $C$ the basic construction of $A$ in $B$, then the Bratteli diagram of $B \subset C$ is the reflection of that of $A \subset B$ [JS97]. Therefore, if there are $n$ $a-$colored edges from $v$ to $w$, then there are $n$ $a-$colored edges from $w$ to $v$.

There is an analogous way to determine the $b-$colored edges that go between $\Gamma_Q^P$ and $\Gamma_Q^M$. We also note that if $p \in P_{\alpha \bar{\alpha}}$ is a minimal projection corresponding to a vertex $v \in \Gamma_Q^{Q_2}$, then the mapping in Example...
Lemma 2.6. Let \( \Gamma \) and \( \Gamma' \) be the principal and dual principal graphs of \( Q \) respectively. Then there are one-to-one correspondences between the following sets of vertices:

\[
\Gamma_+ \leftrightarrow \Gamma^N_N, \quad \Gamma_- \leftrightarrow \Gamma^M_N, \quad \Gamma'_+ \leftrightarrow \Gamma^M_M, \quad \text{and} \quad \Gamma'_- \leftrightarrow \Gamma^N_M
\]

Also observe that rotation by 180° is an anti-isomorphism of each \( P_\gamma \). This induces a one-to-one correspondence \( \Gamma^Q_{Q_1} \leftrightarrow \Gamma^{Q_1}_{Q_2} \). Finally, if the vertices of the principal graphs \( \Gamma_Q \) are weighted according to the traces of their corresponding projections, then it follows by the definition of principal graph that the graph with vertices \( \Gamma^Q_Q \) and \( \Gamma^Q_{Q_1} \) is bipartite with Perron Frobenius eigenvalue \( \delta_a \). Also, the graph with vertices \( \Gamma^P_P \) and \( \Gamma^M_M \) is bipartite with Perron Frobenius eigenvalue \( \delta_b \).

3 \( N = P = M \) planar algebras from intermediate subfactors

The goal of this section is to see that such a \( Q \) as above can be faithfully realized inside an \( N = P = M \) planar algebra \( P \). Much of this section was influenced from discussions with David Penneys and Noah Snyder, and many of the proofs of the following theorems are taken from them. We will first describe how such an algebra arises from an inclusion \( N \subset P \subset M \) of finite index II\(_1\) factors. To start, we consider the following bifinite bimodules:

\[
N L^2(P)_P \text{ and } P L^2(M)_M
\]

and their duals (contragredients)

\[
P L^2(P)_N \text{ and } M L^2(M)_P.
\]

Let \( \alpha \in \Delta \). Since part of the prescribed data for \( \alpha \) is a choice of initial shading, we note that the shading of \( \alpha \), i.e. the shading between any two letters on \( \alpha \) is uniquely determined. Assume that the shading of \( \alpha \) is the sequence \( Q_1 \cdots Q_k \) for \( Q_i = N, P \) or \( M \). We define \( Z_\alpha \) to be the following:

\[
Z_\alpha = Q_1 L^2(Q_1) Q_1 \cap Q_2 \otimes Q_1 \cap Q_2 L^2(Q_2) Q_2 \cap Q_3 \otimes Q_2 \cap Q_3 L^2(Q_3) Q_3 \cap Q_4 \otimes \cdots \otimes Q_{k-1} \cap Q_k L^2(Q_k) Q_k \cap Q_1 \otimes L^2(Q_1) Q_1 \cap Q_1,
\]

and we set \( P_\alpha = \text{Hom}_{Q_1} (L^2(Q_1), Z_\alpha) \) (Notice that \( Q_i \cap Q_{i+1} \) is necessarily \( N, P \), or \( M \)). We note from [Bis97, Con80] that this can be identified with the \( Q_1 - Q_1 \) central vectors of \( Z_\alpha \).

To help understand the planar structure, we let

\[
N(= M_0) \subset M(= M_1) \subset M_2 \subset \cdots \subset M_n \subset \cdots
\]

be the Jones tower for \( N \subset M \), where \( M_n \) is generated by \( M_{n-1} \) and \( e_{n-1} \). Here, \( e_{n-1} \) is the orthogonal projection from \( L^2(M_{n-1}) \) onto \( L^2(M_{n-2}) \). We will define \( e_P \) to be the orthogonal projection from \( L^2(M) \) onto \( L^2(P) \). We will also let \( B = \{ b_i \}_{i=1}^n \) be an orthonormal Pimsner Popa basis for \( M \) over \( N \) where \( n - 1 \) is the largest integer which is bounded above by the index \( [M : N] \). The \( b_i \) are elements in \( M \).
satisfying the following equivalent conditions:

\[ x = \sum_{i=1}^{n} E_N(xb_i)b_i^* \quad \forall x \in M \]

\[ x = \sum_{i=1}^{n} b_iE_N(b_i^*)x \quad \forall x \in M \]

\[ 1 = \sum_{i=1}^{n} b_i e_1 b_i^* , \]

as well as \( E_N(b_i b_j) = \delta_{i,j} \) if \( i \leq n - 1 \) and \( E_N(b_n b_n^*) \) is a projection of trace \([M : N] - (n - 1)\) in \( M \). If we let \( e_P \) be the orthogonal projection from \( L^2(M) \) onto \( L^2(P) \) and \( \{c_i\}_{i=1}^{m} \) be an orthonormal Pimsner-Popa basis of \( P \) over \( N \). Then we have the following lemma.

**Lemma 3.1.** \( e_P = \sum_{i=1}^{m} c_i e_1 c_i^* \).

**Proof.** We compute the 2-norm of \( e_P - \sum_{i=1}^{m} c_i e_1 c_i^* \). Doing so gives:

\[ \|e_P - \sum_{i=1}^{m} c_i e_1 c_i^*\|_2^2 = \text{tr}(e_P) - 2 \sum_{i=1}^{m} \text{tr}(c_i e_1 c_i^*) + \sum_{i,j=1}^{m} \text{tr}(c_i e_1 c_j c_j^*). \]

Since \( e_P \) commutes with the elements \( c_i \), the term in the middle becomes \( 2 \sum_{i=1}^{m} \text{tr}(c_i e_1 c_i^*) \). Using \( c_i e_1 c_j e_1 = E_N(c_i c_j e_1) \), and orthonormality of the basis, the last term becomes \( \sum_{i=1}^{m} \text{tr}(c_i e_1 c_i^*) \). Therefore, we get:

\[ \|e_P - \sum_{i=1}^{m} c_i e_1 c_i^*\|_2^2 = \text{tr}(e_P) - \sum_{i=1}^{m} \text{tr}(c_i e_1 c_i^*) \]

\[ = \text{tr}(e_P) - [M : N]^{-1} \sum_{i=1}^{m} \text{tr}(c_i e_1 c_i^*) \]

\[ = [M : P]^{-1} - [M : N]^{-1}[P : N] = 0 \]

as desired. \( \square \)

We will now show the bimodules \( Z_\alpha \) can be isometrically embedded in \( L^2(M_n) \) for some \( n \). As some notation, we will let \( \delta_Q = [M : Q]^{1/2} \) for \( Q = N, P, \) or \( M \). We also set \( E_1^Q = \delta_Q e_Q \), and \( v_n^Q = E_n E_{n-1} \cdots E_2 E_1^Q \).

**Theorem 3.2.** The map \( \phi : Z_\alpha \to M_k \) given by

\[ \phi(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = x_1 v_1^{Q_1 \cap Q_2} x_2 v_2^{Q_2 \cap Q_3} \cdots v_{k-1}^{Q_{k-1} \cap Q_k} x_k \]

is an isometry.
Proof. Note that the map is well defined as $v^Q$ commutes with $Q$. We proceed by induction on $k$. The result is clearly true for $k = 1$, so assume that it holds for $k - 1$. Using the previous lemma as well as the relation $E_i E_j = E_j E_i$ for $|i - j| \geq 2$, we have:

\[
\left\langle x_1 v_1^{Q_1 \cap Q_2} x_2 v_2^{Q_3 \cap Q_4} \cdots x_k v_k^{Q_{k-1} \cap Q_k}, y_1 v_1^{Q_1 \cap Q_2} y_2 v_2^{Q_2 \cap Q_3} \cdots y_k^{Q_{k-1} \cap Q_k} \right\rangle_{M_k}
\]

\[
= \text{tr}_{M_k} \left( y_k^* (v_{k-1}^{Q_{k-1} \cap Q_k}) \cdots y_2 (v_1^{Q_1 \cap Q_2}) y_1^* x_1 v_1^{Q_1 \cap Q_2} x_2 v_2^{Q_2 \cap Q_3} \cdots v_{k-1}^{Q_{k-1} \cap Q_k} x_k \right)
\]

\[
= \delta_Q^3 \text{tr}_{M_k} \left( y_k^* (v_{k-2}^{Q_{k-2} \cap Q_k}) \cdots y_2 E_{Q_1 \cap Q_2} (y_1^* x_1) E_{k-1} x_{k-1} \cdots v_{k-2}^{Q_{k-2} \cap Q_k} x_k \right)
\]

\[
= \delta_Q^3 [Q_1 \cap Q_2 : N]^{1/2} \cdot \text{tr}_{M_{k-1}} \left( y_k^* (v_{k-2}^{Q_{k-2} \cap Q_k}) \cdots y_2 E_{Q_1 \cap Q_2} (y_1^* x_1) E_{k-1} x_{k-1} \cdots v_{k-2}^{Q_{k-2} \cap Q_k} x_k \right)
\]

\[
= \left\langle E_{Q_1 \cap Q_2} (y_1^* x_1) x_2 \otimes \cdots \otimes Q_{k-1} \cap Q_k x_k, y_2 \otimes \cdots \otimes Q_{k-1} \cap Q_k y_k \right\rangle
\]

\[
= \left\langle x_1 \otimes x_2 \otimes \cdots \otimes Q_{k-1} \cap Q_k x_k, y_1 \otimes y_2 \otimes \cdots \otimes Q_{k-1} \cap Q_k y_k \right\rangle
\]

as desired. \hfill \Box

The map $\phi$ above is clearly a bimodule map, so central vectors get mapped into $N' \cap L^2(M_k)$. Since $N' \cap L^2(M_k) = N' \cap M_k$ is finite dimensional, it follows that each $P_\alpha$ is finite dimensional.

### 3.1 Action of $N - P - M$ tangles on $P_\alpha$

We now describe how the $N - P - M$ planar operad acts on the various $P_\alpha$. Given an $N - P - M$ tangle $T$, we isotope it so that it is in standard form. This means:

1. All of the input and output disks are rectangles and all strings emanate from the top of the rectangles.
2. All the input disks are in different horizontal bands and all maxima and minima of strings are at different vertical levels, and not in the horizontal bands defined by the input disks.
3. The starred intervals of the input disks are all at the bottom-left corner. When we have a diagram of this form, the $*$ is omitted.

One then positions an imaginary horizontal line at the bottom of the tangle, $T$, and then slides it to the top. One starts with the central vector $1_Q \in L^2(Q)$ whenever the bottom of the box is shaded $Q$. The
central vector gets altered as the line crosses either an input box, a maximum on a string, or a minimum on a string. When the line reaches the top, you get the central vector produced by the action of the tangle.

Suppose the horizontal line passes through the $i^{th}$ rectangle (with respect to the isotopy) in the $m_{i}^{th}$ region which is shaded $Q_{m_{i}}$ (reading left to right along the line), and suppose that the vector $v_{i}$ has been assigned to the box. We simply insert $v_{i}$ into the $m_{i}$th slot, i.e.

$$
\sum_{j} x_{1}^{1} \otimes \cdots \otimes x_{m_{i}}^{1} \otimes \cdots \otimes x_{n} \mapsto \sum_{j} x_{1}^{1} \otimes \cdots \otimes v_{i} \otimes \cdots \otimes x_{n}
$$

Now suppose the horizontal line passes through a minimum, and suppose $Y \subset X$ with $X, Y \in \{N, P, M\}$ and $X$ and $Y$ resemble the regions on either side of the minimum. Let $B_{X,Y}$ be a Pimsner-Popa basis for $X$ over $Y$. Then we have the diagrammatic rules:

$$
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\vdots \vdots \\
X \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \vdots \\
Y \\
\end{array}
\end{array}
\mapsto
\begin{array}{c}
\begin{array}{c}
\vdots \vdots \\
X \\
\end{array}
\begin{array}{c}
\vdots \vdots \\
Y \\
\end{array}
\end{array}
\end{align*}
$$

$$
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\vdots \vdots \\
X \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \vdots \\
Y \\
\end{array}
\end{array}
\mapsto
\begin{array}{c}
\begin{array}{c}
\vdots \vdots \\
X \\
\end{array}
\begin{array}{c}
\vdots \vdots \\
Y \\
\end{array}
\end{array}
\end{align*}
$$

Whenever a dotted line passes over a maximum, the following rules apply:

$$
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\vdots \vdots \\
X \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \vdots \\
Y \\
\end{array}
\end{array}
\mapsto
\begin{array}{c}
\begin{array}{c}
\vdots \vdots \\
X \\
\end{array}
\begin{array}{c}
\vdots \vdots \\
Y \\
\end{array}
\end{array}
\end{align*}
$$

Here is an example of a tangle acting on $y_{1} \otimes x \otimes y_{2}$:

$$
\begin{align*}
(y_{1} \otimes x \otimes y_{2}) = \frac{1}{[X : Y]^{1/2}} \cdot \sum_{b \in B_{X,Y}} by_{1} x \otimes y_{2} \otimes b^{*}
\end{align*}
$$

Note also that our rules dictate that a loop with an $X$ on one side and $Y$ on the other counts for a factor $[X : Y]^{1/2}$. As $[M : P]$ and $[P : N]$ are the only two such indices that will appear, we will let $\delta_{a} = [P : N]^{1/2}$ and $\delta_{b} = [M : P]^{1/2}$.

It is a straightforward check to see that each of these maps preserves central vectors. Each map is also locally a bimodule map, hence the action of $T$ will also preserve invariant elements.

Checking that $T$ is well defined up to isotopy involves checking the same (finite number of) relations as in [Jon]. For example, checking

$$
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\vdots \vdots \\
X \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \vdots \\
Y \\
\end{array}
\end{array}
\end{align*}
$$

boils down to checking the relation $x = \sum_{b \in B_{X,Y}} E_{Y}(xb)b^{*}$, which always holds. The key to checking that the action of $T$ is defined up to isotopy is to show that rotation by $2\pi$ is the identity.
Let \( x = \sum_j x_j^1 \otimes x_j^2 \cdots \otimes x_j^n \in P_\alpha \), and let \( T \) be the tangle which is rotation by one-click clockwise, namely

\[
T = \begin{array}{c}
\end{array}
\]

where there are \( n - 1 \) strings that are not bent. By definition, we have:

\[
Z(T)(x) = \begin{cases}
\frac{1}{[Q_n : Q_1]^{1/2}} \sum_{b \in B(Q_n, Q_1)} \sum_j b \otimes x_j^1 \otimes x_j^2 \cdots \otimes x_j^n \otimes x_n x_{n+1} b^* & \text{if } Q_1 \subset Q_n \\
[Q_1 : Q_n]^{1/2} \sum_j 1 \otimes x_j^1 \otimes x_j^2 \cdots \otimes x_j^n x_n E_{Q_n}(x_{n+1}) & \text{if } Q_n \subset Q_1
\end{cases}
\]

To help our computations, we define the following left and right creation operators, \( L_x \) and \( R_x \) for \( x \in Q \). These are given by:

\[
L_x : Z_\alpha \rightarrow L^2(Q)_{Q^c | Q_1} \text{ such that } L_x(x_1 \otimes \cdots \otimes x_{n+1}) = x \otimes x_1 \otimes \cdots \otimes x_{n+1}
\]

\[
R_x : Z_\alpha \rightarrow Z_\alpha \otimes L^2(Q) \text{ such that } R_x(x_1 \otimes \cdots \otimes x_{n+1}) = x_1 \otimes \cdots \otimes x_{n+1} \otimes x
\]

It follows from the definition of the bimodule tensor product that

\[
L_x^*(x_0 \otimes x_1 \otimes \cdots \otimes x_{n+1}) = E_{Q^c | Q_1}(x^* x_0) x_1 \otimes \cdots \otimes x_{n+1} \text{ and }
\]

\[
R_x^*(x_0 \otimes x_1 \otimes \cdots \otimes x_{n+1} \otimes y) = x_0 \otimes x_1 \otimes \cdots \otimes x_{n+1} x_n E_{Q^c | Q_0}(y x^*)
\]

Therefore, we have the following formulae for the rotation tangle, \( T \):

\[
Z(T)(x) = \begin{cases}
\frac{1}{[Q_n : Q_1]^{1/2}} \sum_{b \in B} L_b R_b^*(x) & \text{if } Q_1 \subset Q_n \\
[Q_1 : Q_n]^{1/2} L_1 R_1^*(x) & \text{if } Q_n \subset Q_1
\end{cases}
\]

From Burns’ rotation trick [Bur03] we have the following lemma which is similar to lemmas that appear in [JP11]:

**Lemma 3.3.** Let \( \rho(\alpha) \) be the word formed when the words in \( \alpha \) are cyclically permuted clockwise by one, and let \( y = y_1 \otimes \cdots \otimes y_n \otimes y_{n+1} \in Z_\rho(\alpha) \). Then

\[
\langle T(x), y \rangle = \begin{cases}
\frac{1}{[Q_n : Q_1]^{1/2}} \langle x, y_2 \otimes \cdots \otimes y_{n+1} \otimes y_1 \rangle & \text{if } Q_1 \subset Q_n \\
[Q_1 : Q_n]^{1/2} \langle x, y_2 \otimes \cdots \otimes y_{n+1} \otimes y_1 \rangle & \text{if } Q_n \subset Q_1
\end{cases}
\]
Proof. For the first case, using that $x$ is central, we have
\[
\langle T(x), y \rangle = \frac{1}{[Q_n : Q_1]^{1/2}} \sum_{b \in B} L_b R_b^*(x), y \rangle = \sum_{b \in B} \frac{1}{[Q_n : Q_1]^{1/2}} \langle x, R_b L_b^*(y) \rangle
\]
\[
= \frac{1}{[Q_n : Q_1]^{1/2}} \sum_{b \in B} \langle x, E_{Q_1}(b^* y_1) y_2 \otimes y_3 \otimes \cdots \otimes y_{n+1} \otimes b \rangle
\]
\[
= \frac{1}{[Q_n : Q_1]^{1/2}} \sum_{b \in B} \langle (E_{Q_1}(b^* y_1))^* x, y_2 \otimes y_3 \otimes \cdots \otimes y_{n+1} \otimes b \rangle
\]
\[
= \frac{1}{[Q_n : Q_1]^{1/2}} \sum_{b \in B} \langle x (E_{Q_1}(b^* y_1))^*, y_2 \otimes y_3 \otimes \cdots \otimes y_{n+1} \otimes b \rangle
\]
\[
= \frac{1}{[Q_n : Q_1]^{1/2}} \sum_{b \in B} \langle x, y_2 \otimes y_3 \otimes \cdots \otimes y_{n+1} \otimes b E_{Q_1}(b^* y_1) \rangle
\]
\[
= \frac{1}{[Q_n : Q_1]^{1/2}} \langle x, y_2 \otimes \cdots \otimes y_{n+1} \otimes y_1 \rangle.
\]

For the second case, we have
\[
\langle T(x), y \rangle = [Q_1 : Q_n]^{1/2} \langle L_1 R_1^*(x), y \rangle = [Q_1 : Q_n]^{1/2} \langle x, R_1 L_1^*(y) \rangle
\]
\[
= [Q_1 : Q_n]^{1/2} \langle x, E_{Q_1}(y_1) y_2 \otimes \cdots \otimes y_{n+1} \otimes 1 \rangle
\]
\[
= [Q_1 : Q_n]^{1/2} \langle y_1^* x, y_2 \otimes \cdots \otimes y_{n+1} \otimes 1 \rangle
\]
\[
= [Q_1 : Q_n]^{1/2} \langle x y_1^*, y_2 \otimes \cdots \otimes y_{n+1} \otimes 1 \rangle
\]
\[
= [Q_1 : Q_n]^{1/2} \langle x, y_2 \otimes \cdots \otimes y_{n+1} \otimes y_1 \rangle
\]
as desired. \qed

Corollary 3.4. Rotation by $2\pi$ is the identity.

Proof. The nature of the shading dictates that all index factors cancel when applying the $2\pi$ rotation. The rest follows from the previous lemma. \qed

One can now continue as in [Jon].

3.2 Realizing $Q$ inside $P$

Suppose $Q$ is a planar algebra containing the Fuss-Catalan algebra, so that $Q$ is the planar algebra for a finite index inclusion $N \subset M$ with intermediate subfactor $P$. We note that $Q_{2n,+}$ is the space of $N - N$ central vectors of
\[
\left( N L^2(M)_N \right)^{\otimes n} = \left( N L^2(M)_M \otimes M L^2(M)_N \right)^{\otimes n}.
\]

and $Q_{2n,-}$ is the space of $M - M$ central vectors of $\left( M L^2(M)_N \otimes N L^2(M)_M \right)^{\otimes n}$. Since
\[
N L^2(M)_M = \left( N L^2(P)_P \otimes p L^2(M)_M \right).
\]
it follows that
\[ Q_{2n, +} = P_{(aba)^n/2} \quad \text{and} \quad Q_{2n, -} = P_{(baab)^n/2}. \]

Furthermore, a tangle \( S \) that acts on \( Q \) can be made into an \( N - P - M \) tangle \( S' \) by replacing each string with an \( a \) string cabled to a \( b \) string. This is done so that the shaded regions in \( S \) become the \( M - \)regions in \( S' \) and the unshaded regions in \( S \) become the \( N - \)regions in \( S' \). Notice that this implies that the strings along any disk in \( S' \) read (clockwise from the marked region) as \( aabaabbaa... \) or \( baabbaabb... \). It directly follows from the definitions that if \( x_1, ..., x_n \) are in \( Q \) and \( S \) is as above, then \( Z(S)(f) = Z(S')(f) \) where the left hand side denotes the action of a shaded tangle and the right hand side denotes the action of an \( N - P - M \) tangle.

**Definition 3.5.** If \( Q \) is the planar algebra of an inclusion \( N \subset M \) of finite index \( II_1 \) factors with an intermediate subfactor, \( P \), then we define the \( P \)-construction as above as the \( P \)-augmentation of \( Q \).

### 4 The GJS construction for the \( \sum FC \) potential

Suppose \( Q \) is a subfactor planar algebra containing a copy of the Fuss Catalan planar algebra. For each \( k \geq 0 \), we study the graded algebra \( \text{Gr}_{\pm}^k(Q) = \bigoplus_{n \geq k} Q_{\pm}^n \) as above, and place the following trace on \( \text{Gr}_{\pm}^k(Q) \):

\[
\text{tr}(x) = \frac{1}{(\delta_a \delta_b)^k} \sum_{\text{FC}}^k x
\]

where the shading on the *upper left* corner is \( \pm \).

As in Section 3, we realize \( Q \) inside an augmentation, \( P \), and we consider the algebras \( \text{Gr}_\alpha(P) \) where \( \overline{\alpha} \in \Delta \). If \( \alpha \neq \emptyset \) then the shading after the last letter of \( \alpha \) is uniquely determined and hence we can write,

\[
\text{Gr}_\alpha(P) = \bigoplus_{(\beta, \overline{\sigma} = \Delta)} P_{\overline{\sigma} \beta \alpha}.
\]

\( \text{Gr}_\alpha(P) \) is endowed with a multiplication \( \wedge \) given by

\[
x \wedge y = \alpha \beta \gamma \\
\begin{array}{c}
\\alpha \\beta \\
\\gamma \\
\alpha
\end{array}
\]

adjoint structure

\[
\begin{pmatrix}
\alpha \\
\beta \\
\alpha
\end{pmatrix}^* = \alpha \beta \alpha \\
\begin{pmatrix}
\alpha \\
\beta \\
\alpha
\end{pmatrix}
\]

and normalized trace

\[
\sum \text{CTL} \\
\begin{pmatrix}
\alpha \\
\beta \\
\alpha
\end{pmatrix}
\]
where $\sum CTL$ is the sum of all colored Temperley-Lieb diagrams. In the case where $\alpha = \emptyset$, then we have three such algebras, one for each shading $N$, $P$, and $M$. We therefore form graded algebras $Gr_0^N$, $Gr_0^P$ and $Gr_0^M$ where

$$Gr_0^Q = \bigoplus_{(\beta: \beta \in \Delta^Q)} \mathcal{P}_\beta.$$ 

Using the graded algebras associated to $\mathcal{P}$, we see that we have the following trace preserving inclusions:

$$Gr_{2k}^+(Q) \subset Gr_{\text{aba} \ldots \text{aba}}(\mathcal{P})$$
$$Gr_{2k+1}^+(Q) \subset Gr_{\text{aba} \ldots \text{baa}}(\mathcal{P})$$
$$Gr_{2k}^-(Q) \subset Gr_{\text{baa} \ldots \text{baa}}(\mathcal{P})$$
$$Gr_{2k+1}^-(Q) \subset Gr_{\text{baa} \ldots \text{aba}}(\mathcal{P}).$$

One advantage to working in the $N - P - M$ planar algebra $\mathcal{P}$ is that the map $\Phi : Gr_\alpha(\mathcal{P}) \to Gr_\alpha(\mathcal{P})$ given by

$$\Phi(x) = \sum_{E \in \text{Epi}(CTL)} E \left[ \begin{array}{c} x \\ \end{array} \right]$$

as in [JSW10, BHP12] is a well defined trace preserving isomorphism between $Gr_\alpha(\mathcal{P})$ with the $\sum CTL$ trace and $Gr_\alpha(\mathcal{P})$ with the orthogonalized trace. Therefore, we have proven the following lemma:

**Lemma 4.1.** The potential $\sum CTL$ gives a positive definite trace on $Gr_\alpha(\mathcal{P})$.

Furthermore, by considering either the $a$ or $b \cup$ element, the same analysis as in [JSW10, BHP12] proves the following theorem:

**Theorem 4.2.** Left (and right) multiplication of elements of $Gr_\alpha(\mathcal{P})$ on $L^2(Gr_\alpha(\mathcal{P}))$ is bounded and the associated von Neumann algebra, $M_\alpha = Gr_\alpha(\mathcal{P})''$ is a $II_1$ factor.

We note that if $\gamma$ and $\beta$ are words in $a$ and $b$ such that $\beta \beta \in \Delta$ and $\beta \gamma \beta \gamma \in \Delta$ then we have a unital inclusion of $M_\gamma$ into $M_\beta$, given the extension of

$$x \mapsto \left[ \begin{array}{c} x \\ \beta \\ \gamma \\ \end{array} \right].$$

We therefore have the following theorem, whose proof is exactly the same as the arguments in Section 4 of [JSW10].

**Theorem 4.3.** The following is a Jones’ tower of $II_1$ factors:

$$M_0^Q \subset M_\alpha \subset M_{\alpha \pi^1} \subset \cdots \subset M_{(\alpha \pi)^n} \subset M_{\pi(\alpha \pi)^n}.$$ 

Furthermore, $[M_\alpha : M_0^Q] = \delta_\alpha$, and the Jones projection for $M_0^Q \subset M_\alpha$ is

$$\epsilon_0 = \frac{1}{\delta_\alpha} \left[ \begin{array}{c} \alpha \\ \end{array} \right].$$
5 A semifinite algebra associated to $\mathcal{P}$

As in [GJS11], we will realize the isomorphism class of all of the $M_\alpha$ by examining realizing them as “corners” of a semifinite algebra. To begin, we consider the set semifinite algebra $Gr^Q_\infty$ for $Q = N, P, M$. As a vector space,

$$Gr^Q_\infty = \bigoplus_{\pi \in \Delta, \alpha \in \Delta^Q} P_{\pi \gamma \beta}.$$ 

**Assumption 5.1.** Throughout the rest of this article, the marked regions will now be on the bottom of the boxes.

Pictorially, we realize elements in $Gr^Q_\infty$ as linear combinations of boxes of the form

$$\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\end{array}$$

where the bottom (starred) region is shaded $Q$ (thus the top left corner varies in shading). $Gr^Q_\infty$ comes endowed with the following multiplication:

$$\begin{array}{c}
\kappa \\
\omega \\
\gamma \\
\end{array} \wedge \begin{array}{c}
\alpha \\
\theta \\
\chi \\
\end{array} = \begin{array}{c}
\delta_{\omega, \theta} \\
\gamma' \\
\chi \\
\end{array} \begin{array}{c}
\alpha \\
\theta \\
\gamma' \\
\chi \\
\end{array}$$

and semifinite trace, $\text{Tr}$, which is given by:

$$\text{Tr}(x) = \begin{array}{c}
\Sigma CTL \\
\alpha \\
\pi \\
\end{array}$$

if $x \in Gr(P_\alpha)$ and is zero otherwise. Just as in the analysis of [GJS11, BHP12], we see that $Gr^Q_\infty$ completes to a $II_\infty$ factor, $M^Q_\infty$ when being represented on $L^2(Gr^Q_\infty)$.

Also of importance will be the von Neumann subalgebra $A^Q_\infty \subset M^Q_\infty$ which is generated by all boxes in $G^Q_\infty$ with no strings on top. Notice that there is a normal, faithful, $\text{Tr}$-preserving conditional expectation $E : M^Q_\infty \to A^Q_\infty$ given by

$$E(x) = \begin{array}{c}
\Sigma CTL \\
\alpha \\
\pi \\
\end{array}$$

Furthermore, we have the following lemma, whose proof is identical to that of Lemma 3 of [GJS11]:

**Lemma 5.2.** $A^Q_\infty = \bigoplus_{v \in \Gamma_Q} A_v$ where $\Gamma_Q$ is the $Q$–principal graph of $\mathcal{P}$ and each $A_v$ is a type $I_\infty$ factor.

We now aim to figure out the isomorphism class of the algebras $M_\alpha$. For the remainder of the section, we will assume for simplicity that $Q = N$ and hence we will be finding the isomorphism class of $M_\alpha$. 


such that $s(\alpha) = N$. The other two cases will follow a similar analysis. We first define elements $X_a$ and $X_b$ as follows:

$$X_a = \sum_{\alpha, \pi \in \Delta^N} \alpha \cdot \pi + \alpha \cdot \varnothing$$

and

$$X_b = \sum_{\beta \in \Delta^P} \beta \cdot \varnothing + \beta \cdot \varnothing.$$

Note that $X_a$ and $X_b$ are sums of orthogonally supported operators, and each summand has uniformly bounded operator norm. Therefore, $X_a, X_b \in M_\infty^N$. Furthermore, we have the following lemma:

**Lemma 5.3.** $M_\infty^N$ is generated as a von Neumann algebra by $(A_\infty^N, X_a, X_b)$.

**Proof.** As in the proof of Lemma 7 of [GJS11], all that needs to be shown is that the following diagrams lie in the von Neumann algebra generated by $(A_\infty^N, X_a, X_b)$:

$$\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}$$

If $s(\alpha) = a$, then the exact same method as in the proof of Lemma 4.9 in [BHP12] shows that this element is in the algebra. If $s(\alpha) = b$, the following multiplication produces the diagram:

$$\begin{array}{c}
\frac{1}{\delta_a} \cdot \alpha \cdot \varnothing \cdot \beta \cdot \varnothing
\end{array}$$

A similar argument works for the element

$$\begin{array}{c}
\beta \cdot \varnothing
\end{array}.$$ 

As in [BHP12], we also have the following lemma:

**Lemma 5.4.** $X_a$ and $X_b$ are free with amalgamation over $A_\infty^N$ with respect to the conditional expectation, $E$.

We now define maps $\eta_a$ and $\eta_b$ on $A_\infty^N$ as follows:

$$\eta_c(y) = E(X_c y X_c)$$

for $c = a$ or $b$. Notice that by definition of $E$, $\eta_c$ is a completely positive map of $A_\infty^N$ into itself. Furthermore, it is a straightforward inductive check to note that the formula

$$E(y_0 X_c y_1 X_c \cdots y_{n-1} X_c y_n)$$

$$= \sum_{k=2}^n y_0 \cdot \eta_c(E(y_1 X_c \cdots X_c y_{k-1})) \cdot E(y_k X_c \cdots X_c y_n),$$

as in [GJS11] holds. Pictorially, there are nice expressions for $\eta_c(w)$ for various choices of $w$. To begin, for $i = 1, 2$, let $\alpha_i$ be a word with $\alpha_i \in \Delta^N$, $\beta_i$ be a word with $s(\beta_i) = a$ and $\beta_i \in \Delta^P$, and $\gamma_i$ be a word such...
that \( \gamma_i \in \Delta^M \). Furthermore, suppose \( x \in \mathcal{P}_{\alpha_1 \alpha_2}, y \in \mathcal{P}_{\beta_1 \beta_2} \) and \( z \in \mathcal{P}_{\gamma_1 \gamma_2} \). We then have the following easily verifiable formulae:

\[
\begin{align*}
\eta_a(x) &= \begin{array}{c}
\alpha \\
\hline \\
\alpha_2 \\
\end{array} \\
\eta_b(y) &= \begin{array}{c}
\beta_1 \\
\hline \\
\beta_2 \\
\end{array} \\
\eta_a(z) &= 0 \\
\eta_b(z) &= \begin{array}{c}
\gamma_1 \\
\hline \\
\gamma_2 \\
\end{array}
\end{align*}
\]

With the pictures above, the following useful lemma is easily verified:

**Lemma 5.5.** Let \( \alpha_i \) be a word with \( s(\alpha_i) = N \) and \( \beta_i \) be a word with \( s(\beta_i) = P \) for \( i = 1 \) or \( 2 \). In addition, suppose \( x \in \mathcal{P}_{\alpha_1 \alpha_2} \) and \( y \in \mathcal{P}_{\beta_1 \beta_2} \). We have the following formulae:

\[
x \cdot X_a = X_a \cdot \eta_a(x) \quad \text{and} \quad y \cdot X_b = X_b \cdot \eta_b(y).
\]

This lemma will be used to help describe certain compressions of \( \mathcal{M}^N_{\infty} \).

### 5.1 A suitable compression of \( \mathcal{M}^N_{\infty} \)

To begin, it will be useful to define three projections in \( A^N_{\infty} \)

\[
1_{A^N_{\infty}} = \sum_{\alpha \in \Delta^N} \begin{array}{c}
\alpha \\
\hline \\
\alpha_1 \\
\end{array}, \quad 1_{A^N_{Pa}} = \sum_{\alpha \in \Delta^N} \begin{array}{c}
\alpha \\
\hline \\
\alpha \\
\end{array}, \quad 1_{A^N_{Ma}} = \sum_{\alpha \in \Delta^N} \begin{array}{c}
\alpha \\
\hline \\
\alpha \\
\end{array}.
\]

Note that \( 1_{A^N_{Pa}} + 1_{A^N_{Ma}} + 1_{A^N_{Pa}} \) is the smallest projection dominating the support projections of \( X_a \) and \( X_b \).

Our goal is to better understand what happens when \( \mathcal{M}^N_{\infty} \) is compressed by certain projections. To begin our study, we consider \( \Gamma_N \), the \( N \)-principal graph of \( \mathcal{P} \). For each vertex, \( v \), at the \( N-N \) level of the graph, we choose a minimal projection \( p_v \in A_{\infty} \), and for the vertex, \( * \), we choose the empty \( N- \)shaded diagram. Notice that for each \( v \) we can choose \( p_v \in \mathcal{P}_{\alpha_1 \alpha_2} \) for \( \alpha \in \Delta^N \).

By the definition of the principal graph, we know that there exists a countable index set, \( I \) and partial isometries \( (V_i)_{i \in I} \subset A^N_{\infty} \) such that

\[
V^*_i V_i = \sum_{v \in \Gamma^N_N} p_v \quad \forall i \quad \text{and} \quad \sum_{i \in I} V^*_i V_i = 1_{A^N_{\infty}}.
\]

This necessarily implies that

\[
\sum_{i \in I} \eta_a(V_i)^* \eta_a(V_i) = 1_{A^N_{Pa}} \quad \text{and} \quad \sum_{i \in I} \eta_b(\eta_a(V_i))^* \eta_b(\eta_a(V_i)) = 1_{A^N_{Ma}}.
\]

We define \( R^1 \) by the following formula:

\[
R^1 = \sum_{v \in \Gamma^N_N} (p_v + \eta_a(p_v) + \eta_b(\eta_a(p_v))).
\]

If we set \( Z_i = V_i + \eta_a(V_i) + \eta_b(\eta_a(V_i)) \) then

\[
Z^*_i Z_i = R^1 \quad \text{and} \quad \sum_{i \in I} Z^*_i Z_i = 1_{A^N_{Pa}} + 1_{A^N_{Pa}} + 1_{A^N_{Ma}}.
\]

We have the following lemma regarding compression of \( \mathcal{M}^N_{\infty} \) by \( R^1 \).
Lemma 5.6. As a von Neumann algebra, \( R^1 \mathcal{M}_N^\infty R^1 \) is generated by \( R^1 \mathcal{A}_\infty^N R^1 \), \( R^1 X_a R^1 \), and \( R^1 X_b R^1 \).

Proof. Note that \( \sum_{i,j \in I} Z_i^* X_c Z_j = X_c \) for \( c = a \) or \( b \) by repeated applications of Lemma 5.5. Therefore, every word involving \( X_a \) or \( X_b \) and elements \( x \in \mathcal{A}_\infty \) (whose ending letters are supported under \( R^1 \)) can be replaced by sums of words involving terms of the form \( R^1 X_c R^1 \) and \( R^1 X_c X_d R^1 \) by inserting the relation

\[
\sum_{i \in I} Z_i^* Z_i = 1_{\mathcal{A}_N^\infty} + 1_{\mathcal{A}_a^\infty} + 1_{\mathcal{A}_b^\infty}.
\]

between every letter of the word. \( \square \)

We now investigate the action of compressing \( R^1 \mathcal{M}_N^\infty R^1 \) by subprojections of \( R^1 \). To begin, for each vertex \( w \in \Gamma_N^P \), let \( p_w \) be a minimal projection in \( \mathcal{A}_\infty^N \) corresponding to the vertex \( w \) such that \( p_w \leq \sum_{v \in V} \eta_v(p_v) \). For each edge \( e \) connecting a vertex in \( \Gamma_N^N \) to a vertex in \( \Gamma_N^P \), we let \( s(e) \) and \( t(e) \) be the vertices in \( \Gamma_N^N \) and \( \Gamma_N^P \) respectively which \( e \) connects. We define partial isometries \( \omega_e \in R^1 \mathcal{A}_\infty^N R^1 \) such that

\[
\omega_e^* \omega_e' = \delta_{e,e'} p_{t(e)} \quad \text{and} \quad \sum_{s(e)=v} \omega_{e}^* \omega_{e'}^* = \eta_v(p_v).
\]

Once the \( p_w \) have been chosen, for each vertex \( u \in \Gamma_N^M \), choose a minimal projection \( p_u \) corresponding to \( u \) such that \( p_u \leq \sum_{w \in \Gamma_N^G} \eta_b(p_w) \).

For each edge, \( f \), connecting the \( N-P \) vertices to the \( N-M \) vertices, \( s(f) \) and \( t(f) \) be the vertices in \( \Gamma_N^P \) and \( \Gamma_N^M \) respectively which \( f \) connects. We define partial isometries \( \nu_f \) satisfying:

\[
\nu_f^* \nu_f' = \delta_{f,f'} p_{t(f)} \quad \text{and} \quad \sum_{s(f)=w} \nu_f^* \nu_f^* = \eta_w(p_w).
\]

We now define operators \( X^e_a X^f_b \) by the formulae

\[
X^e_a = p_{s(e)} X_a \omega_e + \omega_e^* X_a p_{s(e)} \quad \text{and} \quad X^f_b = p_{s(f)} X_b \nu_f + \nu_f^* X_b.
\]

We have the following lemma, whose proof is the same as the arguments of Section 4.2 of [BHP12], except easier as there are no loops on the principal graph and only one minimal projection for each vertex.

Lemma 5.7. Set \( R = \sum_{v \in \Gamma} p_v \). Then \( R \mathcal{M}_N^\infty R \) is generated by \( R \mathcal{A}_\infty^N R \) and the elements \( X^e_a X^f_b \) for all \( e \) and \( f \), and each of the elements are free with amalgamation over \( R \mathcal{A}_\infty^N R \) with respect to \( E \).

Note that the algebra \( R \mathcal{A}_\infty^N R \) is simply the bounded functions on the vertices of \( \Gamma \), and the element \( p_{s(e)} X_a \omega_e \) has left support under \( p_{s(e)} \) and right support under \( p_{t(e)} \). Furthermore, if \( \text{Tr}(p_{s(e)}) \geq \text{Tr}(p_{t(e)}) \) then the analysis in [GJS11] shows that \( p_{s(e)} X_a \omega_e p_{s(e)} X_a \omega_e^* \) is a free poisson element with absolutely continuous spectrum in \( p_{t(e)} \mathcal{M}_\infty^N p_{t(e)} \). If \( \text{Tr}(p_{s(e)}) \leq \text{Tr}(p_{t(e)}) \), then \( p_{s(e)} X_a \omega_e (p_{s(e)} X_a \omega_e)^* \) is a free poisson element with absolutely continuous spectrum in \( p_{s(e)} \mathcal{M}_\infty^N p_{s(e)} \). Analogous statements hold for the elements \( p_{s(f)} X_b \nu_f \).
5.2 An amalgamated free product representation for $R\mathcal{M}_N^\infty R$

The work of the previous section shows that

$$R\mathcal{M}_N^\infty R = \mathcal{M}(\Gamma_N, \mu)$$

with $\mathcal{M}(\Gamma, \mu)$ the free graph algebra as in [Har13]. We use this to obtain a formula for $M_0$ when $\Gamma_N$ is finite. Let $e$ be an edge in $\Gamma_N$ connecting $v$ and $w$ with $\text{Tr}(p_v) \geq \text{Tr}(p_w)$. The basic rules for computing free dimension [Dyk93, DR11] show that

$$\text{fdim}(R\mathcal{M}_e R) = 1 - \frac{(\text{Tr}(p_v) - \text{Tr}(p_w))^2 - \sum_{u \neq v, w} \text{Tr}(p_u)^2}{\text{Tr}(R)^2} = 1 - \frac{\sum_{u \in \Gamma} \text{Tr}(p_v)^2 - 2 \text{Tr}(p_v) \text{Tr}(p_w)}{\text{Tr}(R)^2}.$$  

Using the additivity of free dimension, as well as

$$\text{fdim}(\ell^\infty(\Gamma)) = 1 - \frac{\sum_{u \in \Gamma} \text{Tr}(p_v)^2}{\text{Tr}(R)^2},$$

we obtain

$$\text{fdim}(R\mathcal{M}_N^\infty R) = 1 + \frac{-\sum_{u \in \Gamma} \text{Tr}(p_v)^2 + 2 \sum_{g \in E(\Gamma_N)} \text{Tr}(p_{s(g)}) \text{Tr}(p_{t(g)})}{\text{Tr}(R)^2}$$

$$= 1 + \frac{\sum_{u \in \Gamma} \text{Tr}(p_v) \sum_{v \sim u} (\text{Tr}(p_v) - \text{Tr}(p_u))}{\text{Tr}(R)^2}.$$  

Using the Perron-Frobenius condition, this becomes

$$\text{fdim}(R\mathcal{M}_N^\infty R) = 1 + \frac{2I((\delta_a - 1) + (\delta_b - 1))}{\text{Tr}(R)^2}$$

Where $I = \sum_{v \in \Gamma_N} \text{tr}(p_v)^2 = (\sum_{w \in \Gamma_P} \text{tr}(p_w)^2)^2 = \sum_{u \in \Gamma_N} \text{tr}(p_u)^2).$ Therefore, $R\mathcal{M}_N^\infty R$ is an interpolated free group factor with the above parameter. The compression formula for free group factors proves the following lemma

**Lemma 5.8.** $M_0^N \cong L(F_t)$ where $t = 1 + 2I(\delta_a + \delta_b - 2)$.

This gives us the following corollary:

**Corollary 5.9.** The factors $M_\alpha$ have the formula

$$M_\alpha \cong L(F(1 + 2I\delta_a^{-2}(\delta_a + \delta_b - 2)))$$

**Proof.** If $s(\pi) = N$, then it follows from the semifinite algebra $\mathcal{M}_N^\infty$ that $M_\alpha$ is a $\delta_a$ amplification of $M_0^N$. If the shading is different, apply similar analysis to the semifinite algebras $\mathcal{M}_P^P$ and $\mathcal{M}_M^M$.  

We now handle the case where $P$ is infinite depth:

**Lemma 5.10.** If $P$ is infinite-depth, then $M_0^N \cong L(F_\infty)$, and hence $M_\alpha \cong L(F_\infty)$ for all $\alpha$
Proof. Let \( \Gamma_k \) be the graph of \( \Gamma_N \) truncated up to depth \( k \) as in the proof of Theorem 4.2 in [Har13]. As in that proof, we let

\[
B(\Gamma_k) = \{ v \in \Gamma_k; \text{Tr}(p_v) > \sum_{w \sim v \in \Gamma_k} n_{v,w} \text{Tr}(p_w) \}
\]

where \( n_{v,w} \) is the number of edges that connect \( v \) and \( w \). We note that by the Perron Frobenius condition, no vertices in \( \Gamma_{k-1} \) are in \( B(\Gamma_k) \). Following the proof of Theorem 4.2 in [Har13] step-by-step, we arrive at the formula

\[
\text{fdim}(M_N^0) \geq 1 + (\delta_a - 1) \sum_{v \in \Gamma_{k-2}^{\text{N} \cup \text{P}}^{N} \cup \Gamma_N^P} \text{Tr}(p_v)^2 + (\delta_b - 1) \sum_{v \in \Gamma_{k-2}^{\text{N} \cup \text{P}}^{N} \cup \Gamma_N^P} \text{Tr}(p_v)^2
\]

which gets arbitrarily large as \( k \) does. The standard embedding arguments of [Har13] show that \( M_N^0 \approx L(\mathbb{F}_\infty) \). We arrive at the result for the other \( M_a \)’s either by amplification or by examining \( \mathcal{M}(\Gamma_P) \) or \( \mathcal{M}(\Gamma_M) \).

We can use this result to give a complete diagrammatic reproof of the universality result of Popa and Shlyakhtenko regarding the universality of \( L(\mathbb{F}_\infty) \) in subfactor theory.

Corollary 5.11 ([PS03]). Every subfactor planar algebra \( \mathcal{P}' \) is the standard invariant for a finite-index inclusion \( N \subset M \) with \( N \approx L(\mathbb{F}_\infty) \approx \mathcal{M} \).  

Proof. A diagrammatic proof of this fact for \( \mathcal{P}' \) infinite depth was done in [Har13]. If \( \mathcal{P}' \) is finite depth, let \( \mathcal{P}'' \) be the planar algebra for a finite index inclusion \( N' \subset P' \) of II\(_1\) factors. Let \( B \subset C \) be a finite index inclusion of II\(_1\) factors with principal graph \( A_\infty \). We consider the inclusions

\[
N' \otimes B \subset P' \otimes B \subset P' \otimes C.
\]

Let \( Q \) be the planar algebra for \( N \subset M \), and \( \mathcal{P} \) the augmentation of \( Q \) for \( N \subset P \subset M \). We note that \( \mathcal{P}' \) is also the planar algebra for \( N \subset P \), and the planar subalgebra of \( \mathcal{P} \) generated by words whose only color is \( a \) is \( \mathcal{P}' \). From Theorem 4.3, it follows that the standard invariant of \( M_N^0 \subset M_a \) is \( \mathcal{P}' \) and from the above calculation, \( M_N^0 \approx L(\mathbb{F}_\infty) \approx M_a \).

Note that Rădulescu was the first to provide a construction of \( N \subset M \) both isomorphic to \( L(\mathbb{F}_\infty) \) having standard invariant \( \mathcal{P}' \) for \( \mathcal{P}' \) finite depth [Răd94].

5.3 The law of \( \bigcup \in N_0^+ \)

One pleasing feature of the semifinite algebra constructed above is that it gives one a transparent way to find equations of the spectrum of the element \( \bigcup \in N_0 \). This corresponds to the double-cup element

\[
\begin{array}{cc}
\bigcup & \in M_0^N
\end{array}
\]

Picturing \( \bigcup \) as living in \( \mathcal{M}_I^N \), we note that \( \text{Tr}(\bigcup^n) = \text{Tr}(x^n) \) where \( x \) is the element

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2014 Maui and 2015 Qinhuangdao conferences
in honour of Vaughan F. R. Jones’ 60th birthday

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which is supported by
\[ 1_a = \]

Note that \( x \) is expressed as a product of free elements \( 1_a \mathcal{M}^N_\infty 1_a \). Let \( y \) be the element
\[ y = \]

and \( r \) be the element
\[ r = \]

We know that \( \cup_a \) is distributed as a free-poisson element, and its moment generating function is
\[ M_{\cup_a}(z) = -((\delta_a - 1)z - 1) + \sqrt{((\delta_a - 1)z - 1)^2 - 4z}. \]

In the algebra \( 1_a \mathcal{M}^N_\infty 1_a \) with \emph{normalized} trace, \( \tau \), we must have \( \tau(y^n) = \text{Tr}(\cup^n)/\delta_a \) for \( n \geq 1 \) and \( \tau(y^0) = 1 \). Therefore, the moment generating function of \( y \) is
\[ M_y(z) = M_{\cup_a}(z)\delta_a + \frac{\delta_a - 1}{\delta_a}. \]

The tool we will use to calculate the moments of \( x \) is Voiculescu’s \( S \)–transform \([VDN92]\). From \([VDN92]\), it is known that \( S_s \) and \( S_t \) are the \( S \)–transforms for free elements \( s \) and \( t \) in a tracial von Neumann algebra, then
\[ S_{st}(z) = S_{s^{1/2}t^{1/2}}(z) = S_s(z)S_t(z). \]

Furthermore, to compute the \( S \)–transform of an element \( s \), one finds formal power series \( \psi_s, \chi_s \) and \( S_s \) satisfying:
\[ \psi_s(z) = M_s(z) - 1 \quad \chi_s(\psi_s(z)) = z = \psi_s(\chi_s(z)) \quad \text{and} \quad S_s(z) = \frac{(z + 1)\chi_s(z)}{z}. \]

These formulas produce the following expression for the \( S \)–transform of \( x \):
\[ S_x(z) = \frac{(z + 1)^2(z - 1)(\delta_a z - 1)}{((\delta_b - 1)z + \delta_b)((\delta_a - 1)z + 1)}, \]

which can be inverted to give the Cauchy transform of \( x \). We know that the law of a single-colored \( \cup \) (as in \([GJS10]\)) is absolutely continuous with respect to Lebesgue measure and is supported away from the origin. Therefore, the law of \( y \) contains an atom of at the origin of measure \( \frac{\delta_a - 1}{\delta_a} \) at the origin and is absolutely continuous away from the origin. It follows that the law of \( y^{1/2}ry^{1/2} \) has an atom of measure \( \frac{\delta_a - 1}{\delta_a} \) at the origin and is absolutely continuous away from the origin. Therefore, the spectral projection corresponding to \( \{0\} \) for \( y^{1/2}ry^{1/2} \) must be the same as the spectral projection corresponding to \( \{0\} \) for \( y \). From this, we use the polar part of
\[ \]

to conclude that \( \cup \in \mathcal{N}_0^+ \) has law absolutely continuous to Lebesgue measure and supported away from the origin.
5.4 \( N_k \) contains a free group factor

Recall that \( N_k^\pm = Gr_k^\pm(Q)' \). We will use the moment calculation of \( \cup \) above as well as similar elements arising in the semifinite algebra to find a free group factor contained in \( N_k^\pm \). To simplify matters, we note that we need only consider the case where \( Q \) is Fuss Catalan, as any such planar algebra will contain the Fuss Catalan planar algebra.

By the usual amplification tricks, we need need only show that \( N_0^+ \) contains a free group factor. We embed \( Q \) into its augmentation \( P \) as in Example 2.5, which produces an embedding \( Gr_\infty^+(Q) \hookrightarrow Gr_\infty^N(P) \) where \( Gr_\infty^+(Q) \) is realized as the subalgebra of \( Gr_\infty^N(P) \) generated by the Fuss Catalan diagrams. Let \( N_\infty = Gr_\infty^+(Q) \). We have the following lemma:

**Lemma 5.12.** Let \( p_{ab} \) be the following diagram

\[
p_{ab} = \includegraphics{p_ab_diagram.png}
\]

and set

\[
x = \includegraphics{x_diagram.png} \quad \text{and} \quad y = \includegraphics{y_diagram.png}
\]

with \( f^{(2)} \) the second Jones-Wenzl idempotent in Temperley Lieb. Then \( x \) and \( y \) are free in \( p_{ab} M_\infty^N p_{ab} \).

**Proof.** From [BJ97], the projections

\[
p_0 = \includegraphics{p_0_diagram.png} \quad p_1 = \includegraphics{p_1_diagram.png} \quad p_2 = \includegraphics{p_2_diagram.png} \quad p_3 = \includegraphics{p_3_diagram.png} \quad \text{and} \quad p_4 = \includegraphics{p_4_diagram.png}
\]

are inequivalent minimal projections in \( P \), and there exists exactly one edge \( e_i \) which goes between the vertices representing \( p_{i-1} \) and \( p_i \). Therefore, by choosing \( p_0, p_2, \) and \( p_4, \) to line up with our choices of minimal projections lying under \( Q \), it follows that

\[
x = p_2 X^e_b X^e_a X^e_b X^e_a X^e_b X^e_a X^e_b p_2 \quad \text{and} \quad y = p_2 X^e_b X^e_a X^e_b X^e_a X^e_b p_2
\]

so \( x \) and \( y \) are free with amalgamation over \( A_\infty^N \). Since \( p_{ab} = p_2 \) is minimal in \( A_\infty^N \), the result follows.

**Proof of Theorem C.** Clearly, \( N_0^+ \subset M_\infty^N \) so \( N_0 \) is contained in an interpolated free group factor. Conversely, we know that \( p_{ab} N_\infty^N p_{ab} \) contains a copy of \( W^*(x)^* W^*(y) \). Since the law of \( \cup \in N_k \) has no atoms, it follows that

\[
W^*(x) = L(\mathbb{Z}) \oplus \bigoplus_{\delta_0 \delta_1 - 1} \mathbb{C}
\]

where \( q \) is equivalent to \( p_0 \) via the polar part of

\[
\includegraphics{polar_part.png}
\]

It follows from [Dyk93] that

\[
q(W^*(x)^* W^*(y))q = L(\mathbb{Z}) \ast q \left( \left( \bigoplus_{\delta_0 \delta_1 - 1} \mathbb{C} \right) \ast W^*(y) \right) \ast q
\]

which is an interpolated free group factor. By the equivalence of \( q \) and \( p_0 \) in \( N_\infty^N \), and the identity \( N_0^+ = p_0 N_\infty^N p_0 \), it follows that \( N_0^+ \) contains an interpolated free group factor.
Unfortunately, at this point, the author is unable to determine the isomorphism class of the $N_k^{\pm}$. While it is straightforward to show that a suitable compression of the algebra $N_\infty^N$ is generated by products of the form $X_a^e X_b^f$, terms of the form $X_{a_1}^{e_1} X_{a_2}^{e_2}$ and $X_{a}^{e_1} X_{b}^{e_2}$ appear with $e_1 \neq e_2$. The very nature of $N_\infty^N$ makes it difficult to “decouple” this into a free family which still lies in $N_\infty^N$.

References


