THE LINEAR BOUND FOR THE NATURAL WEIGHTED RESOLUTION OF THE HAAR SHIFT

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Abstract. The Hilbert transform has a linear bound in the A_2 characteristic on weighted L^2,

$$\|H\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2},$$

and we extend this linear bound to the nine constituent operators in the natural weighted resolution of the conjugation $M_{w^{\frac{1}{2}}}\mathcal{S}M_{w^{-\frac{1}{2}}}$ induced by the canonical decomposition of a multiplier into paraproducts:

$$M_f = P_f^- + P_f^0 + P_f^+.$$

The main tools used are composition of paraproducts, a "product formula" for Haar coefficients, the Carleson Embedding Theorem, and the linear bound for the square function.

Contents

1. Introduction 83
2. Notation and Preliminary Estimates 85
3. Proof of Theorem 1.2 89
3.1. Estimating Easy Terms 89
3.2. Estimating Hard Terms 91
3.3. An Alternate Approach to Estimating Term (3.10) 99
References 104

1. Introduction

Let $L^2 \equiv L^2(\mathbb{R})$ denote the space of square integrable functions over $\mathbb{R}$. For a weight w, i.e., a positive locally integrable function on $\mathbb{R}$, we set $L^2(w) \equiv L^2(\mathbb{R}; w)$. In particular, we will be interested in A_2 weights, which are defined by finiteness of their A_2 characteristic,

$$[w]_{A_2} \equiv \sup_I \langle w\rangle_I \left\langle w^{-1}\right\rangle_I,$$

where $\langle w\rangle_I$ denotes the average of w over the interval I.

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An operator T is bounded on $L^2(w)$ if and only if $M_{w^{\frac{1}{2}}}TM_{w^{-\frac{1}{2}}}$ - the conjugation of T by the multiplication operator $M_{w^{\pm\frac{1}{2}}}$ - is bounded on L^2. Moreover, the operator norms are equal:

$$\|T\|_{L^2(w)\to L^2(w)} = \left\|M_{w^{\frac{1}{2}}}TM_{w^{-\frac{1}{2}}}\right\|_{L^2\to L^2}.$$

In the case that T is a dyadic operator adapted to a dyadic grid $\mathcal{D}$, it is natural to study weighted norm properties of T by decomposing the multiplication operators $M_{w^{\pm\frac{1}{2}}}$ into their canonical paraproduct decomposition relative to the grid $\mathcal{D}$ and Haar basis $\left\{h_I^0\right\}_{I\in\mathcal{D}}$, i.e.

$$\begin{aligned}
M_{w^{\pm\frac{1}{2}}}f &= \mathsf{P}^{(0,1)}_{\widehat{w^{\pm\frac{1}{2}}}} + \mathsf{P}^{(1,0)}_{\widehat{w^{\pm\frac{1}{2}}}}f + \mathsf{P}^{(0,0)}_{\langle w^{\pm\frac{1}{2}}\rangle}f \\
&\equiv \sum_{I\in\mathcal{D}} \left\langle w^{\pm\frac{1}{2}}, h_I^0\right\rangle_{L^2} \left\langle f, h_I^1\right\rangle_{L^2} h_I^0 \\
&\quad + \sum_{I\in\mathcal{D}} \left\langle w^{\pm\frac{1}{2}}, h_I^0\right\rangle_{L^2} \left\langle f, h_I^0\right\rangle_{L^2} h_I^1 \\
&\quad + \sum_{I\in\mathcal{D}} \left\langle w^{\pm\frac{1}{2}}, h_I^1\right\rangle_{L^2} \left\langle f, h_I^0\right\rangle_{L^2} h_I^0,
\end{aligned}$$

(above h_I^1 is the averaging function) and then decomposing $M_{w^{\frac{1}{2}}}TM_{w^{-\frac{1}{2}}}$ into the nine canonical individual paraproduct composition operators:

$$\begin{aligned}
M_{w^{\frac{1}{2}}}TM_{w^{-\frac{1}{2}}} &= \left(\mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} + \mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}} + \mathsf{P}^{(0,0)}_{\langle w^{\frac{1}{2}}\rangle}\right) T \left(\mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}} + \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} + \mathsf{P}^{(0,0)}_{\langle w^{-\frac{1}{2}}\rangle}\right) \\
&\equiv Q_{T,w}^{(0,1),(0,1)} + Q_{T,w}^{(0,1),(1,0)} + Q_{T,w}^{(0,1),(0,0)} \\
&\quad + Q_{T,w}^{(1,0),(0,1)} + Q_{T,w}^{(1,0),(1,0)} + Q_{T,w}^{(1,0),(0,0)} \\
&\quad + Q_{T,w}^{(0,0),(0,1)} + Q_{T,w}^{(0,0),(1,0)} + Q_{T,w}^{(0,0),(0,0)},
\end{aligned} \tag{1.1}$$

where $Q_{T,w}^{(\varepsilon_1,\varepsilon_2),(\varepsilon_3,\varepsilon_4)} \equiv \mathsf{P}^{(\varepsilon_1,\varepsilon_2)}_{\widehat{w^{\frac{1}{2}}}} T \mathsf{P}^{(\varepsilon_3,\varepsilon_4)}_{\widehat{w^{-\frac{1}{2}}}}$. We refer to this decomposition of the conjugation $M_{w^{\frac{1}{2}}}TM_{w^{-\frac{1}{2}}}$ into nine operators as the natural weighted resolution of the operator T.

The operators we will be interested in have the property that $\|T\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}$. And we will be interested in demonstrating that the operator norms of the $Q_{T,w}^{(\varepsilon_1,\varepsilon_2),(\varepsilon_3,\varepsilon_4)}$ are linear in the A_2 characteristic:

$$\left\|Q_{T,w}^{(\varepsilon_1,\varepsilon_2),(\varepsilon_3,\varepsilon_4)}\right\|_{L^2\to L^2} \lesssim [w]_{A_2}.$$

We wish to apply this general idea to the Haar shift operator. This is the following dyadic model operator

$$M_{w^{\frac{1}{2}}}\mathcal{S}M_{w^{-\frac{1}{2}}} : L^2 \to L^2$$

where $\mathcal{S}$ is a shift operator defined on the Haar basis by $h_I \mapsto h_{I_-} - h_{I_+}$. Because of linearity, it suffices to consider just "half" of the shift operator $\mathcal{S}$ defined on the Haar basis by the operator $\mathcal{S}h_I \equiv h_{I_-}$. Petermichl proved the following result:

Theorem 1.1 (Petermichl, [6]). *Let $w \in A_2$. Then*

$$\|\mathcal{S}\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}\,.$$

This result can be used with an averaging technique, [7], to show that the same estimate persists for the Hilbert transform on $L^2(w)$. The idea of decomposing a general Calderón–Zygmund operator into shifts was further refined by Hytönen in [1, 2] (see also [3, 9, 11]) and was used in his solution of the A_2-Conjecture for all Calderón–Zygmund operators. For the simplest proof of the A_2-Conjecture the interested reader should consult [4].

In this paper we will implement the strategy outlined above to extend this result to the natural weighted resolution of the Haar shift transform and obtain the following result.

Theorem 1.2. *Let $w \in A_2$ and $\mathcal{S}$ the Haar shift on L^2. Then for the resolution of $\mathcal{S}$ into its canonical paraproducts as given in (1.1) we have that each term can be controlled by a linear power of $[w]_{A_2}$. In particular,*

$$\left\| Q_{\mathcal{S},w}^{(\varepsilon_1,\varepsilon_2),(\varepsilon_3,\varepsilon_4)} \right\|_{L^2\to L^2} \lesssim [w]_{A_2}\,.$$

If one could control each term appearing in Theorem 1.2 independent of the Theorem 1.1, this would imply Petermichl's Theorem. But, for one of the terms, we unfortunately need to resort to the estimate in Theorem 1.1, and will point out an interesting question that we are unable to resolve as of this writing. However, our point is to demonstrate that the paraproduct operators arising in the canonical resolution of the Haar shift $\mathcal{S}$ are also bounded, and linearly in the A_2 characteristic.

Finally, we mention that the operators $Q_{\mathcal{I},w}^{(\varepsilon_1,\varepsilon_2),(\varepsilon_3,\varepsilon_4)}$ in the resolution of the identity $\mathcal{I} = M_{w^{\frac12}}\mathcal{I}M_{w^{-\frac12}}$ in (1.1), are all bounded on L^2 <u>if and only if</u> $w \in A_2$, and if so, the operator norm is linear in the A_2 characteristic. The proof of this fact is easy using the techniques in the proof of Theorem 1.2, and amounts to little more than repeating the arguments without the shift involved. Key to this proof working is that the identity is a purely local operator, and this points to the difficulty of Haar shift and its non-local nature.

Throughout this paper $\equiv$ means equal by definition, while $A \lesssim B$ means that there exists an absolute constant C such that $A \le CB$.

2. Notation and Preliminary Estimates

Before proceeding with the proof of Theorem 1.2, we collect a few elementary observations and necessary notation that will be used frequently throughout the remainder of the paper.

To define our paraproducts, let $\mathcal{D}$ denote the usual dyadic grid of intervals on the real line. Define the Haar function h_I^0 and averaging function h_I^1 by

$$h_I^0 \equiv h_I \equiv \frac{1}{\sqrt{|I|}}\left(\mathbf{1}_{I_-} - \mathbf{1}_{I_+}\right) \text{ and } h_I^1 \equiv \frac{1}{|I|}\mathbf{1}_I\,, \qquad I \in \mathcal{D}.$$

The paraproduct operators considered in this paper are the following dyadic operators.

Definition 2.1. *Given a symbol $b = \{b_I\}_{I\in\mathcal{D}}$ and a pair $(\alpha,\beta) \in \{0,1\} \times \{0,1\}$, define the dyadic paraproduct acting on a function f by*

$$\mathsf{P}_b^{(\alpha,\beta)} f \equiv \sum_{I\in\mathcal{D}} b_I \left\langle f, h_I^{\beta} \right\rangle_{L^2} h_I^{\alpha},$$

where h_I^0 is the Haar function associated with I, and h_I^1 is the average function associated with I. The index (α,β) is referred to as the type of $\mathsf{P}_b^{(\alpha,\beta)}$.

For a function b and $I \in \mathcal{D}$ we let

$$\begin{aligned} \widehat{b}(I) &\equiv \left\langle b, h_I^0 \right\rangle_{L^2} \\ \langle b \rangle_I &\equiv \left\langle b, h_I^1 \right\rangle_{L^2}, \end{aligned}$$

denote the corresponding sequences indexed by the dyadic intervals $I \in \mathcal{D}$. With this notation, the canonical paraproduct decomposition of the pointwise multiplier operator M_b is given by

$$M_b = P_{\widehat{b}}^{(0,1)} + P_{\widehat{b}}^{(1,0)} + P_{\langle b\rangle}^{(0,0)}.$$

At points in the argument below we will have to resort to the use of disbalanced Haar functions. To do so, we introduce some additional notation. Given a weight σ on $\mathbb{R}$ we set

$$C_K(\sigma) \equiv \sqrt{\frac{\langle\sigma\rangle_{K_+}\langle\sigma\rangle_{K_-}}{\langle\sigma\rangle_K}} \quad \text{and} \quad D_K(\sigma) \equiv \frac{\widehat{\sigma}(K)}{\langle\sigma\rangle_K}. \tag{2.1}$$

Then we have

$$h_K = C_K(\sigma) h_K^{\sigma} + D_K(\sigma) h_K^1 \tag{2.2}$$

where $\{h_K^{\sigma}\}_{K\in\mathcal{D}}$ is the $L^2(\sigma)$ normalized Haar basis. For an interval $K \in \mathcal{D}$ we let

$$\mathbb{E}_K^{\sigma}(f) \equiv \frac{1}{\sigma(K)} \int_K f\sigma dx$$

where $\sigma(K) = \int_K \sigma dx$. When σ is Lebesgue measure we simply write $\langle f\rangle_K = \mathbb{E}_K(f) \equiv \frac{1}{|K|}\int_K f dx$.

For two functions f and g we have the following "product formula" that appears in higher dimensions in [10]

$$\widehat{fg}(I) = \sum_{J\subsetneq I} \widehat{f}(J)\,\widehat{g}(J)\,\frac{\delta(J,I)}{\sqrt{|I|}} + \widehat{f}(I)\langle g\rangle_I + \widehat{g}(I)\langle f\rangle_I \qquad \forall I \in \mathcal{D},$$

where $\delta(J,I) = 1$ if $J \subset I_-$ and $\delta(J,I) = -1$ if $J \subset I_+$.

For a sequence $a = \{a_I\}_{I\in\mathcal{D}}$ define.

$$\begin{aligned} \|a\|_{\ell^\infty} &\equiv \sup_{I\in\mathcal{D}} |a_I|; \\ \|a\|_{CM} &\equiv \sqrt{\sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\subset I} |a_J|^2}. \end{aligned}$$

The following estimates are well-known.

Lemma 2.1. *We have the characterizations:*

$$\left\|\mathsf{P}_a^{(0,0)}\right\|_{L^2\to L^2} = \|a\|_{\ell^\infty}; \tag{2.3}$$

$$\left\|\mathsf{P}_a^{(0,1)}\right\|_{L^2\to L^2} = \left\|\mathsf{P}_a^{(1,0)}\right\|_{L^2\to L^2} \approx \|a\|_{CM}. \tag{2.4}$$

The Carleson Embedding Theorem is a fundamental tool in this paper and will be used frequently. It is the following:

Theorem 2.2 (Carleson Embedding Theorem)**.** *Let $v \geq 0$, and let $\{\alpha_I\}_{I\in\mathcal{D}}$ be positive constants. The following two statements are equivalent:*

$$\sum_{I\in\mathcal{D}} \alpha_I\, \mathbb{E}_I^v(f)^2 \leq 4C\,\|f\|_{L^2(v)}^2 \quad \forall f \in L^2(v);$$

$$\sup_{I\in\mathcal{D}} v(I)^{-1} \sum_{J\subset I} \alpha_J \langle v\rangle_J^2 \leq C.$$

As a simple application of the Carleson Embedding Theorem, one can prove:

Lemma 2.3. *Let $a = \{a_I\}_{I\in\mathcal{D}}$ be a sequence of non-negative numbers. Then*

$$\left\|\mathsf{P}_a^{(1,1)}\right\|_{L^2\to L^2} \lesssim \left\|a^{\frac{1}{2}}\right\|_{CM}^2. \tag{2.5}$$

Recall that the dyadic square function is given by

$$S\phi(x) \equiv \sqrt{\sum_{I\in\mathcal{D}} \left|\widehat{\phi}(I)\right|^2 h_I^1(x)},$$

and that for any weight $v \geq 0$ we have

$$\|S\phi\|_{L^2(v)}^2 = \sum_{I\in\mathcal{D}} \left|\widehat{\phi}(I)\right|^2 \langle v\rangle_I .$$

Since $\{h_I\}_{I\in\mathcal{D}}$ is an orthonormal basis for $L^2(\mathbb{R})$, it is trivial that $\|Sf\|_{L^2} = \|f\|_{L^2}$. However, in [8] it was shown for $w \in A_2$ that

$$\|Sf\|_{L^2(w)}^2 \lesssim [w]_{A_2}^2 \|f\|_{L^2(w)}^2 .$$

Applying this inequality to $f = w^{-\frac{1}{2}}\mathbf{1}_I$ for $I \in \mathcal{D}$, along with some obvious estimates, yields the following:

$$\sum_{J\subset I} \widehat{w^{-\frac{1}{2}}}(J)^2 \langle w\rangle_J \lesssim [w]_{A_2}^2 |I| \quad \forall I \in \mathcal{D}. \tag{2.6}$$

A trivial consequence of (2.6) is,

$$\sum_{J\subset I} \widehat{w^{-\frac{1}{2}}}(J)^2 \left\langle w^{\frac{1}{2}}\right\rangle_J^2 \lesssim [w]_{A_2}^2 |I| \quad \forall I \in \mathcal{D}, \tag{2.7}$$

since $\left\langle w^{\frac{1}{2}}\right\rangle_J^2 \leq \langle w\rangle_J$. Because of the symmetry of the A_2 condition, we also have these estimates with the roles of w and w^{-1} interchanged. All of these estimates play a fundamental role at various times when applying the Carleson Embedding Theorem.

We also need a modified version of estimate (2.6), but which incorporates a shift in the indices. Define the modified square function S_π by,

$$S_\pi\phi(x) \equiv \sqrt{\sum_{I\in\mathcal{D}} \left|\widehat{\phi}(I)\right|^2 \frac{1}{|I|}\mathbf{1}_{\pi I}(x)}$$

where πI is the dyadic parent of the dyadic interval I. Unfortunately, $S_\pi f$ is not pointwise bounded by Sf ($S_\pi h_K \equiv 1$ on πK but Sh_K vanishes outside K), yet we show in the theorem below that S_π has a linear bound in the characteristic on weighted spaces.

Theorem 2.4. *For any $\phi \in L^2(w)$ we have*

$$\|S_\pi \phi\|_{L^2(w)} \lesssim [w]_{A_2} \|\phi\|_{L^2(w)} .$$

Proof. The proof of this theorem uses the ideas in [8]. Without loss of generality we may assume both w and w^{-1} are bounded so long as these bounds do not enter into our estimates. First note that

$$\|S_\pi\phi\|^2_{L^2(w)} = \sum_{I\in\mathcal{D}} \left|\widehat{\phi}(I)\right|^2 \langle w\rangle_{\pi I} = \left\langle \widetilde{D}_w \phi, \phi\right\rangle_{L^2} ,$$

where $\widetilde{D}_w : L^2 \to L^2$ is the 'discrete multiplier' map that sends $h_I \longrightarrow \langle w\rangle_{\pi I} h_I$. We also have

$$\|\phi\|^2_{L^2(w)} = \langle M_w\phi, \phi\rangle_{L^2} ,$$

where $M_w : L^2 \to L^2$ is the operator of pointwise multiplication by w. We first claim the operator inequality

$$M_w \lesssim [w]_{A_2} \widetilde{D}_w ,$$

which upon taking inverses, is equivalent to

$$\widetilde{D}_w^{-1} \lesssim [w]_{A_2} (M_w)^{-1} = [w]_{A_2} M_{w^{-1}} , \tag{2.8}$$

where $\widetilde{D}_w^{-1} : L^2 \to L^2$ is the map that sends $h_I \longrightarrow \frac{1}{\langle w\rangle_{\pi I}} h_I$. But, (2.8) is equivalent to

$$\sum_{I\in\mathcal{D}} \left|\widehat{\phi}(I)\right|^2 \frac{1}{\langle w\rangle_{\pi I}} \lesssim [w]_{A_2} \|\phi\|^2_{L^2(w^{-1})} .$$

At this point we simply observe that since $|\pi I| = 2|I|$ and $w(I) \le w(\pi I)$, we have

$$\sum_{I\in\mathcal{D}} \left|\widehat{\phi}(I)\right|^2 \frac{1}{\langle w\rangle_{\pi I}} \le 2\sum_{I\in\mathcal{D}} \left|\widehat{\phi}(I)\right|^2 \frac{1}{\langle w\rangle_{I}}$$

and then we invoke the following inequality in [8]:

$$\sum_{I\in\mathcal{D}} \left|\widehat{\phi}(I)\right|^2 \frac{1}{\langle w\rangle_{I}} \lesssim [w]_{A_2} \|\phi\|^2_{L^2(w^{-1})} ,$$

and this proves (2.8).

Now we apply a duality argument to obtain

$$\widetilde{D}_w \lesssim [w]^2_{A_2} M_w , \tag{2.9}$$

which is equivalent to the desired inequality in the statement of the theorem. To see (2.9), we note that

$$\widetilde{D}_w \leq [w]_{A_2} \left(\widetilde{D}_{w^{-1}}\right)^{-1} ,$$

follows from

$$\begin{aligned}\left\langle \widetilde{D}_w \phi, \phi \right\rangle_{L^2} &= \sum_{I\in\mathcal{D}} \left|\widehat{\phi}(I)\right|^2 \langle w\rangle_{\pi I} = \sum_{I\in\mathcal{D}} \left|\widehat{\phi}(I)\right|^2 \frac{\langle w\rangle_{\pi I} \left\langle w^{-1}\right\rangle_{\pi I}}{\left\langle w^{-1}\right\rangle_{\pi I}} \\ &\leq [w]_{A_2} \sum_{I\in\mathcal{D}} \left|\widehat{\phi}(I)\right|^2 \frac{1}{\left\langle w^{-1}\right\rangle_{\pi I}} = [w]_{A_2} \left\langle \left(\widetilde{D}_{w^{-1}}\right)^{-1} \phi, \phi \right\rangle_{L^2} .\end{aligned}$$

Now we continue by applying (2.8) with w replaced by w^{-1} to get

$$\widetilde{D}_w \leq [w]_{A_2} \left(\widetilde{D}_{w^{-1}}\right)^{-1} \leq [w]_{A_2} \left[w^{-1}\right]_{A_2} (M_{w^{-1}})^{-1} = [w]_{A_2}^2 M_w .$$

□

Again, using this Theorem 2.4 with the function $\phi = w^{-\frac{1}{2}} \mathbf{1}_I$ for $I \in \mathcal{D}$ yields

$$\sum_{J\subset I} \widehat{w^{-\frac{1}{2}}}(J)^2 \langle w\rangle_{\pi J} \lesssim [w]_{A_2}^2 |I| \quad \forall I \in \mathcal{D} \tag{2.10}$$

which will be used frequently in the proof below.

3. Proof of Theorem 1.2

Our method of attack on Theorem 1.2 will use the language of paraproducts. We expand the operator $M_{w^{\frac{1}{2}}} \mathcal{S} M_{w^{-\frac{1}{2}}}$ in term of the canonical paraproducts with symbols $w^{\frac{1}{2}}$ and $w^{-\frac{1}{2}}$ to obtain,

$$\left(\mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} + \mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}} + \mathsf{P}^{(0,0)}_{\langle w^{\frac{1}{2}}\rangle}\right) \mathcal{S} \left(\mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}} + \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} + \mathsf{P}^{(0,0)}_{\langle w^{-\frac{1}{2}}\rangle}\right) . \tag{3.1}$$

This results in nine terms and we will study each of these separately to see that they are controlled by (no worse than) the linear characteristic of the weight. Doing so will provide a proof of Theorem 1.2.

In what follows, any time there are 'internal' zeros, the Haar shift $\mathcal{S}$ can be absorbed, and we are left with simply studying a modified (shifted) paraproduct.

3.1. **Estimating Easy Terms.**

There are four easy terms that arise from (3.1), and they are easy because the composition of the paraproducts reduce to classical paraproduct type operators. To motivate our approach, we point out that simple computations (that absorb the Haar shift and are given explicitly below) give

$$\mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}} \approx \mathsf{P}^{(1,1)}_{\widehat{w^{\frac{1}{2}}} \circ \widehat{w^{-\frac{1}{2}}}} ; \tag{3.2}$$

$$\mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(0,0)}_{\langle w^{-\frac{1}{2}}\rangle} \approx \mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}} \circ \langle w^{-\frac{1}{2}}\rangle} ; \tag{3.3}$$

$$\mathsf{P}^{(0,0)}_{\langle w^{\frac{1}{2}}\rangle}\mathcal{S}\mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}} \approx \mathsf{P}^{(0,1)}_{\langle w^{\frac{1}{2}}\rangle\circ\widehat{w^{-\frac{1}{2}}}}; \tag{3.4}$$

$$\mathsf{P}^{(0,0)}_{\langle w^{\frac{1}{2}}\rangle}\mathcal{S}\mathsf{P}^{(0,0)}_{\langle w^{-\frac{1}{2}}\rangle} \approx \mathsf{P}^{(0,0)}_{\langle w^{\frac{1}{2}}\rangle\circ\langle w^{-\frac{1}{2}}\rangle}. \tag{3.5}$$

Here $a\circ b$ is the Schur product of the sequences a and b, and we are letting $\approx$ mean that there has been a shift in one of the basis elements appearing in the definition of the Haar basis (this presents no problem in the analysis of these operators since it simply results in a change in the absolute constants appearing; we will make this rigorous shortly). Notice that (3.3), (3.4) and (3.5) are of the type considered in Lemma 2.1 and so have a complete characterization, while (3.2) can be handled by Lemma 2.3. In each of these characterizations we are of course left with showing that the norm on the symbol for these classical operators can be controlled by the linear power of the characteristic of the weight. We now make these ideas precise.

3.1.1. *Estimating Term* (3.2). Note that a simple computation shows that

$$\mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}}\mathcal{S}\mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}} = \sum_{I\in\mathcal{D}} \widehat{w^{\frac{1}{2}}}(I_-)\widehat{w^{-\frac{1}{2}}}(I)h^1_{I_-}\otimes h^1_I.$$

By the Carleson Embedding Theorem we see that

$$\begin{aligned}
\left|\left\langle \mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}}\mathcal{S}\mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}}\phi,\psi\right\rangle_{L^2}\right| &= \left|\sum_{I\in\mathcal{D}} \widehat{w^{\frac{1}{2}}}(I_-)\widehat{w^{-\frac{1}{2}}}(I)\left\langle\phi,h^1_I\right\rangle_{L^2}\left\langle\psi,h^1_{I_-}\right\rangle_{L^2}\right| \\
&\le \sum_{I\in\mathcal{D}}\left|\widehat{w^{\frac{1}{2}}}(I_-)\widehat{w^{-\frac{1}{2}}}(I)\left\langle\phi,h^1_I\right\rangle_{L^2}\left\langle\psi,h^1_{I_-}\right\rangle_{L^2}\right| \\
&\le \left(\sum_{I\in\mathcal{D}}\left|\widehat{w^{\frac{1}{2}}}(I_-)\widehat{w^{-\frac{1}{2}}}(I)\left\langle\phi\right\rangle^2_I\right|\right)^{\frac{1}{2}}\left(\sum_{I\in\mathcal{D}}\left|\widehat{w^{\frac{1}{2}}}(I_-)\widehat{w^{-\frac{1}{2}}}(I)\left\langle\psi\right\rangle^2_{I_-}\right|\right)^{\frac{1}{2}} \\
&\lesssim \left\|\phi\right\|_{L^2}\left\|\psi\right\|_{L^2}\sup_{J\in\mathcal{D}}\frac{1}{|J|}\sum_{I\subset J}\left|\widehat{w^{\frac{1}{2}}}(I_-)\widehat{w^{-\frac{1}{2}}}(I)\right|.
\end{aligned}$$

However, by Cauchy-Schwarz we then have

$$\begin{aligned}
\frac{1}{|J|}\sum_{I\subset J}\left|\widehat{w^{\frac{1}{2}}}(I_-)\widehat{w^{-\frac{1}{2}}}(I)\right| &\le \frac{1}{|J|}\left\|w^{\frac{1}{2}}\mathbf{1}_J\right\|_{L^2}\left\|w^{-\frac{1}{2}}\mathbf{1}_J\right\|_{L^2} \\
&= \left(\left\langle w\right\rangle_J\left\langle w^{-1}\right\rangle_J\right)^{\frac{1}{2}} \le [w]^{\frac{1}{2}}_{A_2},
\end{aligned}$$

which gives

$$\left\|\mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}}\mathcal{S}\mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}}\right\|_{L^2\to L^2} \lesssim [w]^{\frac{1}{2}}_{A_2} \le [w]_{A_2},$$

yielding the desired linear bound for term (3.2).

3.1.2. *Estimating Terms* (3.3) *and* (3.4). These terms have are symmetric and so we focus only on the first term. In this case term (3.3) reduces to a classical paraproduct operator with a symbol given by the product of two sequences as indicated by the notation $\approx$. Indeed, we have the identity,

$$\mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(0,0)}_{\langle w^{-\frac{1}{2}} \rangle} = \sum_{I \in \mathcal{D}} \widehat{w^{\frac{1}{2}}}(I_-) \langle w^{-\frac{1}{2}} \rangle_I h^1_{I_-} \otimes h_I .$$

Now by the Carleson Embedding Theorem applied to ψ we have that

$$\begin{aligned} \left| \left\langle \mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(0,0)}_{\langle w^{-\frac{1}{2}} \rangle} \phi, \psi \right\rangle_{L^2} \right| &= \left| \sum_{I \in \mathcal{D}} \widehat{w^{\frac{1}{2}}}(I_-) \langle w^{-\frac{1}{2}} \rangle_I \left\langle \phi, h_I \right\rangle_{L^2} \left\langle \psi, h^1_{I_-} \right\rangle_{L^2} \right| \\ &\le \left\| \phi \right\|_{L^2} \left(\sum_{I \in \mathcal{D}} \left| \widehat{w^{\frac{1}{2}}}(I_-) \langle w^{-\frac{1}{2}} \rangle_I \right|^2 \langle \psi \rangle^2_{I_-} \right)^{\frac{1}{2}} \\ &\lesssim \left\| \phi \right\|_{L^2} \left\| \psi \right\|_{L^2} \sup_{J \in \mathcal{D}} \frac{1}{|J|} \sum_{I \subset J} \left| \widehat{w^{\frac{1}{2}}}(I_-) \langle w^{-\frac{1}{2}} \rangle_I \right|^2 . \end{aligned}$$

But

$$\sup_{J \in \mathcal{D}} \frac{1}{|J|} \sum_{I \subset J} \left| \widehat{w^{\frac{1}{2}}}(I_-) \langle w^{-\frac{1}{2}} \rangle_I \right|^2 \le \sup_{J \in \mathcal{D}} \frac{1}{|J|} \sum_{I \subset J} \widehat{w^{\frac{1}{2}}}(I_-)^2 \langle w^{-1} \rangle_I \lesssim [w]^2_{A_2} ,$$

where we have used the linear bound for the square function in (2.10) in the last inequality. Thus,

$$\left\| \mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(0,0)}_{\langle w^{-\frac{1}{2}} \rangle} \right\|_{L^2 \to L^2} \lesssim [w]_{A_2}$$

which gives the desired linear bound in terms of the A_2 characteristic.

3.1.3. *Estimating Term* (3.5). Term (3.5) reduces to a standard Haar multiplier. Indeed, we have the identity

$$\mathsf{P}^{(0,0)}_{\langle w^{\frac{1}{2}} \rangle} \mathcal{S} \mathsf{P}^{(0,0)}_{\langle w^{-\frac{1}{2}} \rangle} = \sum_{I \in \mathcal{D}} \langle w^{\frac{1}{2}} \rangle_I \langle w^{-\frac{1}{2}} \rangle_{I_-} h_{I_-} \otimes h_I ,$$

where $(h_J \otimes h_K) f \equiv \langle f, h_K \rangle h_J$, and then

$$\left\| \mathsf{P}^{(0,0)}_{\langle w^{\frac{1}{2}} \rangle} \mathcal{S} \mathsf{P}^{(0,0)}_{\langle w^{-\frac{1}{2}} \rangle} \right\|_{L^2 \to L^2} = \sup_{I \in \mathcal{D}} \langle w^{\frac{1}{2}} \rangle_I \langle w^{-\frac{1}{2}} \rangle_{I_-} \lesssim \sup_{I \in \mathcal{D}} \langle w \rangle_I^{\frac{1}{2}} \left\langle w^{-1} \right\rangle_I^{\frac{1}{2}} = [w]^{\frac{1}{2}}_{A_2} \le [w]_{A_2} .$$

This gives the desired linear estimate for (3.5).

3.2. **Estimating Hard Terms.** There are five remaining terms to be controlled. These include the four difficult terms,

$$\mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}} \approx \mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} \mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}} ; \tag{3.6}$$

$$\mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(0,0)}_{\langle w^{-\frac{1}{2}} \rangle} \approx \mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} \mathsf{P}^{(0,0)}_{\langle w^{-\frac{1}{2}} \rangle} ; \tag{3.7}$$

$$\mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} \approx \mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}} \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} ; \tag{3.8}$$

$$\mathsf{P}^{(0,0)}_{\langle w^{\frac{1}{2}} \rangle} \mathcal{S} \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} \approx \mathsf{P}^{(0,0)}_{\langle w^{\frac{1}{2}} \rangle} \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} . \tag{3.9}$$

To estimate terms (3.6) and (3.8) we will rely on disbalanced Haar functions adapted to the weight w and w^{-1}. For these terms we will also proceed by computing the norm of the operators in question by using duality. Key to this will be the application of the Carleson Embedding Theorem. The proof of the necessary estimates for these terms are carried out in subsection 3.2.1. Terms (3.7) and (3.9) will be handled via a similar method; their analysis is handled in subsection 3.2.2.

The remaining very difficult term is the one for which the Haar shift can not be absorbed into one of the paraproducts. Namely, we need to control the following expression,

$$\mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}}\mathcal{S}\mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}}. \tag{3.10}$$

In this situation, we will explicitly compute $\left\langle \mathcal{S}h_J^1, h_L^1\right\rangle_{L^2}$ and see that the shift presents no problem in how the operator behaves, nor in estimating its norm in terms of the A_2 characteristic $[w]_{A_2}$. This is carried out in subsection 3.2.3.

3.2.1. *Estimating Terms* (3.6) *and* (3.8)*.* Once again these two terms are symmetric and so it suffices to prove the desired estimate only for (3.6). We need to show that

$$\left\Vert \mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}}\mathcal{S}\mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}}\right\Vert_{L^2} \lesssim [w]_{A_2}\,. \tag{3.11}$$

Proceeding by duality, we fix $\phi,\psi\in L^2$ and consider

$$\begin{aligned}
\left\langle \mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}}\mathcal{S}\mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}}\phi,\psi\right\rangle_{L^2} &= \left\langle \mathcal{S}\mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}}\phi, \mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}}\psi\right\rangle_{L^2}\\
&= \sum_{J,K\in\mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J)\left\langle \phi\right\rangle_J \widehat{w^{\frac{1}{2}}}(K)\widehat{\psi}(K)\left\langle h_{J_-},h_K^1\right\rangle_{L^2}\\
&= \sum_{J\in\mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J)\left\langle \phi\right\rangle_J\left(\sum_{K\in\mathcal{D}}\widehat{w^{\frac{1}{2}}}(K)\widehat{\psi}(K)\left\langle h_{J_-},h_K^1\right\rangle_{L^2}\right)\\
&= \sum_{J\in\mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J)\left\langle \phi\right\rangle_J\left(\frac{1}{\left\vert J_-\right\vert^{\frac{1}{2}}}\sum_{K\subsetneq J_-}\widehat{w^{\frac{1}{2}}}(K)\widehat{\psi}(K)\delta\left(K,J_-\right)\right)\\
&= \sum_{J\in\mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J)\left\langle \phi\right\rangle_J\left(\widehat{\psi w^{\frac{1}{2}}}(J_-)-\widehat{w^{\frac{1}{2}}}(J_-)\left\langle \psi\right\rangle_{J_-}-\widehat{\psi}(J_-)\left\langle w^{\frac{1}{2}}\right\rangle_{J_-}\right)\\
&\equiv T_1+T_2+T_3.
\end{aligned}$$

The fifth equality follows by an application of the product formula for Haar coefficients. We need to show that each of these terms has the desired estimate

$$|T_j|\lesssim [w]_{A_2}\left\Vert \phi\right\Vert_{L^2}\left\Vert \psi\right\Vert_{L^2}$$

since this will imply that

$$\left\Vert \mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}}\mathcal{S}\mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}}\right\Vert_{L^2\to L^2} = \sup_{\phi,\psi\in L^2}\left\vert\left\langle \mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}}\mathcal{S}\mathsf{P}^{(0,1)}_{\widehat{w^{-\frac{1}{2}}}}\phi,\psi\right\rangle_{L^2}\right\vert \lesssim [w]_{A_2}$$

which implies the desired estimate on the norm of the operator.

Consider term T_3. This term can be controlled by

$$\begin{aligned} |T_3| &\leq \left| \sum_{J\in\mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J) \langle\phi\rangle_J \widehat{\psi}(J_-) \left\langle w^{\frac{1}{2}} \right\rangle_{J_-} \right| \\ &\leq \|\psi\|_{L^2} \left(\sum_{J\in\mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J)^2 \left\langle w^{\frac{1}{2}} \right\rangle_{J_-}^2 \langle\phi\rangle_{J_-}^2 \right)^{\frac{1}{2}} \\ &\lesssim [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2}, \end{aligned}$$

with the last estimate following by an application of the Carleson Embedding Theorem using (2.7). For the term T_2, we have

$$\begin{aligned} |T_2| &\leq \left| \sum_{J\in\mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J)\widehat{w^{\frac{1}{2}}}(J_-) \langle\psi\rangle_J \langle\phi\rangle_{J_-} \right| \\ &\leq \left(\sum_{J\in\mathcal{D}} \left| \widehat{w^{-\frac{1}{2}}}(J)\widehat{w^{\frac{1}{2}}}(J_-) \right| \langle\psi\rangle_J^2 \right)^{\frac{1}{2}} \left(\sum_{J\in\mathcal{D}} \left| \widehat{w^{-\frac{1}{2}}}(J)\widehat{w^{\frac{1}{2}}}(J_-) \right| \langle\phi\rangle_{J_-}^2 \right)^{\frac{1}{2}} \\ &\lesssim [w]_{A_2}^{\frac{1}{2}} \|\phi\|_{L^2} \|\psi\|_{L^2}. \end{aligned}$$

Again we have used here two applications of the Carleson Embedding Theorem, one for ϕ and one for ψ, since we have that

$$\begin{aligned} \sum_{K\subset J} \left| \widehat{w^{-\frac{1}{2}}}(K)\widehat{w^{\frac{1}{2}}}(K_-) \right| &\leq \left(\sum_{K\subset J} \widehat{w^{-\frac{1}{2}}}(K)^2 \right)^{\frac{1}{2}} \left(\sum_{K\subset J} \widehat{w^{\frac{1}{2}}}(K_-)^2 \right)^{\frac{1}{2}} \\ &\leq \sqrt{w^{-1}(J)w(J)} = \sqrt{\frac{w^{-1}(J)w(J)}{|J|^2}}\,|J| \\ &\leq [w]_{A_2}^{\frac{1}{2}} |J|. \end{aligned}$$

Finally, we prove the estimate for term T_1, which requires the use of disbalanced Haar functions. We expand the terms in T_1 using Haar functions with respect to two different disbalanced bases. In particular, using (2.2) we have

$$\begin{aligned} \left\langle \psi w^{\frac{1}{2}}, h_{J_-} \right\rangle_{L^2} &= C_{J_-}(w) \left\langle \psi w^{\frac{1}{2}}, h_{J_-}^w \right\rangle_{L^2} + D_{J_-}(w) \left\langle \psi w^{\frac{1}{2}} \right\rangle_{J_-} \\ &= C_{J_-}(w) \left\langle \psi w^{-\frac{1}{2}}, h_{J_-}^w \right\rangle_{L^2(w)} + D_{J_-}(w) \langle w\rangle_{J_-} \mathbb{E}_{J_-}^w \left(\psi w^{-\frac{1}{2}}\right) \end{aligned}$$

and

$$\begin{aligned} \left\langle w^{-\frac{1}{2}}, h_J \right\rangle_{L^2} &= C_J(w^{-1}) \left\langle w^{-\frac{1}{2}}, h_J^{w^{-1}} \right\rangle_{L^2} + D_J(w^{-1}) \left\langle w^{-\frac{1}{2}} \right\rangle_J \\ &= C_J(w^{-1}) \left\langle w^{\frac{1}{2}}, h_J^{w^{-1}} \right\rangle_{L^2(w^{-1})} + D_J(w^{-1}) \left\langle w^{-\frac{1}{2}} \right\rangle_J. \end{aligned}$$

Then we can write the term T_1 as a sum of four terms, namely

$$T_1 = S_1 + S_2 + S_3 + S_4$$

with

$$
\begin{aligned}
S_1 &= \sum_{K\in\mathcal{D}} \langle\phi\rangle_K\, C_{K_-}(w)C_K(w^{-1}) \left\langle w^{\frac12}, h_K^{w^{-1}}\right\rangle_{L^2(w^{-1})} \left\langle \psi w^{-\frac12}, h_{K_-}^{w}\right\rangle_{L^2(w)} \\
S_2 &= \sum_{K\in\mathcal{D}} \langle\phi\rangle_K\, C_{K_-}(w)D_K(w^{-1}) \left\langle \psi w^{-\frac12}, h_{K_-}^{w}\right\rangle_{L^2(w)} \left\langle w^{-\frac12}\right\rangle_K \\
S_3 &= \sum_{K\in\mathcal{D}} \langle\phi\rangle_K\, D_{K_-}(w)C_K(w^{-1}) \left\langle w\right\rangle_{K_-} \mathbb{E}_{K_-}^{w}\left(\psi w^{-\frac12}\right) \left\langle w^{\frac12}, h_K^{w^{-1}}\right\rangle_{L^2(w^{-1})} \\
S_4 &= \sum_{K\in\mathcal{D}} \langle\phi\rangle_K\, D_{K_-}(w)D_K(w^{-1}) \left\langle w^{-\frac12}\right\rangle_K \left\langle w\right\rangle_{K_-} \mathbb{E}_{K_-}^{w}\left(\psi w^{-\frac12}\right).
\end{aligned}
$$

We need to show that each term can be estimated by a constant times $[w]_{A_2}\|\phi\|_2\|\psi\|_2$, which would then imply

$$
|T_1| \lesssim [w]_{A_2}\|\phi\|_2\|\psi\|_2
$$

as required. We now proceed to prove the necessary estimates.

Consider the term S_1. We have

$$
\begin{aligned}
|S_1| &\le \left|\sum_{K\in\mathcal{D}} \langle\phi\rangle_K\, C_{K_-}(w)C_K(w^{-1}) \left\langle w^{\frac12}, h_K^{w^{-1}}\right\rangle_{L^2(w^{-1})} \left\langle \psi w^{-\frac12}, h_{K_-}^{w}\right\rangle_{L^2(w)}\right| \\
&\le \left\|\psi w^{-\frac12}\right\|_{L^2(w)} \left(\sum_{K\in\mathcal{D}} \langle\phi\rangle_K^2\, C_{K_-}(w)^2 C_K(w^{-1})^2 \left\langle w^{-\frac12}, h_K^{w}\right\rangle_{L^2(w)}^2\right)^{\frac12} \\
&= \|\psi\|_{L^2} \left(\sum_{K\in\mathcal{D}} \langle\phi\rangle_K^2\, \frac{\langle w\rangle_{K_{-+}}\langle w\rangle_{K_{--}}}{\langle w\rangle_{K_-}} \frac{\left\langle w^{-1}\right\rangle_{K_+}\left\langle w^{-1}\right\rangle_{K_-}}{\left\langle w^{-1}\right\rangle_K} \left\langle w^{\frac12}, h_K^{w^{-1}}\right\rangle_{L^2(w^{-1})}^2\right)^{\frac12} \\
&\lesssim [w]_{A_2}^{\frac12}\|\psi\|_{L^2} \left(\sum_{K\in\mathcal{D}} \langle\phi\rangle_K^2 \left\langle w^{\frac12}, h_K^{w^{-1}}\right\rangle_{L^2(w^{-1})}^2\right)^{\frac12} \\
&\lesssim [w]_{A_2}^{\frac12}\|\phi\|_{L^2}\|\psi\|_{L^2}.
\end{aligned}
$$

Here the last inequality follows by the Carleson Embedding Theorem since

$$
\sum_{K\subset J} \left\langle w^{\frac12}, h_K^{w^{-1}}\right\rangle_{L^2(w^{-1})}^2 \le |J| \quad \forall J\in\mathcal{D}.
$$

Turning to term S_2 we have

$$
\begin{aligned}
|S_2| &= \left| \sum_{K\in\mathcal{D}} \langle \phi\rangle_K C_{K_-}(w) D_K(w^{-1}) \left\langle \psi w^{-\frac{1}{2}}, h_{J_-}^w \right\rangle_{L^2(w)} \left\langle w^{-\frac{1}{2}} \right\rangle_K \right| \\
&\leq \left(\sum_{K\in\mathcal{D}} \left\langle \psi w^{-\frac{1}{2}}, h_{K_-}^w \right\rangle_{L^2(w)}^2 \frac{\langle w\rangle_{K_{-+}} \langle w\rangle_{K_{--}}}{\langle w\rangle_{K_-}} \left\langle w^{-\frac{1}{2}} \right\rangle_K^2 \right)^{\frac{1}{2}} \left(\sum_{K\in\mathcal{D}} \frac{\widehat{w^{-1}}(K)^2}{\langle w^{-1}\rangle_K^2} \langle \phi\rangle_K^2 \right)^{\frac{1}{2}} \\
&\lesssim [w]_{A_2}^{\frac{1}{2}} \left\| \psi w^{-\frac{1}{2}} \right\|_{L^2(w)} \left(\sum_{K\in\mathcal{D}} \frac{\widehat{w^{-1}}(K)^2}{\langle w^{-1}\rangle_K^2} \langle \phi\rangle_K^2 \right)^{\frac{1}{2}} \\
&\leq [w]_{A_2}^{\frac{1}{2}} \|\psi\|_{L^2} \left(\sum_{K\in\mathcal{D}} \frac{\widehat{w^{-1}}(K)^2}{\langle w^{-1}\rangle_K^2} \langle \phi\rangle_K^2 \right)^{\frac{1}{2}} \\
&\leq [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2}
\end{aligned}
$$

with the last estimate following from the Carleson Embedding Theorem, once we prove the following estimate:

$$
\sum_{K\subset J} \frac{\widehat{w^{-1}}(K)^2}{\langle w^{-1}\rangle_K^2} \lesssim [w]_{A_2} |J| \quad \forall J\in\mathcal{D}. \tag{3.12}
$$

To prove (3.12) recall the following two estimates in [6, Lemma 5.2 and Lemma 5.3], translated to the notation of this paper:

$$
\begin{aligned}
\sum_{K\subset J} \frac{\widehat{w^{-1}}(K)^2}{\langle w^{-1}\rangle_K} &\lesssim [w]_{A_2} w^{-1}(J) \quad \forall J\in\mathcal{D} \\
\sum_{K\subset J} \frac{\widehat{w^{-1}}(K)^2}{\langle w^{-1}\rangle_K^3} &\lesssim w(J) \quad \forall J\in\mathcal{D}.
\end{aligned}
$$

These two estimates coupled with a simple application of Cauchy-Schwarz then proves (3.12). Indeed,

$$
\begin{aligned}
\sum_{K\subset J} \frac{\widehat{w^{-1}}(K)^2}{\langle w^{-1}\rangle_K^2} &= \sum_{K\subset J} \frac{\widehat{w^{-1}}(K)^2}{\langle w^{-1}\rangle_K} 1 \frac{1}{\langle w^{-1}\rangle_K} \\
&\leq \sqrt{\sum_{K\subset J} \frac{\widehat{w^{-1}}(K)^2}{\langle w^{-1}\rangle_K} 1^2} \sqrt{\sum_{K\subset J} \frac{\widehat{w^{-1}}(K)^2}{\langle w^{-1}\rangle_K} \frac{1}{\langle w^{-1}\rangle_K^2}} \\
&\lesssim \sqrt{[w]_{A_2} w^{-1}(J)} \sqrt{w(J)} \\
&= \sqrt{[w]_{A_2}} \sqrt{w(J) w^{-1}(J)} \leq [w]_{A_2} |J| .
\end{aligned}
$$

For the term S_3 we have

$$
\begin{aligned}
|S_3| &= \left|\sum_{K\in\mathcal{D}} \langle\phi\rangle_K\, D_{K_-}(w)C_K(w^{-1})\,\langle w\rangle_{K_-}\,\mathbb{E}^w_{K_-}\left(\psi w^{-\frac12}\right)\left\langle w^{\frac12},h_K^{w^{-1}}\right\rangle_{L^2(w^{-1})}\right| \\
&= \left|\sum_{K\in\mathcal{D}} \langle\phi\rangle_K\, \frac{\widehat{w}(K_-)}{\langle w\rangle_{K_-}}\,\langle w\rangle_{K_-}\,\mathbb{E}^w_{K_-}\left(\psi w^{-\frac12}\right)\sqrt{\frac{\langle w^{-1}\rangle_{K_+}\langle w^{-1}\rangle_{K_-}}{\langle w^{-1}\rangle_K}}\left\langle w^{\frac12},h_K^{w^{-1}}\right\rangle_{L^2(w^{-1})}\right| \\
&= \left|\sum_{K\in\mathcal{D}} \langle\phi\rangle_K\, \widehat{w}(K_-)\mathbb{E}^w_{K_-}\left(\psi w^{-\frac12}\right)\sqrt{\frac{\langle w^{-1}\rangle_{K_+}\langle w^{-1}\rangle_{K_-}}{\langle w^{-1}\rangle_K}}\left\langle w^{\frac12},h_K^{w^{-1}}\right\rangle_{L^2(w^{-1})}\right| \\
&\le \left(\sum_{K\in\mathcal{D}} \langle\phi\rangle_K^2\left\langle w^{\frac12},h_K^{w^{-1}}\right\rangle_{L^2(w^{-1})}^2\right)^{\frac12}\left(\sum_{K\in\mathcal{D}}\frac{\langle w^{-1}\rangle_{K_+}\langle w^{-1}\rangle_{K_-}}{\langle w^{-1}\rangle_K}\widehat{w}(K_-)^2\mathbb{E}^w_K\left(\psi w^{-\frac12}\right)^2\right)^{\frac12} \\
&\lesssim \|\phi\|_{L^2}\left(\sum_{K\in\mathcal{D}}\frac{\langle w^{-1}\rangle_{K_+}\langle w^{-1}\rangle_{K_-}}{\langle w^{-1}\rangle_K}\widehat{w}(K_-)^2\mathbb{E}^w_{K_-}\left(\psi w^{-\frac12}\right)^2\right)^{\frac12} \\
&\lesssim [w]_{A_2}\,\|\psi\|_{L^2}\,\|\phi\|_{L^2}\,.
\end{aligned}
$$

For the last inequality we used the Carleson Embedding Theorem twice, first applied to ϕ as above, and then to ψ upon noting that

$$
\sum_{K\subset J}\frac{\langle w^{-1}\rangle_{K_+}\langle w^{-1}\rangle_{K_-}}{\langle w^{-1}\rangle_K}\widehat{w}(K_-)^2 \lesssim \sum_{K\subset J}\langle w^{-1}\rangle_K\,\widehat{w}(K_-)^2\lesssim [w]_{A_2}^2\,w(J)\quad \forall J\in\mathcal{D}
$$

via the linear bound for the modified square function (2.10).

Finally consider the term S_4. We have

$$
\begin{aligned}
|S_4| &= \left|\sum_{K\in\mathcal{D}} \langle\phi\rangle_K\, D_{K_-}(w)D_K(w^{-1})\left\langle w^{-\frac12}\right\rangle_K\langle w\rangle_{K_-}\,\mathbb{E}^w_{K_-}\left(\psi w^{-\frac12}\right)\right| \\
&= \left|\sum_{K\in\mathcal{D}} \langle\phi\rangle_K\, \frac{\widehat{w}(K_-)}{\langle w\rangle_{K_-}}\frac{\widehat{w^{-1}}(K)}{\langle w^{-1}\rangle_K}\left\langle w^{-\frac12}\right\rangle_K\langle w\rangle_{K_-}\,\mathbb{E}^w_{K_-}\left(\psi w^{-\frac12}\right)\right| \\
&= \left|\sum_{K\in\mathcal{D}} \langle\phi\rangle_K\, \frac{\widehat{w^{-1}}(K)\widehat{w}(K_-)}{\langle w^{-1}\rangle_K}\left\langle w^{-\frac12}\right\rangle_K\mathbb{E}^w_{K_-}\left(\psi w^{-\frac12}\right)\right| \\
&\le \left(\sum_{K\in\mathcal{D}} \langle\phi\rangle_K^2\,\frac{\widehat{w^{-1}}(K)\widehat{w}(K_-)}{\langle w^{-1}\rangle_K}\left\langle w^{-\frac12}\right\rangle_K^2\right)^{\frac12}\left(\sum_{K\in\mathcal{D}}\frac{\widehat{w^{-1}}(K)\widehat{w}(K_-)}{\langle w^{-1}\rangle_K}\mathbb{E}^w_{K_-}\left(\psi w^{-\frac12}\right)^2\right)^{\frac12} \\
&\le \left(\sum_{K\in\mathcal{D}} \langle\phi\rangle_K^2\,\widehat{w^{-1}}(K)\widehat{w}(K_-)\right)^{\frac12}\left(\sum_{K\in\mathcal{D}}\frac{\widehat{w^{-1}}(K)\widehat{w}(K_-)}{\langle w^{-1}\rangle_K}\mathbb{E}^w_{K_-}\left(\psi w^{-\frac12}\right)^2\right)^{\frac12}.
\end{aligned}
$$

Now note that in [6] the following estimates are proved

$$\begin{aligned}\sum_{K\subset J}\widehat{w^{-1}}(K)\widehat{w}(K_-) &\lesssim [w]_{A_2}|J| \quad \forall J\in\mathcal{D}\\ \sum_{K\subset J}\frac{\widehat{w^{-1}}(K)\widehat{w}(K_-)}{\langle w^{-1}\rangle_K} &\lesssim [w]_{A_2}w(J) \quad \forall J\in\mathcal{D}.\end{aligned}$$

The Carleson Embedding Theorem applied to ϕ and ψ then gives that

$$|S_4|\lesssim [w]_{A_2}\|\psi\|_{L^2}\|\phi\|_{L^2}$$

as desired.

3.2.2. *Estimating Terms* (3.7) *and* (3.9). Once again there is symmetry between these terms, so we focus only on (3.7).

Fix $\phi,\psi\in L^2$. We compute

$$\left\langle \mathsf{P}^{(0,1)}_{\widehat{w^{\frac12}}}\mathcal{S}\mathsf{P}^{(0,0)}_{\left\langle w^{-\frac12}\right\rangle}\phi,\psi\right\rangle_{L^2} = \left\langle \mathcal{S}\mathsf{P}^{(0,0)}_{\left\langle w^{-\frac12}\right\rangle}\phi,\mathsf{P}^{(1,0)}_{\widehat{w^{\frac12}}}\psi\right\rangle_{L^2}$$

$$\begin{aligned}&= \sum_{J,K\in\mathcal{D}}\widehat{\phi}(J)\left\langle w^{-\frac12}\right\rangle_J\widehat{\psi}(K)\widehat{w^{\frac12}}(K)\left\langle h^1_K,h_{J_-}\right\rangle_{L^2}\\ &= \sum_{J\in\mathcal{D}}\widehat{\phi}(J)\left\langle w^{-\frac12}\right\rangle_J\left(\frac{1}{|J_-|^{\frac12}}\sum_{K\subsetneq J_-}\widehat{\psi}(K)\widehat{w^{\frac12}}(K)\delta(J_-,K)\right)\\ &= \sum_{J\in\mathcal{D}}\widehat{\phi}(J)\left\langle w^{-\frac12}\right\rangle_J\left(\widehat{\psi w^{\frac12}}(J_-)-\widehat{\psi}(J_-)\left\langle w^{\frac12}\right\rangle_{J_-}-\widehat{w^{\frac12}}(J_-)\langle\psi\rangle_{J_-}\right)\\ &\equiv T_1+T_2+T_3.\end{aligned}$$

We show that each term satisfies $|T_j|\lesssim [w]_{A_2}\|\phi\|_{L^2}\|\psi\|_{L^2}$, which would imply that

$$\left\|\mathsf{P}^{(0,1)}_{\widehat{w^{\frac12}}}\mathcal{S}\mathsf{P}^{(0,0)}_{\left\langle w^{-\frac12}\right\rangle}\right\|_{L^2\to L^2} = \sup_{\phi,\psi\in L^2}\left|\left\langle \mathsf{P}^{(0,1)}_{\widehat{w^{\frac12}}}\mathcal{S}\mathsf{P}^{(0,0)}_{\left\langle w^{-\frac12}\right\rangle}\phi,\psi\right\rangle_{L^2}\right|\lesssim [w]_{A_2}.$$

proving the desired norm on the operator in question.

The term T_2 is easiest since we have

$$\begin{aligned}|T_2| &= \left|\sum_{J\in\mathcal{D}}\widehat{\phi}(J)\left\langle w^{-\frac12}\right\rangle_J\widehat{\psi}(J_-)\left\langle w^{\frac12}\right\rangle_{J_-}\right|\\ &\lesssim [w]^{\frac12}_{A_2}\sum_{J\in\mathcal{D}}\left|\widehat{\phi}(J)\widehat{\psi}(J_-)\right|\\ &\le [w]^{\frac12}_{A_2}\|\phi\|_{L^2}\|\psi\|_{L^2}\end{aligned}$$

using the definition of A_2, Cauchy-Schwarz and Parseval. For term T_3, we find

$$\begin{aligned} |T_3| &= \left| \sum_{J\in\mathcal{D}} \widehat{\phi}(J) \left\langle w^{-\frac{1}{2}} \right\rangle_J \widehat{w^{\frac{1}{2}}}(J_-) \left\langle \psi \right\rangle_{J_-} \right| \\ &\leq \left\| \phi \right\|_{L^2} \left(\sum_{J\in\mathcal{D}} \left\langle w^{-\frac{1}{2}} \right\rangle_J^2 \widehat{w^{\frac{1}{2}}}(J_-)^2 \left\langle \psi \right\rangle_{J_-}^2 \right)^{\frac{1}{2}} \\ &\lesssim [w]_{A_2} \left\| \phi \right\|_{L^2} \left\| \psi \right\|_{L^2} \end{aligned}$$

with the last estimate following from the Carleson Embedding Theorem and (2.10) (which shows that the Carleson Embedding Theorem applies).

For term T_1, we require disbalanced Haar functions again. Note that it is possible to write

$$\begin{aligned} T_1 &\equiv \sum_{J\in\mathcal{D}} \widehat{\phi}(J) \left\langle w^{-\frac{1}{2}} \right\rangle_J \widehat{\psi w^{\frac{1}{2}}}(J_-) \\ &= \sum_{J\in\mathcal{D}} \widehat{\phi}(J) \left\langle w^{-\frac{1}{2}} \right\rangle_J \left\langle \psi w^{\frac{1}{2}}, C_{J_-}(w) h_{J_-}^w + D_{J_-}(w) h_{J_-}^1 \right\rangle_{L^2} \\ &= \sum_{J\in\mathcal{D}} \widehat{\phi}(J) \left\langle w^{-\frac{1}{2}} \right\rangle_J C_{J_-}(w) \left\langle \psi w^{\frac{1}{2}}, h_{J_-}^w \right\rangle_{L^2} + \sum_{J\in\mathcal{D}} \widehat{\phi}(J) \left\langle w^{-\frac{1}{2}} \right\rangle_J D_{J_-}(w) \left\langle \psi w^{\frac{1}{2}}, h_{J_-}^1 \right\rangle_{L^2} \\ &= \sum_{J\in\mathcal{D}} \widehat{\phi}(J) \left\langle w^{-\frac{1}{2}} \right\rangle_J C_{J_-}(w) \left\langle \psi w^{-\frac{1}{2}}, h_{J_-}^w \right\rangle_{L^2(w)} \\ &\quad + \sum_{J\in\mathcal{D}} \widehat{\phi}(J) \left\langle w^{-\frac{1}{2}} \right\rangle_J D_{J_-}(w) \left\langle w \right\rangle_{J_-} \mathbb{E}_{J_-}^w \left(\psi w^{-\frac{1}{2}} \right) \\ &\equiv S_1 + S_2. \end{aligned}$$

For term S_1 observe that

$$\left\langle w^{-\frac{1}{2}} \right\rangle_J C_{J_-}(w) \lesssim \left\langle w^{-\frac{1}{2}} \right\rangle_J \sqrt{\left\langle w \right\rangle_J} \leq \sqrt{\left\langle w^{-1} \right\rangle_J \left\langle w \right\rangle_J} \leq [w]_{A_2}^{\frac{1}{2}},$$

and so

$$|S_1| \lesssim [w]_{A_2}^{\frac{1}{2}} \left\| \phi \right\|_{L^2} \left\| \psi w^{-\frac{1}{2}} \right\|_{L^2(w)} = [w]_{A_2}^{\frac{1}{2}} \left\| \phi \right\|_{L^2} \left\| \psi \right\|_{L^2}.$$

For term S_2 we have

$$
\begin{aligned}
|S_2| &= \left| \sum_{J\in\mathcal{D}} \widehat{\phi}(J) \left\langle w^{-\frac{1}{2}} \right\rangle_J D_{J_-}(w) \left\langle w \right\rangle_{J_-} \mathbb{E}^w_{J_-} \left(\psi w^{-\frac{1}{2}} \right) \right| \\
&\leq \|\phi\|_{L^2} \left(\sum_{J\in\mathcal{D}} \left\langle w^{-\frac{1}{2}} \right\rangle_J^2 D_{J_-}(w)^2 \left\langle w \right\rangle_{J_-}^2 \mathbb{E}^w_{J_-} \left(\psi w^{-\frac{1}{2}} \right)^2 \right)^{\frac{1}{2}} \\
&= \|\phi\|_{L^2} \left(\sum_{J\in\mathcal{D}} \left\langle w^{-\frac{1}{2}} \right\rangle_J^2 \frac{\widehat{w}(J_-)^2}{\left\langle w \right\rangle_{J_-}^2} \left\langle w \right\rangle_{J_-}^2 \mathbb{E}^w_{J_-} \left(\psi w^{-\frac{1}{2}} \right)^2 \right)^{\frac{1}{2}} \\
&= \|\phi\|_{L^2} \left(\sum_{J\in\mathcal{D}} \left\langle w^{-\frac{1}{2}} \right\rangle_J^2 \widehat{w}(J_-)^2 \mathbb{E}^w_{J_-} \left(\psi w^{-\frac{1}{2}} \right)^2 \right)^{\frac{1}{2}} \\
&\lesssim [w]_{A_2} \|\phi\|_{L^2} \|\psi\|_{L^2}
\end{aligned}
$$

with the last estimate following from the Carleson Embedding Theorem and the linear bound for the modified square function (2.10).

3.2.3. *Estimating Term* (3.10). We are left with analyzing $\mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}}$ as an operator on L^2. But, by the analysis above we have from (3.1) that

$\mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} = M_{w^{\frac{1}{2}}} \mathcal{S} M_{w^{-\frac{1}{2}}} - (3.2) - (3.3) - (3.4) - (3.5) - (3.6) - (3.7) - (3.8) - (3.9).$

And, using all the estimates above, we find that

$$
\left\| \mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} \right\|_{L^2\to L^2} \lesssim [w]_{A_2} + \left\| M_{w^{\frac{1}{2}}} \mathcal{S} M_{w^{-\frac{1}{2}}} \right\|_{L^2\to L^2}.
$$

Now using Petermichl's result, Theorem 1.1, (see also [1–3]) we have that $\left\| M_{w^{\frac{1}{2}}} \mathcal{S} M_{w^{-\frac{1}{2}}} \right\|_{L^2\to L^2} \lesssim [w]_{A_2}$. This combines to give:

$$
\left\| \mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} \right\|_{L^2\to L^2} \lesssim [w]_{A_2}
$$

as required.

We pose a question that we are unable to resolve as of this writing. Resolution of this question would provide an alternate proof of the linear bound of the Hilbert transform.

Question 3.1. *Can one give a direct proof that*

$$
\left\| \mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} \right\|_{L^2\to L^2} \lesssim [w]_{A_2}
$$

without resorting to the use of Theorem 1.1? A direct proof of the above estimate would provide an alternate proof of Theorem 1.1.

3.3. **An Alternate Approach to Estimating Term (3.10).** In this final section we show that the tools of this paper are almost sufficient to provide a direct proof of (3.10).

Fix $\phi, \psi \in L^2$. Now, observe that we have

$$\begin{aligned}\left\langle \mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} \phi, \psi \right\rangle_{L^2} &= \left\langle \mathcal{S} \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} \phi, \mathsf{P}^{(1,0)}_{\widehat{w^{\frac{1}{2}}}} \psi \right\rangle_{L^2} \\ &= \sum_{J,L \in \mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J) \widehat{w^{\frac{1}{2}}}(L) \widehat{\phi}(J) \widehat{\psi}(L) \left\langle \mathcal{S} h^1_J, h^1_L \right\rangle_{L^2}.\end{aligned}$$

We then further specialize to the case when $J \cap L = \emptyset$ and when $J \cap L \neq \emptyset$. The case when $J \cap L \neq \emptyset$ can be handled via the techniques of this paper. The case when $J \cap L = \emptyset$ requires a new idea.

3.3.1. *Estimating Term* (3.10)*:* $J \cap L \neq \emptyset$. We can then split this into three sums, when $L = J$, when $L \subsetneq J$ and when $J \subsetneq L$. Doing so, we see that

$$\begin{aligned}\left\langle \mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}} \mathcal{S} \mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}} \phi, \psi \right\rangle_{L^2} &= \sum_{J \in \mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J) \widehat{w^{\frac{1}{2}}}(J) \widehat{\phi}(J) \widehat{\psi}(J) \left\langle \mathcal{S} h^1_J, h^1_J \right\rangle_{L^2} \\ &\quad + \sum_{J \in \mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J) \widehat{\phi}(J) \left(\sum_{L \subsetneq J} \widehat{w^{\frac{1}{2}}}(L) \widehat{\psi}(L) \left\langle \mathcal{S} h^1_J, h^1_L \right\rangle_{L^2} \right) \\ &\quad + \sum_{L \in \mathcal{D}} \widehat{w^{\frac{1}{2}}}(L) \widehat{\psi}(L) \left(\sum_{J \subsetneq L} \widehat{w^{-\frac{1}{2}}}(J) \widehat{\phi}(J) \left\langle \mathcal{S} h^1_J, h^1_L \right\rangle_{L^2} \right) \\ &= T_1 + T_2 + T_3.\end{aligned}$$

Consider now term T_1 and observe

$$\begin{aligned}|T_1| &= \left| \sum_{J \in \mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J) \widehat{w^{\frac{1}{2}}}(J) \widehat{\phi}(J) \widehat{\psi}(J) \left\langle \mathcal{S} h^1_J, h^1_J \right\rangle_{L^2} \right| \\ &\leq \sum_{J \in \mathcal{D}} \left| \widehat{w^{-\frac{1}{2}}}(J) \widehat{w^{\frac{1}{2}}}(J) \widehat{\phi}(J) \widehat{\psi}(J) \right| \left\| h^1_J \right\|^2_{L^2} \\ &\leq \sum_{J \in \mathcal{D}} \left| \frac{\widehat{w^{-\frac{1}{2}}}(J) \widehat{w^{\frac{1}{2}}}(J)}{|J|} \widehat{\phi}(J) \widehat{\psi}(J) \right| \\ &\leq [w]^{\frac{1}{2}}_{A_2} \left\| \phi \right\|_{L^2} \left\| \psi \right\|_{L^2}.\end{aligned}$$

Here we make an obvious estimate using the definition of A_2, Cauchy-Schwarz and that $\mathcal{S} : L^2 \to L^2$ has norm at most one.

There is a symmetry between terms T_2 and T_3, so we only handle term T_2. We first make a computation of $\left\langle \mathcal{S} h^1_J, h^1_L \right\rangle_{L^2}$ when $L \subsetneq J$. Set

$$\mathfrak{s}(J) \equiv \sqrt{2} \sum_{K \in \mathcal{D}: K_- \supsetneq J} \frac{1}{|K|}.$$

Letting $h_K(J) \equiv h_K(c(J))$ wherein $c(J)$ is the center of the dyadic interval J and using the definition of $\mathcal{S}$ it is a straightforward computation to show that

$$\begin{aligned}\left\langle \mathcal{S}h_J^1, h_L^1\right\rangle_{L^2} &= \sum_{K\supsetneq J} h_K(J)\left\langle h_L^1, h_{K_-}\right\rangle_{L^2}\\ &= \sum_{\substack{K_-\supsetneq L\\ K\supsetneq J}} h_K(J)h_{K_-}(L)\\ &= \sum_{\substack{K_-\supsetneq L\\ K\supsetneq J}} \frac{\delta(J,K)}{|K|^{\frac12}}\frac{\delta(L,K_-)}{|K_-|^{\frac12}}\\ &= \sqrt{2}\sum_{\substack{K_-\supsetneq L\\ K\supsetneq J}} \frac{\delta(J,K)\delta(L,K_-)}{|K|}\\ &= \sqrt{2}\sum_{K\in\mathcal{D}:K_-\supsetneq J}\frac{1}{|K|}\equiv \mathfrak{s}(J).\end{aligned}$$

We have used that $|K_-| = \frac12 |K|$, and that $\delta(J,K) = \delta(L,K_-)$ since $L \subsetneq J \subsetneq K$. Note that we have

$$\mathfrak{s}(J)\le \sqrt{2}\sum_{K\supsetneq J}\frac{1}{|K|}=\frac{\sqrt{2}}{|J|}.$$

Now define the function $v_J \equiv w^{\frac12}\left(\mathbf{1}_{J_-}-\mathbf{1}_{J_+}\right)$ and observe

$$\widehat{v_J}(L)=\begin{cases}\widehat{w^{\frac12}}(L) & : \quad L\subset J_-\\ -\widehat{w^{\frac12}}(L) & : \quad L\subset J_+.\end{cases}$$

Then using the product formula we have

$$\begin{aligned}\sum_{L\subsetneq J}\widehat{w^{\frac12}}(L)\widehat{\psi}(L)\left\langle \mathcal{S}h_J^1, h_L^1\right\rangle_{L^2} &= \mathfrak{s}(J)\sum_{L\subsetneq J}\widehat{w^{\frac12}}(L)\widehat{\psi}(L)\\ &= \mathfrak{s}(J)\left(\sum_{L\subset J_-}\widehat{w^{\frac12}}(L)\widehat{\psi}(L)-\sum_{L\subset J_+}(-\widehat{w^{\frac12}}(L))\widehat{\psi}(L)\right)\\ &= \mathfrak{s}(J)\,|J|^{\frac12}\,\frac{1}{|J|^{\frac12}}\sum_{L\subsetneq J}\widehat{\psi}(L)\widehat{v_J}(L)\delta(L,J)\\ &= \mathfrak{s}(J)\,|J|^{\frac12}\left(\widehat{v_J\psi}(J)-\widehat{\psi}(J)\left\langle v_J\right\rangle_J-\widehat{v_J}(J)\left\langle\psi\right\rangle_J\right).\end{aligned}$$

However, simple computations show that

$$\begin{aligned}\left\langle v_J\right\rangle_J &= |J|^{-\frac12}\,\widehat{w^{\frac12}}(J)\\ \widehat{v_J}(J) &= |J|^{\frac12}\left\langle w^{\frac12}\right\rangle_J\\ \widehat{\psi v_J}(J) &= |J|^{\frac12}\left\langle \psi w^{\frac12}\right\rangle_J.\end{aligned}$$

Thus, we have

$$\sum_{L\subsetneq J}\widehat{w^{\frac12}}(L)\widehat{\psi}(L)\left\langle \mathcal{S}h_J^1, h_L^1\right\rangle_{L^2}=\mathfrak{s}(J)\,|J|\left\langle\psi w^{\frac12}\right\rangle_J-\mathfrak{s}(J)\widehat{w^{\frac12}}(J)\widehat{\psi}(J)-\mathfrak{s}(J)\,|J|\left\langle w^{\frac12}\right\rangle_J\left\langle\psi\right\rangle_J.$$

This then yields that

$$\begin{aligned}
T_2 &= \sum_{J\in\mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J)\widehat{\phi}(J)\left(\sum_{L\subsetneq J}\widehat{w^{\frac{1}{2}}}(L)\widehat{\psi}(L)\left\langle \mathcal{S}h_J^1, h_L^1\right\rangle_{L^2}\right)\\
&= \sum_{J\in\mathcal{D}} \widehat{w^{-\frac{1}{2}}}(J)\widehat{\phi}(J)\left(\mathfrak{s}(J)\left|J\right|\left\langle \psi w^{\frac{1}{2}}\right\rangle_J - \mathfrak{s}(J)\widehat{w^{\frac{1}{2}}}(J)\widehat{\psi}(J) - \mathfrak{s}(J)\left|J\right|\left\langle w^{\frac{1}{2}}\right\rangle_J\left\langle\psi\right\rangle_J\right)\\
&\equiv S_1+S_2+S_3.
\end{aligned}$$

Using the estimate that $\mathfrak{s}(J)\lesssim\left|J\right|^{-1}$, it is immediate to see that

$$\left|S_2\right|\leq [w]_{A_2}^{\frac{1}{2}}\left\|\phi\right\|_{L^2}\left\|\psi\right\|_{L^2}.$$

It is also an easy application of the Carleson Embedding Theorem, again using the linear bound for the square function and that $\mathfrak{s}(J)\left|J\right|\lesssim 1$, to see that

$$\left|S_3\right|\lesssim\left\|\phi\right\|_{L^2}\left(\sum_{J\in\mathcal{D}}\widehat{w^{-\frac{1}{2}}}(J)^2\left\langle w^{\frac{1}{2}}\right\rangle_J^2\left\langle\psi\right\rangle_J^2\right)^{\frac{1}{2}}\lesssim [w]_{A_2}\left\|\phi\right\|_{L^2}\left\|\psi\right\|_{L^2}.$$

Finally, note that for term S_1, by an application of Cauchy-Schwarz and again that $\mathfrak{s}(J)\left|J\right|\lesssim 1$, we have

$$\begin{aligned}
\left|S_1\right| &= \left|\sum_{J\in\mathcal{D}}\mathfrak{s}(J)\left|J\right|\widehat{w^{-\frac{1}{2}}}(J)\widehat{\phi}(J)\left\langle\psi w^{\frac{1}{2}}\right\rangle_J\right|\\
&= \left|\sum_{J\in\mathcal{D}}\mathfrak{s}(J)\left|J\right|\widehat{w^{-\frac{1}{2}}}(J)\widehat{\phi}(J)\left\langle w\right\rangle_J\mathbb{E}_J^w\left(\psi w^{-\frac{1}{2}}\right)\right|\\
&\lesssim \left\|\phi\right\|_{L^2}\left(\sum_{J\in\mathcal{D}}\widehat{w^{-\frac{1}{2}}}(J)^2\left\langle w\right\rangle_J^2\mathbb{E}_J^w\left(\psi w^{-\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}.
\end{aligned}$$

We claim that the following Carleson estimate holds:

$$\sum_{K\subset L}\widehat{w^{-\frac{1}{2}}}(K)^2\left\langle w\right\rangle_K^2\lesssim [w]_{A_2}^2\, w(L)\quad\forall L\in\mathcal{D}. \tag{3.13}$$

Assuming (3.13) holds, we can apply the Carleson Embedding Theorem to conclude that

$$\left|S_1\right|\lesssim [w]_{A_2}\left\|\phi\right\|_{L^2}\left\|\psi w^{-\frac{1}{2}}\right\|_{L^2(w)} = [w]_{A_2}\left\|\phi\right\|_{L^2}\left\|\psi\right\|_{L^2}.$$

Then combining the estimates on S_j with $j=1,2,3$ gives that

$$\left|T_2\right|\lesssim [w]_{A_2}\left\|\phi\right\|_{L^2}\left\|\psi\right\|_{L^2}.$$

Further, combining the estimates on T_j for $j=1,2,3$ gives that

$$\sup_{\phi,\psi\in L^2}\left|\left\langle \mathsf{P}^{(0,1)}_{\widehat{w^{\frac{1}{2}}}}\mathcal{S}\mathsf{P}^{(1,0)}_{\widehat{w^{-\frac{1}{2}}}}\phi,\psi\right\rangle_{L^2}\right|\lesssim [w]_{A_2}\left\|\phi\right\|_{L^2}\left\|\psi\right\|_{L^2},$$

which is the desired estimate.

Thus it only remains to demonstrate (3.13). Fix L and start with a standard Calderón–Zygmund stopping time argument on the function w.

Let $Q_1^0 \equiv L$. Fix $\gamma > 1$ large and let $\mathcal{G}_0 = \left\{Q_1^0\right\}$ be the ground zero generation. Then let $\mathcal{G}_1 = \mathcal{G}_1\left(Q_1^0\right) = \left\{Q_j^1\right\}_{j\in\Gamma_1^0}$ be the first generation of maximal dyadic subintervals Q_j^1 of L such that

$$\langle w\rangle_{Q_j^1} > \gamma \langle w\rangle_{Q_1^0}, \qquad j \in \Gamma_1^0.$$

Then for each $i \in \Gamma_1^0$ define $\mathcal{G}_2\left(Q_i^1\right) = \left\{Q_j^2\right\}_{j\in\Gamma_i^1}$ to be the collection of maximal dyadic subintervals Q_j^2 of Q_i^1 such that

$$\langle w\rangle_{Q_j^2} > \gamma \langle w\rangle_{Q_i^1}, \qquad j \in \Gamma_i^1.$$

Set $\mathcal{G}_2 = \bigcup\limits_{Q_i^1\in\mathcal{G}_1} \mathcal{G}_2\left(Q_i^1\right)$ to be the second generation of subintervals of L. Continue by recursion to define the k^{th} generation $\mathcal{G}_k$ of subintervals of L by $\mathcal{G}_k \equiv \bigcup\limits_{Q_i^{k-1}\in\mathcal{G}_{k-1}} \mathcal{G}_k\left(Q_i^{k-1}\right)$ where $\mathcal{G}_k\left(Q_i^{k-1}\right) = \left\{Q_j^k\right\}_j$ is the collection of maximal dyadic subintervals Q_j^k of Q_i^{k-1} such that

$$\langle w\rangle_{Q_j^k} > \gamma \langle w\rangle_{Q_i^{k-1}}, \qquad j \in \Gamma_i^{k-1}.$$

Then define the corona

$$\mathcal{C}_{Q_j^k} = \left\{K \subset Q_j^k : K \not\subset Q_i^{k+1} \text{ for any } Q_i^{k+1} \in \mathcal{G}_{k+1}\right\},$$

and denote by $\mathcal{G} = \bigcup\limits_{k=0}^{\infty} \mathcal{G}_k$ the collection of all stopping intervals in L. Within each corona $\mathcal{C}_G$ for $G \in \mathcal{G}$, we get

$$\begin{aligned}
\sum_{K\in\mathcal{C}_G} \widehat{w^{-\frac{1}{2}}}(K)^2 \langle w\rangle_K^2 &\lesssim \gamma \left(\sum_{K\in\mathcal{C}_G} \widehat{w^{-\frac{1}{2}}}(K)^2\right) \langle w\rangle_G^2 \\
&\leq \gamma \left(\int_G \left|w^{-\frac{1}{2}} - \left\langle w^{-\frac{1}{2}}\right\rangle_G\right|^2\right) \langle w\rangle_G^2 \\
&\leq \gamma \int_G \langle w\rangle_G^2\, w^{-1}.
\end{aligned}$$

Summing up over all the coronas $\mathcal{C}_G$ for $G \in \mathcal{G}$ with $\gamma = 2$ we get

$$\begin{aligned}
\sum_{K\subset L} \widehat{w^{-\frac{1}{2}}}(K)^2 \langle w\rangle_K^2 &\lesssim \sum_{G\in\mathcal{G}} \int_G \langle w\rangle_G^2 w^{-1} \\
&= \sum_{k,j} \left|Q_j^k\right|_{w^{-1}} \left(\frac{1}{\left|Q_j^k\right|}\int_{Q_j^k} w\right)^2 \\
&= \int_{\mathbb{R}} \sum_{k,j} \left(\frac{1}{\left|Q_j^k\right|}\int_{Q_j^k} w\right)^2 \mathbf{1}_{Q_j^k}(x)\, w^{-1}(x)\, dx \\
&\approx \int_{\mathbb{R}} \left(\sup_{k.j:\ x\in Q_j^k} \frac{1}{\left|Q_j^k\right|}\int_{Q_j^k} w\right)^2 w^{-1}(x)\, dx \\
&\lesssim \int_L |Mw|^2 w^{-1} \lesssim [w]_{A_2}^2 \int_L w^2 w^{-1} = [w]_{A_2}^2 \int_L w,
\end{aligned}$$

which gives (3.13). Note that we used that the sequence of numbers

$$\left\{\frac{1}{\left|Q_j^k\right|}\int_{Q_j^k} w\right\}_{k,j:\ x\in Q_j^k}$$

is super-geometric in the sense that consecutive terms have ratio exceeding $\gamma = 2$, and so the sum of their squares is essentially the square of the largest one.

3.3.2. *Estimating Term* (3.10)*:* $J \cap L = \emptyset$. We must consider this case since the Haar shift is not a completely local operator. It instead sees some long range interaction, and we need to account for this since there can be terms for which $L \cap J = \emptyset$ and $\left\langle \mathcal{S}h_J^1, h_L^1\right\rangle_{L^2} \neq 0$. A simple example occurs when L and J are dyadic brothers. Fortunately, in the case of the identity operator, this term does not appear and the proof would terminate.

The tools used in this paper currently appear to be unable to resolve this term. We thus pose a refined version of Question 3.1

Question 3.2. *Is the following estimate*

$$\left\| \sum_{J,L\in\mathcal{D}:J\cap L=\emptyset} \widehat{w^{-\frac{1}{2}}}(J)\widehat{w^{\frac{1}{2}}}(L)\widehat{\phi}(J)\widehat{\psi}(L) \left\langle \mathcal{S}h_J^1, h_L^1\right\rangle_{L^2} \right\|_{L^2\to L^2} \lesssim [w]_{A_2}$$

true?

We again know that this estimate is true by using Theorem 1.1, but are interested in a direct resolution of Question 3.2. A direct proof of the estimate above would provide a new proof of Theorem 1.1.

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