NPM planar algebras and the Guionnet-Jones-Shlyakhtenko construction

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1 Introduction

In [Jon83] Jones initiated the study of modern subfactor theory by defining the index of a subfactor. A stronger combinatorial invariant, called the standard invariant, was later developed and aximoitized by Ocneanu's paragroups [Ocn88], Popa's λ-lattices [Pop95], and Jones' planar algebras [Jon].

Given λ-lattice, Popa constructed a II_1-subfactor whose standard invariant is exactly the given λ-lattice [Pop95]. Later, in in work with Shlyakhtenko, it was shown that the factors in Popa's construction can be made to be isomorphic to the free group factor on infinitely many generators, $L(\mathbb{F}_\infty)$ [PS03]. Guionnet, Jones, and Shlyakhtenko gave a diagrammatic (planar algebraic) proof of Popa's result [GJS10]. In the finite depth case, they showed that the factors involved in the construction are interpolated free group factors [GJS11]. The author later showed that in the infinite depth case, the factors involved in the construction are all isomorphic to $L(\mathbb{F}_\infty)$ [Har13].

This article initially appeared as the final chapter of the author's graduate thesis, and is based on a problem posed by Vaughan Jones. The problem is as follows: Given a subfactor planar algebra, $\mathcal{Q}$, one can consider the algebras $\mathrm{Gr}_k^{\pm}(\mathcal{Q})$ as defined in [GJS10], and place the following "toy potential" on $\mathcal{Q}$:

$$\mathrm{tr}(x) = \begin{array}{c}\boxed{\sum V}\\ | \\ \boxed{x}\end{array}$$

where V is a rotationally invariant set of elements in $\mathcal{Q}$. If one is fortunate, tr is positive definite on $\mathcal{Q}$ and left multiplication is bounded on $L^2(\mathrm{Gr}(\mathcal{Q}))$. To this end, it is an interesting problem to study the von Neumann algebras, $N_k^{\pm}(= \mathrm{Gr}_k^{\pm}(\mathcal{Q})'')$ associated to $\mathcal{Q}$ and V.

The case that will be considered here is the case where $\mathcal{Q}$ is the standard invariant for a subfactor $N \subset M$ that contains an intermediate subfactor, P. As such, it follows that $\mathcal{Q}$ contains the Fuss Catalan planar algebra as a sub planar algebra [BJ97]. Therefore, we can consider the following potential on $\mathrm{Gr}_0^{+}(\mathcal{Q})$:

$$\mathrm{tr}(x) = \begin{array}{c}\boxed{\sum FC}\\ | \\ \boxed{x}\end{array}$$

where $\sum FC$ represents the sum of all Fuss Catalan diagrams. Recall that Fuss Catalan diagrams are planar two-colored string diagrams satisfying the following condition: If the colors of the strings are a and b, then the colors of strings intersecting the boundary form the pattern, $aabbaabbaabb...$

To attack this problem, it is natural to introduce a new planar algebra, $\mathcal{P}$ which will be called an $N-P-M$ planar algebra. A pleasing feature of this planar algebra is that the boundary conditions of the input disks can be taken to be any word in aa and bb. The planar algebra $\mathcal{Q}$ will be realized as a subalgebra of $\mathcal{P}$ by replacing each strand with an a strand next to a b strand so that the pattern formed by the input disks in the $\mathcal{Q}-$tangles is of the form $aabbaabbaabb....$ Whenever $\mathcal{Q}$ is itself the Fuss Catalan planar algebra, we can take $\mathcal{P}$ to be generated the set of all string diagrams having boundary conditions any word in aa and bb.

The purpose of the $N-P-M$ planar algebra is that it gives one a natural way to decouple the strings a and b and treat them as free generators in an appropriate sense (see Section 5 for how this is done). We form algebras M_α for α a suitable word in a and b which are the $N-P-M$ analogues of the algebras $M_k^\pm$ from [GJS10], and will use a semifinite algebra as in [GJS11] to find the isomorphism class of the algebras M_α. More precisely, we will prove the following theorem:

Theorem A. *Let α be a word in a and b where a appears n times, and b apears m times. Define $\delta_\alpha = \delta_a^n \cdot \delta_b^m$. M_α is a II_1 factor and is isomorphic to $L(\mathbb{F}(1+2I\delta_\alpha^{-2}(\delta_a+\delta_b-2)))$ for $\mathcal{P}$ finite depth. Here, and $I=\sum_{v\in\Gamma_N^N}\mu(v)^2$ with μ the Perron Frobenius weighting on the principal graphs on $\mathcal{P}$.*

This formula has some interest, because it contains information about the inclusions $N\subset P$ and $P\subset M$ (δ_a and δ_b respectively) as well as the larger inclusion $N\subset M$ (the global index I). Just as in the case for the GJS algebras, we will also prove the following theorem:

Theorem B. *$M_\alpha\cong L(\mathbb{F}_\infty)$ when $\mathcal{P}$ is infinite depth.*

By taking a tensor product of planar algebras, one obtains the following diagrammatic corollary

Corollary ([PS03]). *Given any standard invariant, $\mathcal{Q}$, there exist II_1-factors $N\subset M$ having standard invariant $\mathcal{Q}$ and both isomorphic to $L(\mathbb{F}_\infty)$.*

In addition to understanding the algebraic structure of the N_k, we will make use of the semifinite algebra to show that the law of $\cup\in\mathbb{N}_0^+$ has a nice expression in terms of known laws. Unfortunately, the author has not yet been able to identify the isomorphism classes of the algebras $N_k^\pm$, however the semifinite algebra will show that there is evidence that the algebras $N_k^\pm$ are free group factors:

Theorem C. *The von Neumann algebras, $N_k^\pm$, are each contained in a free group factor and contain a free group factor.*

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2 $N-P-M$ planar algebras

The object that is at the heart of all of these computations is an $N-P-M$ planar algebra which can be thought of an augmentation of such a $\mathcal{Q}$ as above. Given parameters a and b, we denote $\mathcal{W}$ as the set of finite words on aa and bb. We define what it means for a planar tangle to be an $N-P-M$ tangle:

Definition 2.1. A planar tangle is said to be an $N-P-M$ tangle if its regions are shaded by three colors, N, P, and M such that the following conditions are met:

- A region colored N only borders regions colored P
- A region colored M only borders regions colored P
- A region colored P can border regions colored N or M but not P.

We will denote the set of $N-P-M$ tangles as the $N-P-M$ planar operad. All tangles will be drawn so that the internal and external disks are rectangles. In order for smooth isotopy to make sense, the rectangles have their corners smoothed out.

Any string serving as the boundary string of a region colored N will be called an a string and any string bordering a region colored M will be called a b string. We note that the conditions on the regions show that if α is a word in a and b, then α serves as a word of strings intersecting a disk if and only if $\alpha \in \Delta$ where Δ is the set

$$\Delta = \{awa : w \in \mathcal{W}\} \cup \{bwb : w \in \mathcal{W}\} \cup \mathcal{W}$$

The following notation will be useful:

Notation 2.2. We denote Δ^Q to be the set of all words in Δ where the shading before the first letter is Q (for $Q = N,\ P$, or M). If $\alpha \in \Delta$, we define $s(\alpha)$ to be the first letter of the word α. Whenever a word is mentioned, part of the data is its initial region (hence the choice of region between every pair of letters), not just its letters. If a tangle, T, has boundary condition α on its outer disk, we will call T a planar α tangle. Any internal rectangle with boundary condition α will be called an α rectangle.

We remark that just as for shaded planar algebras, there is a natural gluing operation. Namely, if we have two planar tangles S, T satisfying the following boundary condition:

- Some internal rectangle D_S of S has boundary data which agrees with T, i.e. the shadings along the boundaries of T and D_S agree when counting clockwise from the marked point.

then we may compose S and T to get the planar tangle $S \circ_{D_S} T$ by taking S union the interior of T, removing the boundary of D_S, smoothing the strings, and removing any closed loops.

Given $\alpha \in \Delta$, we let $\overline{\alpha}$ be α read in the opposite order. When a string appears with a label α, then the string is meant to be a band of strings having colors ordered by the word α. The strings are read in the order of top to bottom and left to right. Also, unless otherwise marked, all marked regions of rectangles will be assumed to be on the top-left corner of the box. In addition, whenever there is a box written without a tangle, it is assumed that the box is placed in a larger tangle whose boundary data agrees with the boundary data for the box and whose marked region is the same as the marked region of the box.

We now define an $N-P-M$ planar algebra:

Definition 2.3. An $N-P-M$ planar algebra consists of the following data:

- Given parameters a and b as above, there is a finite dimensional complex vector space $\mathcal{P}_\alpha$ for every nonempty word, $\alpha \in \Delta$. There are three vector spaces $\mathcal{P}_\emptyset^N$, $\mathcal{P}_\emptyset^P$, and $\mathcal{P}_\emptyset^M$ in the case when α is empty. These are one dimensional complex vector spaces.

- An action of planar tangles by multilinear maps, i.e., for each planar α tangle T, whose rectangles $D_i(T)$ are α_i rectangles, there is a multilinear map

$$Z_T\colon \prod_{i\in I} P_{\alpha_i} \to P_\alpha$$

satisfying the following axioms:

Isotopy: If θ is an orientation preserving diffeomorphism of $\mathbb{R}^2$, then $Z_{\theta(T)} = Z_T$. That is, let T^0 be the interior of T, and let $f \in \prod_{D\subset T^0} P_{\alpha_D}$. Then

$$Z_{\theta(T)}(f_\theta) = Z_T(f)$$

where $f_\theta(\theta(D)) = f(D)$.

Naturality: For S, T composable tangles, $Z(S \circ_D T) = Z(S) \circ_D Z(T)$, where the composition on the right hand side is the composition of multilinear maps.

- $\mathcal{P}$ is unital [Jon11]: Let S be an $N-P-M$ tangle with no input disks and boundary condition $\alpha \in \Delta$. Then, there is an element $Z(S) \in P_\alpha$ so that the following holds:

  Let S be a tangle a nonempty set of internal disks such that S can be glued into the internal disk D^S of T. Then
$$Z(T \circ S) = Z(T) \circ Z_S.$$
  Here $(Z(T) \circ Z_S)(f) = \tilde{f}$ where

$$\tilde{f}(D) = \begin{cases} f(D) \text{ if } D \neq D^S \\ Z(S) \text{ if } D = D^S \end{cases}$$

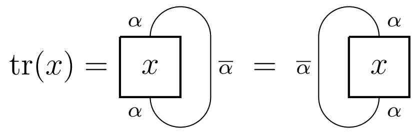
  This condition allows isotopy classes of such an S to be elements of P_α. This action allows us to identify the empty diagrams (shaded N, P, and M) with the scalar $1 \in \mathbb{C}$. We make this assumption in the rest of this chapter. The naturality axiom, combined with this identification, forces closed strings with parameters a and b to be replaced by scalars δ_a and δ_b respectively.

- There is a conjugate linear involution, $* : \mathcal{P}_\alpha \to \mathcal{P}_{\overline{\alpha}}$. It is compatible with reflection of tangles i.e., if $\overline{T}$ is a tangle which is produced by an orientation reversing diffeomorphism, φ, of T, then we have
$$(Z_T(f))^* = Z_{\overline{T}}(\overline{f})$$
  where $\overline{f}(\varphi(D)) = f(D)^*$.

- Each $\mathcal{P}_\alpha$ comes equipped with the positive definite sesquilinear form:

$$\langle x, y\rangle = \boxed{x} \overset{\alpha}{—} \boxed{y^*}$$

- $\mathcal{P}$ is <u>spherical</u>, i.e. for all $\alpha \in \Delta$ and all $x \in P_{\alpha\overline{\alpha}}$, we have



  This says that we can think of our planar tangles as living in a sphere instead of a plane.

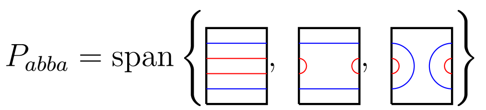
Remark 2.4. In viewing the action of a tangle, the letter Z will often be omitted.

A-priori, it is not clear that an $N - P - M$ planar algebra should exist. The following example shows that this is the case. For the rest of this paper, an a string will be colored blue and a b string will be colored red.

Example 2.5. Let $\delta_a, \delta_b \in \{2\cos(\pi/n) : n = 3, 4, 5, ...\} \cup [2, \infty)$ and define P_α by

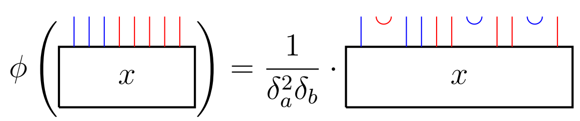
$$P_\alpha = \mathrm{span}\{\text{planar string diagrams with with boundary condition } \alpha\}$$

i.e., P_α is the $\mathbb{C}-$linear span of isotopy classes of α tangles with no input disks and no loops. For example,



The action of $N - P - M$ tangles is as follows: All string diagrams are inserted into the necessary input disks. The result of this operation is a new string diagram with except with some loops. These loops are replaced with a parameter δ_a or δ_b, depending on the color of a loop. The adjoint operation is the conjugate linear extension of reflection of diagrams.

It is straightforward to check that $\mathcal{P}$ satisfies all of the axioms of an $N - P - M$ planar algebra except positive definiteness. Given $\alpha \in \Delta$, we form the word α' which is a word of colors that can appear in a Fuss Catalan diagram, and is obtained from α by inserting the minimal numbers of aa's or bb's between letters in α. For example, if $\alpha = aaabbaabbbba$, then $\alpha' = a(bb)aabbaabb(aa)bba$. We then define a map $\phi : P_\alpha \to P_{\alpha'}$ which is given by inserting a cup of the appropriate color whenever that color has been inserted into α, and then dividing by $\delta_a^m \cdot \delta_b^n$. Here, aa was inserted m times and bb was inserted n times. For example,

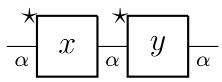


This map is easily seen to be preserve the desired sesquilinear form, and we know that this form is positive semidefinite on the Fuss Catalan algebras, with positive definiteness in the case $\delta_a, \delta_b \geq 2$, from [BJ97]. Therefore, after taking a quotient in the case that δ_a or δ_b is less than 2, this example produces an $N - P - M$ planar algebra.

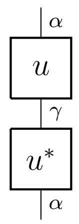
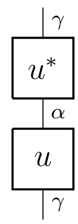
By unitality, this planar algebra is represented in every $N - P - M$ planar algebra.

2.1 Principal graphs of $N - P - M$ planar algebras

We first remark that if $\overline{\alpha}\alpha \in \Delta$, then the axioms of an $N - P - M$ planar algebra show that $P_{\alpha\overline{\alpha}}$ is a finite dimensional C^* algebra with multiplication given by

$$x \cdot y =$$  .

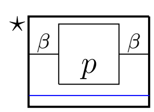
Let $p \in P_{\alpha\overline{\alpha}}$ and $q \in P_{\gamma\overline{\gamma}}$ be projections. Then we say p is equivalent to q if there is a $u \in P_{\alpha\overline{\gamma}}$ so that

 $= p$ and  $= q.$

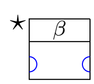
To an $N - P - M$ planar algebra $\mathcal{P}$, there are three principal graphs associated to $\mathcal{P}$, Γ_N, Γ_P, and Γ_M. We will call Γ_Q the $Q-$principal graph of $\mathcal{P}$. Each Γ_Q has three sets of vertices, Γ_Q^N, Γ_Q^P and Γ_Q^M. They are described by the following procedure:

The vertices $v \in \Gamma_{Q_1}^{Q_2}$ correspond to equivalence classes of minimal projections p_v in the finite-dimensional C^* algebra $P_{\alpha\overline{\alpha}}$ for some α depending on v where $\alpha\overline{\alpha} \in \Delta^{Q_1}$ and $\overline{\alpha}\alpha \in \Delta^{Q_2}$. There are $a-$colored edges connecting the vertices Γ_Q^N to the vertices Γ_Q^P as well as $b-$colored edges connecting the the vertices Γ_Q^P to Γ_Q^M. The $a-$colored edges are defined as follows:

Suppose $v \in \Gamma_Q^N$ and $w \in \Gamma_Q^P$, let $p \in P_{\beta\overline{\beta}}$ be equivalent to p_v. It follows that the element

$$i_a(p) =$$ 

is a projection in $P_{\beta aa\overline{\beta}}$. We draw n $a-$colored edges between v and w if n is the maximal number such that there exist orthogonal projections $q_1, \ldots, q_n \in P_{\beta aa\overline{\beta}}$ which are each equivalent to p_w and satisfy $\sum_{i=1}^n q_i \leq i_a(p)$. We can also get edges from w to v in a similar manner. In principle, the construction of the $a-$edges leads to oriented edges, however, the presence of the Jones basic construction shows that the edges can be unoriented. More precisely, consider the projection

$$e = \frac{1}{\delta_a}$$  $\in P_{\beta aaaa\overline{\beta}},$

and z be its central support. We note that $P_{\beta\overline{\beta}}$ unitally includes into $P_{\beta aaaa\overline{\beta}}$ by applying the map i_a twice. It is also a straightforward check to see that the mapping $P_{\beta\overline{\beta}} \to P_{\beta aaaa\overline{\beta}}$ given by $x \mapsto i_a(i_a(x))e$ is an isometry, and $ei_a(y)e = i_a(E_{P_{\beta\overline{\beta}}}(y))e$ for $y \in P_{\beta aa\overline{\beta}}$. Therefore, from [JS97] it follows $zP_{\beta aaaa\overline{\beta}}z$ is isomorphic to the basic construction of $P_{\beta\overline{\beta}}$ in $P_{\beta aa\overline{\beta}}$. If A, B, and C are finite dimensional C^* algebras with C the basic construction of A in B, then the Bratteli diagram of $B \subset C$ is the reflection of that of $A \subset B$ [JS97]. Therefore, if there are n $a-$colored edges from v to w, then there are n $a-$colored edges from w to v.

There is an analogous way to determine the $b-$colored edges that go between Γ_Q^P and Γ_Q^M. We also note that if $p \in P_{\alpha\overline{\alpha}}$ is a minimal projection corresponding to a vertex $v \in \Gamma_{Q_1}^{Q_2}$, then the mapping in Example

2.5 shows that p is equivalent to a minimal projection in P_w where $w = abbaabbaa...$, $aabbaabb...$, $bbaabbaa...$ or $baabbaa....$ In particular, if $\mathcal{Q}$ is as in the introduction, then we have established the following proposition once it is shown that we can consider $\mathcal{Q}$ inside an augmentation $\mathcal{P}$.

Lemma 2.6. *Let Γ and Γ' be the principal and dual principal graphs of $\mathcal{Q}$ respectively. Then there are one-to-one correspondences between the following sets of vertices:*

$$\Gamma_+ \leftrightarrow \Gamma_N^N,\ \Gamma_- \leftrightarrow \Gamma_N^M,\ \Gamma'_+ \leftrightarrow \Gamma_M^M,\ \textit{and}\ \Gamma'_- \leftrightarrow \Gamma_M^N$$

Also observe that rotation by $180°$ is an anti-isomorphism of each P_γ. This induces a one-to-one correspondence $\Gamma_{Q_1}^{Q_2} \leftrightarrow \Gamma_{Q_2}^{Q_1}$. Finally, if the vertices of the principal graphs Γ_Q are weighted according to the traces of their corresponding projections, then it follows by the definition of principal graph that the graph with vertices Γ_Q^N and Γ_Q^P is bipartite with Perron Frobenius eigenvalue δ_a. Also, the graph with vertices Γ_Q^P and Γ_Q^M is bipartite with Perron Frobenius eigenvalue δ_b.

3 $N-P-M$ planar algebras from intermediate subfactors

The goal of this section is to see that such a $\mathcal{Q}$ as above can be faithfully realized inside an $N-P-M$ planar algebra $\mathcal{P}$. Much of this section was influenced from discussions with David Penneys and Noah Snyder, and many of the proofs of the following theorems are taken from them. We will first describe how such an algebra arises from an inclusion $N \subset P \subset M$ of finite index II_1 factors. To start, we consider the following bifinite bimodules:

$$_NL^2(P)_P \text{ and } {}_PL^2(M)_M$$

and their duals (contragredients)

$$_PL^2(P)_N \text{ and } {}_ML^2(M)_P.$$

Let $\alpha \in \Delta$. Since part of the prescribed data for α is a choice of initial shading, we note that the shading of α, i.e. the shading between any two letters on α is uniquely determined. Assume that the shading of α is the sequence $Q_1 \cdots Q_k$ for $Q_i = N,\ P$ or M. We define Z_α to be the following:

$$Z_\alpha = {}_{Q_1}L^2(Q_1)_{Q_1\cap Q_2} \underset{Q_1\cap Q_2}{\otimes} {}_{Q_1\cap Q_2}L^2(Q_2)_{Q_2\cap Q_3} \cdots \underset{Q_{k-1}\cap Q_k}{\otimes} {}_{Q_{k-1}\cap Q_k}L^2(Q_k)_{Q_k\cap Q_1} \underset{Q_k\cap Q_1}{\otimes} L^2(Q_1)_{Q_k\cap Q_1}.$$

and we set $P_\alpha = \mathrm{Hom}_{Q_1-Q_1}(L^2(Q_1), Z_\alpha)$ (Notice that $Q_i \cap Q_{i+1}$ is necessarily N, P, or M). We note from [Bis97, Con80] that this can be identified with the $Q_1 - Q_1$ central vectors of Z_α.

To help understand the planar structure, we let

$$N(= M_0) \subset M(= M_1) \subset M_2 \subset \cdots \subset M_n \subset \cdots$$

be the Jones tower for $N \subset M$, where M_n is generated by M_{n-1} and e_{n-1}. Here, e_{n-1} is the orthogonal projection from $L^2(M_{n-1})$ onto $L^2(M_{n-2})$. We will define e_P to be the orthogonal projection from $L^2(M)$ onto $L^2(P)$. We will also let $B = \{b_i\}_{i=1}^n$ be an orthonormal Pimsner Popa basis for M over N where $n-1$ is the largest integer which is bounded above by the index $[M : N]$. The b_i are elements in M

satisfying the following equivalent conditions:

$$x = \sum_{i=1}^{n} E_N(xb_i)b_i^* \ \forall x \in M$$
$$x = \sum_{i=1}^{n} b_i E_N(b_i^* x) \ \forall x \in M$$
$$1 = \sum_{i=1}^{n} b_i e_1 b_i^*,$$

as well as $E_N(b_i b_j^*) = \delta_{i,j}$ if $i \leq n-1$ and $E_N(b_n b_n^*)$ is a projection of trace $[M : N] - (n-1)$ in M. If we let e_P be the orthogonal projection from $L^2(M)$ onto $L^2(P)$ and $\{c_i\}_{i=1}^m$ be an orthonormal Pimsner-Popa basis of P over N. Then we have the following lemma.

Lemma 3.1. $e_P = \sum_{i=1}^m c_i e_1 c_i^*$.

Proof. We compute the 2-norm of $e_P - \sum_{i=1}^m c_i e_1 c_i^*$. Doing so gives:

$$\|e_p - \sum_{i=1}^{m} c_i e_1 c_i^* \|_2^2 = \mathrm{tr}(e_P) - 2\sum_{i=1}^{m} \mathrm{tr}(c_i e_1 c_i^* e_P) + \sum_{i,j=1}^{m} \mathrm{tr}(c_i e_1 c_i^* c_j e_1 c_j^*).$$

Since e_P commutes with the elements c_i, the term in the middle becomes $2\sum_{i=1}^m \mathrm{tr}(c_i e_1 c_i^*)$. Using $e_1 c_i^* c_j e_1 = E_N(c_i^* c_j e_1)$, and orthonormality of the basis, the last term becomes $\sum_{i=1}^m \mathrm{tr}(c_i e_1 c_i^*)$. Therefore, we get:

$$\begin{aligned}\|e_p - \sum_{i=1}^{m} c_i e_1 c_i^* \|_2^2 &= \mathrm{tr}(e_P) - \sum_{i=1}^{m} \mathrm{tr}(c_i e_1 c_i^*) \\ &= \mathrm{tr}(e_P) - [M:N]^{-1} \sum_{i=1}^{m} \mathrm{tr}(c_i c_i^*) \\ &= [M:P]^{-1} - [M:N]^{-1}[P:N] = 0\end{aligned}$$

as desired. □

We will now show the bimodules Z_α can be isometrically embedded in $L^2(M_n)$ for some n. As some notation, we will let $\delta_Q = [M : Q]^{1/2}$ for $Q = N$, P, or M. We also set $E_1^Q = \delta_Q e_Q$, and $v_n^Q = E_n E_{n-1} \cdots E_2 E_1^Q$.

Theorem 3.2. *The map $\phi : Z_\alpha \to M_k$ given by*

$$\phi(x_1 \underset{Q_1 \cap Q_2}{\otimes} x_2 \underset{Q_2 \cap Q_3}{\otimes} \cdots \underset{Q_{k-1} \cap Q_k}{\otimes} x_k) = x_1 v_1^{Q_1 \cap Q_2} x_2 v_2^{Q_2 \cap Q_3} \cdots v_{k-1}^{Q_{k-1} \cap Q_k} x_k$$

is an isometry.

Proof. Note that the map is well defined as v_r^Q commutes with Q. We proceed by induction on k. The result is clearly true for $k=1$, so assume that it holds for $k-1$. Using the previous lemma as well as the relation $E_iE_j = E_jE_i$ for $|i-j| \geq 2$, we have:

$$
\begin{aligned}
&\left\langle x_1 v_1^{Q_1\cap Q_2} x_2 v_2^{Q_2\cap Q_3} \cdots v_{k-1}^{Q_{k-1}\cap Q_k} x_k,\ y_1 v_1^{Q_1\cap Q_2} y_2 v_2^{Q_2\cap Q_3} \cdots v_{k-1}^{Q_{k-1}\cap Q_k} y_k \right\rangle_{M_k}\\
&= \operatorname{tr}_{M_k}\left(y_k^*(v_{k-1}^{Q_{k-1}\cap Q_k})^* \cdots (v_2^{Q_2\cap Q_3})^* y_2^* (v_1^{Q_1\cap Q_2})^* y_1^* x_1 v_1^{Q_1\cap Q_2} x_2 v_2^{Q_2\cap Q_3} \cdots v_{k-1}^{Q_{k-1}\cap Q_k} x_k \right)\\
&= \delta_Q^2 \operatorname{tr}_{M_k}\left(y_k^*(v_{k-1}^{Q_{k-1}\cap Q_k})^* \cdots y_2^* E_{Q_1\cap Q_2}(y_1^* x_1) e_{Q_1\cap Q_2} x_2 \cdots v_{k-1}^{Q_{k-1}\cap Q_k} x_k \right)\\
&= \delta_Q^2 \operatorname{tr}_{M_k}\left(y_k^*(v_{k-2}^{Q_{k-1}\cap Q_k})^* E_{k-1} \cdots y_2^* E_{Q_1\cap Q_2}(y_1^* x_1) e_{Q_1\cap Q_2} x_2 \cdots E_{k-1} v_{k-2}^{Q_{k-1}\cap Q_k} x_k \right)\\
&= \frac{\delta_Q^2}{[M:N]^{1/2}}\cdot\\
&\operatorname{tr}_{M_k}\left(y_k^*(v_{k-2}^{Q_{k-1}\cap Q_k})^* \cdots y_2^* E_{Q_1\cap Q_2}(y_1^* x_1) E_{k-1} \cdots E_2 \sum_{i=1}^m (b_i E_1 b_1^*) E_2 \cdots E_{k-1} x_2 \cdots v_{k-2}^{Q_{k-1}\cap Q_k} x_k \right)\\
&= \frac{\delta_Q^2 [Q_1\cap Q_2 : N]}{[M:N]^{1/2}} \cdot \operatorname{tr}_{M_k}\left(y_k^*(v_{k-2}^{Q_{k-1}\cap Q_k})^* \cdots y_2^* E_{Q_1\cap Q_2}(y_1^* x_1) E_{k-1} x_2 \cdots v_{k-2}^{Q_{k-1}\cap Q_k} x_k \right)\\
&= \operatorname{tr}_{M_{n-1}}\left(y_k^*(v_{k-2}^{Q_{k-1}\cap Q_k})^* \cdots y_2^* E_{Q_1\cap Q_2}(y_1^* x_1) x_2 \cdots v_{k-2}^{Q_{k-1}\cap Q_k} x_k \right)\\
&= \left\langle E_{Q_1\cap Q_2}(y_1^* x_1) x_2 \underset{Q_2\cap Q_3}{\otimes} \cdots \underset{Q_{k-1}\cap Q_k}{\otimes} x_k,\ y_2 \underset{Q_2\cap Q_3}{\otimes} \cdots \underset{Q_{k-1}\cap Q_k}{\otimes} y_k \right\rangle\\
&= \left\langle x_1 \underset{Q_1\cap Q_2}{\otimes} x_2 \underset{Q_2\cap Q_3}{\otimes} \cdots \underset{Q_{k-1}\cap Q_k}{\otimes} x_k,\ y_1 \underset{Q_1\cap Q_2}{\otimes} y_2 \underset{Q_2\cap Q_3}{\otimes} \cdots \underset{Q_{k-1}\cap Q_k}{\otimes} y_k \right\rangle
\end{aligned}
$$

as desired. □

The map ϕ above is clearly a bimodule map, so central vectors get mapped into $N' \cap L^2(M_k)$. Since $N' \cap L^2(M_k) = N' \cap M_k$ is finite dimensional, it follows that each P_α is finite dimensional.

3.1 Action of $N-P-M$ tangles on P_α

We now describe how the $N-P-M$ planar operad acts on the various P_α. Given an $N-P-M$ tangle T, we isotope it so that it is in standard form. This means:

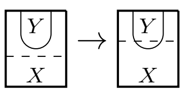
1. All of the input and output disks are rectangles and all strings emanate from the top of the rectangles.
2. All the input disks are in different horizontal bands and all maxima and minima of strings are at different vertical levels, and not in the horizontal bands defined by the input disks.
3. The starred intervals of the input disks are all at the bottom-left corner. When we have a diagram of this form, the $\star$ is omitted.

One then positions an imaginary horizontal line at the bottom of the tangle, T, and then slides it to the top. One starts with the central vector $1_Q \in L^2(Q)$ whenever the bottom of the box is shaded Q. The central vector gets altered as the line crosses either an input box, a maximum on a string, or a minimum on a string. When the line reaches the top, you get the central vector produced by the action of the tangle.

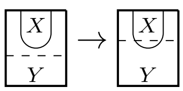
Suppose the horizontal line passes through the i^{th} rectangle (with respect to the isotopy) in the m_i^{th} region which is shaded Q_{m_i} (reading left to right along the line), and suppose that the vector v_i has been assigned to the box. We simply insert v_i into the m_i^{th} slot, i.e.

$$\begin{aligned}\sum_j x_1^j \otimes \cdots \otimes x_{m_i}^j \otimes \cdots \otimes x_n \mapsto & \sum_j x_1^j \otimes \cdots \otimes x_{m_i}^j v_i \otimes \cdots \otimes x_n \\ = & \sum_j x_1^j \otimes \cdots \otimes v_i x_{m_i}^j \otimes \cdots \otimes x_n\end{aligned}$$

Now suppose the horizontal line passes through a minimum, and suppose $Y \subset X$ with $X, Y \in \{N, P, M\}$ and X and Y resemble the regions on either side of the minimum. Let $B_{X,Y}$ be a Pimsner-Popa basis for X over Y. Then we have the diagrammatic rules:

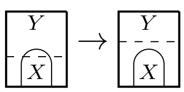


$$x \mapsto \frac{1}{[X:Y]^{1/2}} \sum_{b \in B_{X,Y}} xb \underset{Y}{\otimes} 1_Y \underset{Y}{\otimes} b^* = \sum_{b \in B_{X,Y}} b \underset{Y}{\otimes} 1 \underset{Y}{\otimes} b^* x$$

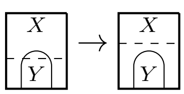


$$x \mapsto x \underset{Y}{\otimes} 1_X \underset{Y}{\otimes} 1_Y = 1_Y \underset{Y}{\otimes} 1_X \underset{Y}{\otimes} x$$

Whenever a dotted line passes over a maximum, the following rules apply:

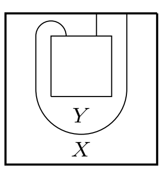


$$y_1 \underset{Y}{\otimes} x \underset{Y}{\otimes} y_2 \mapsto [X:Y]^{1/2} y_1 E_Y(x) y_2$$



$$x_1 \underset{Y}{\otimes} y \underset{Y}{\otimes} x_2 \mapsto x_1 y x_2$$

Here is an example of a tangle acting on $y_1 \underset{Y}{\otimes} x \underset{Y}{\otimes} y_2$:

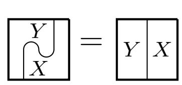


$$(y_1 \underset{Y}{\otimes} x \underset{Y}{\otimes} y_2) = \frac{1}{[X:Y]^{1/2}} \cdot \sum_{b \in B_{X,Y}} b y_1 x \underset{Y}{\otimes} y_2 \underset{Y}{\otimes} b^*$$

Note also that our rules dictate that a loop with an X on one side and Y on the other counts for a factor $[X:Y]^{1/2}$. As $[M:P]$ and $[P:N]$ are the only two such indices that will appear, we will let $\delta_a = [P:N]^{1/2}$ and $\delta_b = [M:P]^{1/2}$.

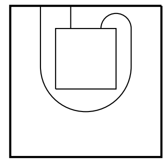
It is a straightforward check to see that each of these maps preserves central vectors. Each map is also locally a bimodule map, hence the action of T will also preserve invariant elements.

Checking that T is well defined up to isotopy involves checking the same (finite number of) relations as in [Jon]. For example, checking



boils down to checking the relation $x = \sum_{b \in B_{X,Y}} E_Y(xb)b^*$, which always holds. The key to checking that the action of T is defined up to isotopy is to show that rotation by 2π is the identity.

Let $x = \sum_j x_1^j \underset{Q_1 \cap Q_2}{\otimes} x_2^j \cdots \underset{Q_n \cap Q_1}{\otimes} x_{n+1}^j \in P_\alpha$, and let T be the tangle which is rotation by one-click clockwise, namely

$T =$  .

where there are $n-1$ strings that are not bent. By definition, we have:

$$Z(T)(x) = \begin{cases} \frac{1}{[Q_n:Q_1]^{1/2}} \sum_{b \in B(Q_n,Q_1)} \sum_j b \underset{Q_1}{\otimes} x_1 \underset{Q_1 \cap Q_2}{\otimes} x_2 \cdots \underset{Q_{n-1} \cap Q_n}{\otimes} x_n x_{n+1} b^* & \text{if } Q_1 \subset Q_n \\ [Q_1 : Q_n]^{1/2} \sum_j 1 \underset{Q_n}{\otimes} x_1 \underset{Q_1 \cap Q_2}{\otimes} x_2 \cdots \underset{Q_{n-1} \cap Q_n}{\otimes} x_n E_{Q_n}(x_{n+1}) & \text{if } Q_n \subset Q_1 \end{cases}$$

To help our computations, we define the following left and right creation operators, L_x and R_x for $x \in Q$. These are given by:

$$L_x : Z_\alpha \to L^2(Q) \underset{Q \cap Q_1}{\otimes} Z_\alpha \text{ such that } L_x(x_1 \otimes \cdots \otimes x_{n+1}) = x \otimes x_1 \otimes \cdots \otimes x_{n+1}$$
$$R_x : Z_\alpha \to Z_\alpha \underset{Q \cap Q_1}{\otimes} L^2(Q) \text{ such that } R_x(x_1 \otimes \cdots \otimes x_{n+1}) = x_1 \otimes \cdots \otimes x_{n+1} \otimes x$$

It follows from the definition of the bimodule tensor product that

$$L_x^*(x_0 \otimes x_1 \otimes \cdots \otimes x_{n+1}) = E_{Q \cap Q_1}(x^* x_0) x_1 \otimes \cdots \otimes x_{n+1} \text{ and}$$
$$R_x^*(\otimes x_1 \otimes \cdots \otimes x_{n+1} \otimes y) = x_1 \otimes \cdots \otimes x_n \otimes x_{n+1} E_{Q_1 \cap Q_0}(y x^*)$$

Therefore, we have the following formulae the rotation tangle, T:

$$Z(T)(x) = \begin{cases} \frac{1}{[Q_n:Q_1]^{1/2}} \sum_{b \in B} L_b R_b^*(x) & \text{if } Q_1 \subset Q_n \\ [Q_1 : Q_n]^{1/2} L_1 R_1^*(x) & \text{if } Q_n \subset Q_1 \end{cases}$$

From Burns' rotation trick [Bur03] we have the following lemma which is similar to lemmas that appear in [JP11]:

Lemma 3.3. *Let $\rho(\alpha)$ be the word formed when the words in α are cyclically permuted clockwise by one, and let $y = y_1 \otimes \cdots \otimes y_n \otimes y_{n+1} \in Z_{\rho(\alpha)}$. Then*

- $\langle T(x), y \rangle = \dfrac{1}{[Q_n : Q_1]^{1/2}} \langle x, y_2 \otimes \cdots \otimes y_{n+1} \otimes y_1 \rangle$ *if* $Q_1 \subset Q_n$
- $\langle T(x), y \rangle = [Q_1 : Q_n]^{1/2} \langle x, y_2 \otimes \cdots \otimes y_{n+1} \otimes y_1 \rangle$ *if* $Q_n \subset Q_1$

Proof. For the first case, using that x is central, we have

$$\begin{aligned}\langle T(x), y\rangle = \frac{1}{[Q_n : Q_1]^{1/2}}\langle \sum_{b\in B} L_b R_b^*(x), y\rangle &= \sum_{b\in B}\frac{1}{[Q_n : Q_1]^{1/2}}\langle x, R_b L_b^*(y)\rangle \\ &= \frac{1}{[Q_n : Q_1]^{1/2}}\sum_{b\in B}\langle x, E_{Q_1}(b^* y_1) y_2 \otimes y_3 \otimes \cdots \otimes y_{n+1}\otimes b\rangle \\ &= \frac{1}{[Q_n : Q_1]^{1/2}}\sum_{b\in B}\langle (E_{Q_1}(b^* y_1))^* x, y_2 \otimes y_3 \otimes \cdots \otimes y_{n+1}\otimes b\rangle \\ &= \frac{1}{[Q_n : Q_1]^{1/2}}\sum_{b\in B}\langle x(E_{Q_1}(b^* y_1))^*, y_2 \otimes y_3 \otimes \cdots \otimes y_{n+1}\otimes b\rangle \\ &= \frac{1}{[Q_n : Q_1]^{1/2}}\sum_{b\in B}\langle x, y_2 \otimes y_3 \otimes \cdots \otimes y_{n+1}\otimes bE_{Q_1}(b^* y_1)\rangle \\ &= \frac{1}{[Q_n : Q_1]^{1/2}}\langle x, y_2 \otimes \cdots \otimes y_{n+1}\otimes y_1\rangle.\end{aligned}$$

For the second case, we have

$$\begin{aligned}\langle T(x), y\rangle = [Q_1 : Q_n]^{1/2}\langle L_1 R_1^*(x), y\rangle &= [Q_1 : Q_n]^{1/2}\langle x, R_1 L_1^*(y)\rangle \\ &= [Q_1 : Q_n]^{1/2}\langle x, E_{Q_1}(y_1) y_2 \otimes \cdots \otimes y_{n+1}\otimes 1\rangle \\ &= [Q_1 : Q_n]^{1/2}\langle x, y_1 y_2 \otimes \cdots \otimes y_{n+1}\otimes 1\rangle \\ &= [Q_1 : Q_n]^{1/2}\langle y_1^* x, y_2 \otimes \cdots \otimes y_{n+1}\otimes 1\rangle \\ &= [Q_1 : Q_n]^{1/2}\langle x y_1^*, y_2 \otimes \cdots \otimes y_{n+1}\otimes 1\rangle \\ &= [Q_1 : Q_n]^{1/2}\langle x, y_2 \otimes \cdots \otimes y_{n+1}\otimes y_1\rangle\end{aligned}$$

as desired. □

Corollary 3.4. *Rotation by* 2π *is the identity.*

Proof. The nature of the shading dictates that all index factors cancel when applying the 2π rotation. The rest follows from the previous lemma. □

One can now continue as in [Jon].

3.2 Realizing $\mathcal{Q}$ inside $\mathcal{P}$

Suppose $\mathcal{Q}$ is a planar algebra containing the Fuss-Catalan algebra, so that $\mathcal{Q}$ is the planar algebra for a finite index inclusion $N \subset M$ with intermediate subfactor P. We note that $Q_{2n,+}$ is the space of $N-N$ central vectors of

$$({}_N L^2(M)_N)^{\otimes_N^n} = ({}_N L^2(M)_M \underset{M}{\otimes} {}_M L^2(M)_N)^{\otimes_N^n}.$$

and $Q_{2n,-}$ is the space of $M-M$ central vectors of $({}_M L^2(M)_N \underset{N}{\otimes} {}_N L^2(M)_M)^{\otimes_M^n}$. Since

$${}_N L^2(M)_M = {}_N L^2(P)_P \underset{P}{\otimes} {}_P L^2(M)_M,$$

it follows that

$$Q_{2n,+} = P_{(abba)^{n/2}} \text{ and } Q_{2n,-} = P_{(baab)^{n/2}}.$$

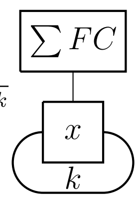
Furthermore, a tangle S that acts on $\mathcal{Q}$ can be made into an $N-P-M$ tangle S' by replacing each string with an a string cabled to a b string. This is done so that the shaded regions in S become the $M-$regions in S' and the unshaded regions in S become the $N-$regions in S'. Notice that this implies that the strings along any disk in S' read (clockwise from the marked region) as $abbaabbaa...$ or $baabbaabb...$. It directly follows from the definitions that if $x_1, ..., x_n$ are in $\mathcal{Q}$ and S is as above, then $Z(S)(f) = Z(S')(f)$ where the left hand side denotes the action of a shaded tangle and the right hand side denotes the action of an $N-P-M$ tangle.

Definition 3.5. If $\mathcal{Q}$ is the planar algebra of an inclusion $N \subset M$ of finite index II_1 factors with an intermediate subfactor, P, then we define the $\mathcal{P}$ constructed as above as the <u>$P-$augmentation of $\mathcal{Q}$</u>.

4 The GJS construction for the $\sum FC$ potential

Suppose $\mathcal{Q}$ is a subfactor planar algebra containing a copy of the Fuss Catalan planar algebra. For each $k \geq 0$, we study the graded algebra $\mathrm{Gr}_k^{\pm}(\mathcal{Q}) = \bigoplus_{n \geq k} \mathcal{Q}_n^{\pm}$ as above, and place the following trace on $\mathrm{Gr}_k^{\pm}(\mathcal{Q})$:

$$\mathrm{tr}(x) = \frac{1}{(\delta_a \delta_b)^k}$$



.

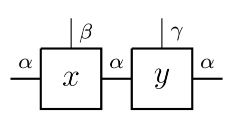
where the shading on the *upper left* corner is $\pm$.

As in Section 3, we realize $\mathcal{Q}$ inside an augmentation, $\mathcal{P}$, and we consider the algebras $\mathrm{Gr}_\alpha(\mathcal{P})$ where $\overline{\alpha}\alpha \in \Delta$. If $\alpha \neq \emptyset$ then the shading after the last letter of α is uniquely determined and hence we can write,

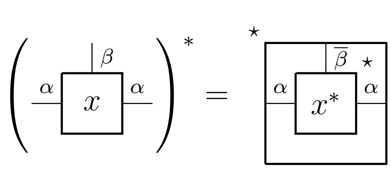
$$\mathrm{Gr}_\alpha(\mathcal{P}) = \oplus_{(\beta:\overline{\alpha}\beta\alpha\in\Delta)} \mathcal{P}_{\overline{\alpha}\beta\alpha}.$$

$\mathrm{Gr}_\alpha(\mathcal{P})$ is endowed with a multiplication $\wedge$ given by

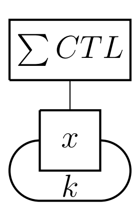
$x \wedge y =$



adjoint structure



and normalized trace



where $\sum CTL$ is the sum of all colored Temperley-Lieb diagrams. In the case where $\alpha = \emptyset$, then we have three such algebras, one for each shading N, P, and M. We therefore form graded algebras Gr_0^N, Gr_0^P and Gr_0^M where

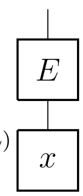
$$\mathrm{Gr}_0^Q = \bigoplus_{(\beta:\beta\in\Delta^Q)} \mathcal{P}_\beta.$$

Using the graded algebras associated to $\mathcal{P}$, we see that we have the following trace preserving inclusions:

$$\begin{aligned}
\mathrm{Gr}_{2k}^+(\mathcal{Q}) &\subset \mathrm{Gr}_{abba\cdots abba}(\mathcal{P})\\
\mathrm{Gr}_{2k+1}^+(\mathcal{Q}) &\subset \mathrm{Gr}_{abba\cdots baab}(\mathcal{P})\\
\mathrm{Gr}_{2k}^-(\mathcal{Q}) &\subset \mathrm{Gr}_{baab\cdots baab}(\mathcal{P})\\
\mathrm{Gr}_{2k+1}^-(\mathcal{Q}) &\subset \mathrm{Gr}_{baab\cdots abba}(\mathcal{P}).
\end{aligned}$$

One advantage to working in the $N-P-M$ planar algebra $\mathcal{P}$ is that the map $\Phi : \mathrm{Gr}_\alpha(\mathcal{P}) \to \mathrm{Gr}_\alpha(\mathcal{P})$ given by

$$\Phi(x) = \sum_{E\in\mathrm{Epi}(CTL)}$$



as in [JSW10, BHP12] is a well defined trace preserving isomorphism between $\mathrm{Gr}_\alpha(\mathcal{P})$ with the $\sum CTL$ trace and $\mathrm{Gr}_\alpha(\mathcal{P})$ with the orthogonalized trace. Therefore, we have proven the following lemma:

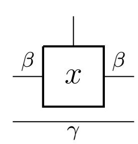
Lemma 4.1. *The potential $\sum CTL$ gives a positive definite trace on $\mathrm{Gr}_\alpha(\mathcal{P})$.*

Furthermore, by considering either the a or $b \cup$ element, the same analysis as in [JSW10, BHP12] proves the following theorem:

Theorem 4.2. *Left (and right) multiplication of elements of $Gr_\alpha(\mathcal{P})$ on $L^2(Gr_\alpha(\mathcal{P}))$ is bounded and the associated von Neumann algebra, $M_\alpha = Gr_\alpha(\mathcal{P})''$ is a II_1 factor.*

We note that if γ and β are words in a and b such that $\overline{\beta}\beta \in \Delta$ and $\overline{\beta\gamma}\beta\gamma \in \Delta$ then we have a unital inclusion of M_γ into $M_{\beta\gamma}$ given the extension of

$$x \mapsto$$



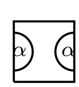
We therefore have the following theorem, whose proof is exactly the same as the arguments in Section 4 of [JSW10].

Theorem 4.3. *The following is a Jones' tower of II_1 factors:*

$$M_0^Q \subset M_\alpha \subset M_{\alpha\overline{\alpha}} \subset \cdots \subset M_{(\alpha\overline{\alpha})^n} \subset M_{\overline{\alpha}(\alpha\overline{\alpha})^n}.$$

Furthermore, $[M_\alpha : M_0^Q] = \delta_\alpha$, and the Jones projection for $M_0^Q \subset M_\alpha$ is

$$e_0 = \frac{1}{\delta_\alpha}$$



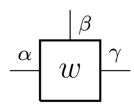
5 A semifinite algebra associated to $\mathcal{P}$

As in [GJS11], we will realize the isomorphism class of all of the M_α by examining realizing them as “corners" of a semifinite algebra. To begin, we consider the set semifinite algebra Gr_∞^Q for $Q = N,\ P,\ M$. As a vector space,

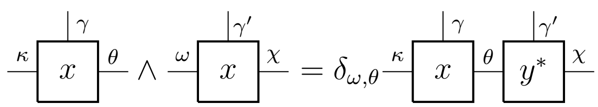
$$\mathrm{Gr}_\infty^Q = \bigoplus_{\substack{\alpha,\gamma,\beta \\ \overline{\alpha}\gamma\beta\in\Delta,\,\alpha\in\Delta^Q}} \mathcal{P}_{\overline{\alpha}\gamma\beta}.$$

Assumption 5.1. Throughout the rest of this article, the marked regions will now be on the **bottom** of the boxes.

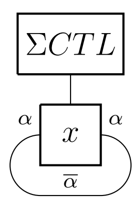
Pictorially, we realize elements in Gr_∞^Q as linear combinations of boxes of the form



where the *bottom* (starred) region is shaded Q (thus the top left corner varies in shading). Gr_∞^Q comes endowed with the following multiplication:

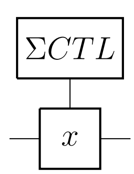


and semifinite trace, Tr, which is given by:

$\mathrm{Tr}(x) =$ 

if $x \in \mathrm{Gr}(\mathcal{P}_\alpha)$ and is zero otherwise. Just as in the analysis of [GJS11, BHP12], we see that Gr_∞^Q completes to a II_∞ factor, $\mathcal{M}_\infty^Q$ when being represented on $L^2(\mathrm{Gr}_\infty^Q)$.

Also of importance will be the von Neumann subalgebra $\mathcal{A}_\infty^Q \subset \mathcal{M}_\infty^Q$ which is generated by all boxes in $\mathcal{G}_\infty^Q$ with no strings on top. Notice that there is a normal, faithful, Tr-preserving conditional expectation $E : \mathcal{M}_\infty^Q \to \mathcal{A}_\infty^Q$ given by

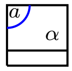
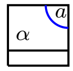
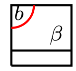
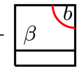
$E(x) =$  .

Furthermore, we have the following lemma, whose proof is identical to that of Lemma 3 of [GJS11]:

Lemma 5.2. $\mathcal{A}_\infty^Q = \bigoplus_{v\in\Gamma_Q} \mathcal{A}_v$ *where* Γ_Q *is the* $Q-$*principal graph of* $\mathcal{P}$ *and each* $\mathcal{A}_v$ *is a type* I_∞ *factor.*

We now aim to figure out the isomorphism class of the algebras $\mathcal{M}_\alpha$. For the remainder of the section, we will assume for simplicity that $Q = N$ and hence we will be finding the isomorphism class of M_α

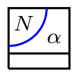
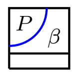
such that $s(\overline{\alpha}) = N$. The other two cases will follow a similar analysis. We first define elements X_a and X_b as follows:

$X_a = \sum_{\substack{\alpha \\ \alpha,\overline{\alpha}\in\Delta^N}}$  $+$  and $X_b = \sum_{\substack{\beta \\ \beta\in\Delta^P\,\overline{\beta}\in\Delta^N \\ s(\overline{\beta})=a}}$  $+$  .

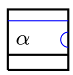
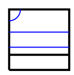
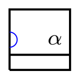
Note that X_a and X_b are sums of orthogonally supported operators, and each summand has uniformly bounded operator norm. Therefore, $X_a, X_b \in \mathcal{M}_\infty^N$. Furthermore, we have the following lemma:

Lemma 5.3. *$\mathcal{M}_\infty^N$ is generated as a von Neumann algebra by $(\mathcal{A}_\infty^N, X_a, X_b)$.*

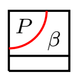
Proof. As in the proof of Lemma 7 of [GJS11], all that needs to be shown is that the following diagrams lie in the von Neumann algebra generated by $(\mathcal{A}_\infty^N, X_a, X_b)$:

 and 

If $s(\alpha) = a$, then the exact same method as in the proof of Lemma 4.9 in [BHP12] shows that this element is in the algebra. if $s(\alpha) = b$, the following multiplication produces the diagram:

$\frac{1}{\delta_a} \cdot$  $\cdot$  $\cdot$ .

A similar argument works for the element

.

∎

As in [BHP12], we also have the following lemma:

Lemma 5.4. *X_a and X_b are free with amalgamation over $\mathcal{A}_\infty^N$ with respect to the conditional expectation, E.*

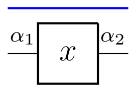
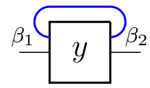
We now define maps η_a and η_b on $\mathcal{A}_\infty^N$ as follows:

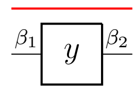
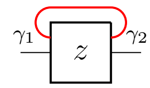
$$\eta_c(y) = E(X_c y X_c)$$

for $c = a$ or b. Notice that by definition of E, η_c is a completely positive map of $\mathcal{A}_I^N$ into itself. Furthermore it is a straightforward inductive check to note that the formula

$$E(y_0 X_c y_1 X_c \cdots y_{n-1} X_c y_n) \\ = \sum_{k=2}^{n} y_0 \cdot \eta_c(E(y_1 X_c \cdots X_c y_{k-1})) \cdot E(y_k X_c \ldots X_c y_n),$$

as in [GJS11] holds. Pictorially, there are nice expressions for $\eta_c(w)$ for various choices of w. To begin, for $i = 1, 2$, let α_i be a word with $\alpha_i \in \Delta^N$, β_i be a word with $s(\beta_i) = a$ and $\beta_i \in \Delta^P$, and γ_i be a word such that $\gamma_i \in \Delta^M$. Furthermore, suppose $x \in \mathcal{P}_{\overline{\alpha_1}\alpha_2}$, $y \in \mathcal{P}_{\overline{\beta_1}\beta_2}$ and $z \in \mathcal{P}_{\overline{\gamma_1}\gamma_2}$. We then have the following easily verifiable formulae:

$\eta_a(x) =$  $\eta_a(y) =$  $\eta_a(z) = 0$

$\eta_b(x) = 0$ $\eta_b(y) =$  $\eta_b(z) =$ 

With the pictures above, the following useful lemma is easily verified:

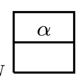
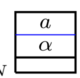
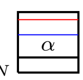
Lemma 5.5. *Let α_i be a word with $s(\alpha_i) = N$ and β_i be a word with $s(\beta_i) = P$ for $i = 1$ or 2. In addition, suppose $x \in \mathcal{P}_{\overline{\alpha_1}\alpha_2}$ and $y \in \mathcal{P}_{\overline{\beta_1}\beta_2}$. We have the following formulae:*

$$x \cdot X_a = X_a \cdot \eta_a(x) \text{ and } y \cdot X_b = X_b \cdot \eta_b(y).$$

This lemma will be used to help describe certain compressions of $\mathcal{M}_\infty^N$.

5.1 A suitable compression of $\mathcal{M}_\infty^N$

To begin, it will be useful to define three projections in $\mathcal{A}_\infty^N$

$1_{\mathcal{A}_N^N} = \sum_{\alpha \in \Delta^N}$  $1_{\mathcal{A}_{P_a}^N} = \sum_{\alpha \in \Delta^N}$  $1_{\mathcal{A}_{M_a}^N} = \sum_{\alpha \in \Delta^N}$  .

Note that $1_{\mathcal{A}_N^N} + 1_{\mathcal{A}_{P_a}^N} + 1_{\mathcal{A}_{M_a}^N}$ is the smallest projection dominating the support projections of X_a and X_b.

Our goal is to better understand what happens when $\mathcal{M}_\infty^N$ is compressed by certain projections. To begin our study, we consider Γ_N, the $N-$principal graph of $\mathcal{P}$. For each vertex, v, at the $N-N$ level of the graph, we choose a minimal projection $p_v \in \mathcal{A}_\infty$, and for the vertex, $*$, we choose the empty $N-$shaded diagram. Notice that for each v we can choose $p_v \in \mathcal{P}_{\overline{\alpha}\alpha}$ for $\alpha \in \Delta^N$.

By the definition of the principal graph, we know that there exists a countable index set, I and partial isometries $(V_i)_{i\in I} \subset \mathcal{A}_\infty^N$ such that

$$V_iV_i^* = \sum_{v \in \Gamma_N^N} p_v \; \forall i \text{ and } \sum_{i \in I} V_i^*V_i = 1_{\mathcal{A}_N^N}.$$

This necessarily implies that

$$\sum_{i\in I} \eta_a(V_i)^*\eta_a(V_i) = 1_{\mathcal{A}_{P_a}^N} \text{ and } \sum_{i \in I} \eta_b(\eta_a(V_i))^*\eta_b(\eta_a(V_i)) = 1_{\mathcal{A}_{M_a}^N}$$

We define R^1 by the following formula:

$$R^1 = \sum_{v\in\Gamma_N^N} (p_v + \eta_a(p_v) + \eta_b(\eta_a(p_v))).$$

If we set $Z_i = V_i + \eta_a(V_i) + \eta_b(\eta_a(V_i))$ then

$$Z_iZ_i^* = R^1 \text{ and } \sum_{i\in I} Z_i^*Z_i = 1_{\mathcal{A}_N^N} + 1_{\mathcal{A}_{P_a}^N} + 1_{\mathcal{A}_{M_a}^N}.$$

We have the following lemma regarding compression of $\mathcal{M}_\infty^N$ by R^1.

Lemma 5.6. *As a von Neumann algebra, $R^1\mathcal{M}_\infty^N R^1$ is generated by $R^1\mathcal{A}_\infty^N R^1$, $R^1X_aR^1$, and $R^1X_bR^1$.*

Proof. Note that$\sum_{i,j\in I} Z_i^* X_c Z_j = X_c$ for $c = a$ or b by repeated applications of Lemma 5.5. Therefore, every word involving X_a or X_b and elements $x \in \mathcal{A}_\infty$ (whose ending letters are supported under R^1) can be replaced by sums of words involving terms of the form $R^1X_cR^1$ and R^1xR^1 by inserting the relation

$$\sum_{i\in I} Z_i^* Z_i = 1_{\mathcal{A}_N^N} + 1_{\mathcal{A}_{P_a}^N} + 1_{\mathcal{A}_{M_a}^N}.$$

between every letter of the word. □

We now investigate the action of compressing $R^1\mathcal{M}_\infty^N R^1$ by subprojections of R^1. To begin, for each vertex $w \in \Gamma_N^P$, let p_w be a minimal projection in $\mathcal{A}_\infty^N$ corresponding to the vertex w such that $p_w \leq \sum_{v\in V} \eta_a(p_v)$. For each edge e connecting a vertex in Γ_N^N to a vertex in Γ_N^P, we let $s(e)$ and $t(e)$ be the vertices in Γ_N^N and Γ_N^P respectively which e connects. We define partial isometries $\omega_e \in R^1\mathcal{A}_\infty^N R^1$ such that

$$\omega_e^*\omega_{e'} = \delta_{e,e'} p_{t(e)} \text{ and } \sum_{s(e)=v} \omega_e\omega_e^* = \eta_a(p_v).$$

Once the p_w have been chosen, for each vertex $u \in \Gamma_N^M$, choose a minimal projection p_u corresponding to u such that $p_u \leq \sum_{w\in\Gamma_N^P} \eta_b(p_w)$.

For each edge, f, connecting the $N-P$ vertices to the $N-M$ vertices, $s(f)$ and $t(f)$ be the vertices in Γ_N^P and Γ_N^M respectively which f connects. We define partial isometries ν_f satisfying:

$$\nu_f^*\nu_{f'} = \delta_{f,f'} p_{t(f)} \text{ and } \sum_{s(f)=w} \nu_f\nu_f^* = \eta_a(p_w)$$

We now define operators X_a^e X_b^f by the formulae

$$X_a^e = p_{s(e)}X_a\omega_e + \omega_e^*X_ap_{s(e)} \text{ and } X_b^f = p_{s(f)}X_b\nu_f + \nu_e^*X_b.$$

We have the following lemma, whose proof is the same as the arguments of Section 4.2 of [BHP12], except easier as there are no loops on the principal graph and only one minimal projection for each vertex.

Lemma 5.7. *Set $R = \sum_{v\in\Gamma} p_v$. Then $R\mathcal{M}_\infty^N R$ is generated by $R\mathcal{A}_\infty^N R$ and the elements X_a^e X_b^f for all e and f, and each of the elements are free with amalgamation over $R\mathcal{A}_\infty^N R$ with respect to E.*

Note that the algebra $R\mathcal{A}_\infty^N R$ is simply the bounded functions on the vertices of Γ, and the element $p_{s(e)}X_a\omega_e$ has left support under $p_{s(e)}$ and right support under $p_{t(e)}$. Furthermore, if $\mathrm{Tr}(p_{s(e)}) \geq \mathrm{Tr}(p_{t(e)})$ then the analysis in [GJS11] shows that $(p_{s(e)}X_a\omega_e)^*p_{s(e)}X_a\omega_e$ is a free poisson element with absolutely continuous spectrum in $p_{t(e)}\mathcal{M}_\infty^N p_{t(e)}$. If $\mathrm{Tr}(p_{s(e)}) \leq \mathrm{Tr}(p_{t(e)})$, then $p_{s(e)}X_a\omega_e(p_{s(e)}X_a\omega_e)^*$ is a free poisson element with absolutely continuous spectrum in $p_{s(e)}\mathcal{M}_N^\infty p_{s(e)}$. Analogous statements hold for the elements $p_{s(f)}X_b\nu_f$.

5.2 An amalgamated free product representation for $R\mathcal{M}_N^\infty R$

The work of the previous section shows that

$$R\mathcal{M}_\infty^N R = \mathcal{M}(\Gamma_N, \mu)$$

with $\mathcal{M}(\Gamma, \mu)$ the free graph algebra as in [Har13]. We use this to obtain a formula for M_0 when Γ_N is finite. Let e be an edge in Γ_N connecting v and w with $\mathrm{Tr}(p_v) \geq \mathrm{Tr}(p_w)$. The basic rules for computing free dimension [Dyk93, DR11] show that

$$\mathrm{fdim}(R\mathcal{M}_e R) = 1 - \frac{(\mathrm{Tr}(p_v) - \mathrm{Tr}(p_w))^2 - \sum_{u\neq v,w} \mathrm{Tr}(p_u)^2}{\mathrm{Tr}(R)^2} = 1 - \frac{\sum_{u\in\Gamma} \mathrm{Tr}(p_v)^2 - 2\,\mathrm{Tr}(p_v)\,\mathrm{Tr}(p_w)}{\mathrm{Tr}(R)^2}.$$

Using the additivity of free dimension, as well as

$$\mathrm{fdim}(\ell^\infty(\Gamma)) = 1 - \frac{\sum_{u\in\Gamma} \mathrm{Tr}(p_v)^2}{\mathrm{Tr}(R)^2},$$

we obtain

$$\begin{aligned}\mathrm{fdim}(R\mathcal{M}_\infty^N R) &= 1 + \frac{-\sum_{u\in\Gamma} \mathrm{Tr}(p_v)^2 + 2\sum_{g\in E(\Gamma_N)} \mathrm{Tr}(p_{s(g)})\,\mathrm{Tr}(p_{t(g)})}{\mathrm{Tr}(R)^2} \\ &= 1 + \frac{\sum_{u\in\gamma} \mathrm{Tr}(p_u) \sum_{v\sim u}(\mathrm{Tr}(p_v) - \mathrm{Tr}(p_u))}{\mathrm{Tr}(R)^2}.\end{aligned}$$

Using the Perron-Frobenius condition, this becomes

$$\mathrm{fdim}(R\mathcal{M}_\infty^N R) = 1 + \frac{2I((\delta_a - 1) + (\delta_b - 1))}{\mathrm{Tr}(R)^2}$$

Where $I = \sum_{v\in\Gamma_N^N} \mathrm{tr}(p_v)^2 (= \sum_{w\in\Gamma_N^P} \mathrm{tr}(p_w)^2 = \sum_{u\in\Gamma_N^M} \mathrm{tr}(p_u)^2)$. Therefore, $R\mathcal{M}_\infty^N R$ is an interpolated free group factor with the above parameter. The compression formula for free group factors proves the following lemma

Lemma 5.8. *$M_0^N \cong L(\mathbb{F}_t)$ where $t = 1 + 2I(\delta_a + \delta_b - 2)$.*

This gives us the following corollary:

Corollary 5.9. *The factors M_α have the formula*

$$M_\alpha \cong L(\mathbb{F}(1 + 2I\delta_\alpha^{-2}(\delta_a + \delta_b - 2)))$$

Proof. If $s(\overline{\alpha}) = N$, then it follows from the semifinte algebra $\mathcal{M}_\infty^N$ that M_α is a δ_α amplification of M_0^N. If the shading is different, apply similar analysis to the semifinite algebras $\mathcal{M}_\infty^P$ and $\mathcal{M}_\infty^M$. □

We now handle the case where $\mathcal{P}$ is infinite depth:

Lemma 5.10. *If $\mathcal{P}$ is infinite-depth, then $\mathcal{M}_0^N \cong L(\mathbb{F}_\infty)$, and hence $\mathcal{M}_\alpha \cong L(\mathbb{F}_\infty)$ for all α*

Proof. Let Γ_k be the graph of Γ_N truncated up to depth k as in the proof of Theorem 4.2 in [Har13]. As in that proof, we let

$$B(\Gamma_k) = \{v \in \Gamma_k; \operatorname{Tr}(p_v) > \sum_{w \sim v \in \Gamma_k} n_{v,w} \operatorname{Tr}(p_w)\}$$

where $n_{v,w}$ is the number of edges that connect v and w. We note that by the Perron Frobenius condition, no vertices in Γ_{k-1} are in $B(\Gamma_k)$. Following the proof of Theorem 4.2 in [Har13] step-by-step, we arrive at the formula

$$\operatorname{fdim}(M_0^N) \geq 1 + (\delta_a - 1) \sum_{\substack{v \in \Gamma_{k-2} \text{ and} \\ v \in \Gamma_N^N \cup \Gamma_N^P}} \operatorname{Tr}(p_v)^2 + (\delta_b - 1) \sum \sum_{\substack{v \in \Gamma_{k-2} \text{ and} \\ v \in \Gamma_N^P \cup \Gamma_N^M}} \operatorname{Tr}(p_v)^2$$

which gets arbitrarily large as k does. The standard embedding arguments of [Har13] show that $M_0^N \cong L(\mathbb{F}_\infty)$. We arrive at the result for the other M_α's either by amplification or by examining $\mathcal{M}(\Gamma_P)$ or $\mathcal{M}(\Gamma_M)$. □

We can use this result to give a complete diagrammatic reproof of the universality result of Popa and Shlyakhtenko regarding the universality of $L(\mathbb{F}_\infty)$ in subfactor theory.

Corollary 5.11 ([PS03]). *Every subfactor planar algebra $\mathcal{P}'$ is the standard invariant for a finite-index inclusion $\mathcal{N} \subset \mathcal{M}$ with $\mathcal{N} \cong L(\mathbb{F}_\infty) \cong \mathcal{M}$.*

Proof. A diagrammatic proof of this fact for $\mathcal{P}'$ infinite depth was done in [Har13]. If $\mathcal{P}'$ is finite depth, let $\mathcal{P}'$ be the planar algebra for a finite index inclusion $N' \subset P'$ of II_1 factors. Let $B \subset C$ be a finite index inclusion of II_1 factors with principal graph A_∞. We consider the inclusions

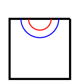
$$N' \otimes B \subset P' \otimes B \subset P' \otimes C.$$

Let $\mathcal{Q}$ be the planar algebra for $N \subset M$, and $\mathcal{P}$ the augmentation of $\mathcal{Q}$ for $N \subset P \subset M$. We note that $\mathcal{P}'$ is also the planar algebra for $N \subset P$, and the planar subalgebra of $\mathcal{P}$ generated by words whose only color is a is $\mathcal{P}'$. From Theorem 4.3, it follows that the standard invariant of $M_\emptyset^N \subset M_a$ is $\mathcal{P}'$ and from the above calculation, $M_\emptyset^N \cong L(\mathbb{F}_\infty) \cong M_a$ □

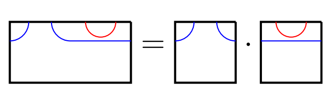
Note that Rădulescu was the first to provide a construction of $\mathcal{N} \subset \mathcal{M}$ both isomorphic to $L(\mathbb{F}_\infty)$ having standard invariant $\mathcal{P}'$ for $\mathcal{P}'$ finite depth [Răd94].

5.3 The law of $\cup \in N_0^+$

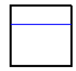
One pleasing feature of the semifinite algebra constructed above is that it gives one a transparent way to find equations of the spectrum of the element $\cup \in N_0$. This corresponds to the double-cup element

 $\in M_\emptyset^N$

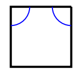
Picturing $\cup$ as living in $\mathcal{M}_I^N$, we note that $\operatorname{Tr}(\cup^n) = \operatorname{Tr}(x^n)$ where x is the element

$x =$ 

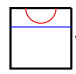
which is supported by

$$1_a =$$ 

Note that x is expressed as a product of free elements $1_a \mathcal{M}_\infty^N 1_a$. Let y be the element

$$y =$$ 

and r be the element

$$r =$$ .

We know that $\cup_a$ is distributed as a free-poisson element, and its moment generating function is

$$M_{\cup_a}(z) = \frac{-((\delta_a - 1)z - 1) + \sqrt{((\delta_a - 1)z - 1)^2 - 4z}}{2z}.$$

In the algebra $1_a \mathcal{M}_\infty^N 1_a$ with *normalized* trace, τ, we must have $\tau(y^n) = \mathrm{Tr}(\cup^n)/\delta_a$ for $n \geq 1$ and $\tau(y^0) = 1$. Therefore, the moment generating function of y is

$$M_y(z) = M_{\cup_a}(z)\delta_a + \frac{\delta_a - 1}{\delta_a}.$$

The tool we will use to calculate the moments of x is Voiculescu's $S-$transform [VDN92]. From [VDN92], it is known that is S_s and S_t are the $S-$ transforms for free elements s and t in a tracial von Neumann algebra, then

$$S_{st}(z) = S_{s^{1/2}ts^{1/2}}(z) = S_s(z)S_t(z).$$

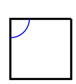
Furthermore, to compute the $S-$transform of an element s, one finds formal power series ψ_s, χ_s and S_s satisfying:

$$\psi_s(z) = M_s(z) - 1 \quad \chi_s(\psi_s(z)) = z = \psi_s(\chi_s(z)) \text{ and } S_s(z) = \frac{(z+1)\chi_s(z)}{z}$$

These formulas produce the following expression for the $S-$transform of x:

$$S_x(z) = \frac{(z+1)^2(z-1)(\delta_a z - 1)}{((\delta_b - 1)z + \delta_b)((\delta_a - 1)z + 1)},$$

which can be inverted to give the Cauchy transform of x. We know that the law of a single-colored $\cup$ (as in [GJS10]) is absolutely continuous with respect to Lebesgue measure and is supported away from the origin. Therefore, the law of y contains an atom of at the origin of measure $\frac{\delta_a - 1}{\delta_a}$ at the origin and is absolutely continuous away from the origin. It follows that the law of $y^{1/2}ry^{1/2}$ has an atom of measure $\frac{\delta_a - 1}{\delta_a}$ at the origin and is absolutely continuous away from the origin. Furthermore, the spectral projection corresponding to $\{0\}$ for $y^{1/2}ry^{1/2}$ must be the same as the spectral projection corresponding to $\{0\}$ for y. From this, we use the polar part of



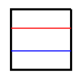
to conclude that $\cup \in \mathcal{N}_0^+$ has law absolutely continuous to Lebesgue measure and supported away from the origin.

5.4 N_k contains a free group factor

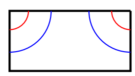
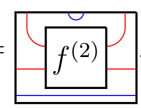
Recall that $N_k^{\pm} = \mathrm{Gr}_k^{\pm}(\mathcal{Q})''$. We will use the moment calculation of $\cup$ above as well as similar elements arising in the semifinite algebra to find a free group factor contained in $N_k^{\pm}$. To simplify matters, we note that we need only consider the case where $\mathcal{Q}$ is Fuss Catalan, as any such planar algebra will contain the Fuss Catalan planar algebra.

By the usual amplification tricks, we need need only show that N_0^+ contains a free group factor. We embed $\mathcal{Q}$ into its augmentation $\mathcal{P}$ as in Example 2.5, which produces an embedding $\mathrm{Gr}_\infty^+(\mathcal{Q}) \hookrightarrow \mathrm{Gr}_\infty^N(\mathcal{P})$ where $\mathrm{Gr}_\infty^+(\mathcal{Q})$ is realized as the subalgebra of $\mathrm{Gr}_\infty^N(\mathcal{P})$ generated by the Fuss Catalan diagrams. Let $\mathcal{N}_\infty = \mathrm{Gr}_\infty^+(\mathcal{Q})$. We have the following lemma:

Lemma 5.12. *Let p_{ab} be the following diagram*

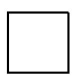
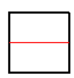
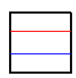
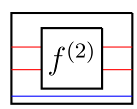
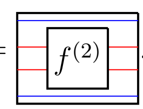
$p_{ab} =$ 

and set

$x =$  *and* $y =$ .

with $f^{(2)}$ the second Jones-Wenzl idempotent in Temperely Lieb. Then x and y are free in $p_{ab}\mathcal{M}_\infty^N p_{ab}$.

Proof. From [BJ97], the projections

$p_0 =$  $p_1 =$  $p_2 =$  $p_3 =$  and $p_4 =$ .

are inequivalent minimal projections in $\mathcal{P}$, and there exists exactly one edge e_i which goes between the vertices representing p_{i-1} and p_i. Therefore, by choosing p_0, p_2, and p_4, to line up with our choices of minimal projections lying under Q, It follows that

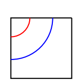
$$x = p_2 X_b^{e_2} X_a^{e_1} X_a^{e_1} X_b^{e_2} p_2 \text{ and } y = p_2 X_b^{e_3} X_a^{e_4} X_a^{e_4} X_b^{e_3} p_2$$

so x and y are free with amalgamation over $\mathcal{A}_\infty^N$. Since $p_{ab} = p_2$ is minimal in $\mathcal{A}_\infty^N$, the result follows. □

Proof of Theorem C. Clearly, $N_0^+ \subset M_\emptyset^N$ so N_0 is contained in an interpolated free group factor. Conversely, we know that $p_{ab}\mathcal{N}_\infty^N p_{ab}$ contains a copy of $W^*(x) * W^*(y)$. Since the law of $\cup \in \mathcal{N}_k$ has no atoms, it follows that

$$W^*(x) = \underset{1}{\overset{q}{L(\mathbb{Z})}} \oplus \underset{\delta_a\delta_b - 1}{\mathbb{C}}$$

where q is equivalent to p_0 via the polar part of

.

It follows from [Dyk93] that

$$q(W^*(x) * W^*(y))q = L(\mathbb{Z}) * q\left(\left(\underset{1}{\overset{q}{\mathbb{C}}} \oplus \underset{\delta_a\delta_b-1}{\mathbb{C}}\right) * W^*(y)\right) q$$

which is an interpolated free group factor. By the equivalence of q and p_0 in $\mathcal{N}_\infty^N$, and the identity $N_0^+ = p_0\mathcal{N}_\infty^N p_0$, it follows that N_0^+ contains an interpolated free group factor. □

Unfortunately, at this point, the author is unable to determine the isomorphism class of the $N_k^{\pm}$. While it is straightforward to show that a suitable compression of the algebra $\mathcal{N}_\infty^N$ is generated by products of the form $X_a^e X_b^f$, terms of the form

$$X_a^{e_1} X_b^f \text{ and } X_a^{e_2} X_b^f$$

appear with $e_1 \neq e_2$. The very nature of $\mathcal{N}_\infty^N$ makes it difficult to "decouple" this into a free family which still lies in $\mathcal{N}_\infty^N$.

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