BOOLEAN ALGEBRAS OF PROJECTIONS IN LOCALLY CONVEX SPACES

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INTRODUCTION

The object of this paper is to report recent work with Werner Ricker on algebras of scalar-type spectral operators in locally convex spaces. The details will appear elsewhere [3]. Our motivation lies in the penetrating study by W.G. Bade [1], [2] (see also [4]) of operator algebras generated by Boolean algebras of projections in Banach spaces. At the heart of Bade's work lies the Bartle - Dunford - Schwartz theory of Banach space-valued measures. Consequently, a natural avenue of approach in the present setting is via the study of spectral measures in locally convex spaces, a study initiated by Schaefer [11] and further developed by Walsh [12] and, more recently, by Ricker [9]. In the study of operator algebras, there are, however, new difficulties which arise in the locally convex setting. In the resolution of these difficulties, a decisive role is played by the notion of closed vector measure, introduced by Kluvánek [5].

1. BOOLEAN ALGEBRAS OF PROJECTIONS

To fix notation, X will denote a locally convex space which is assumed to be quasi-complete. $L(X)$ will denote the space of continuous linear operators on X equipped with the topology of pointwise convergence on X. $L(X)$ is assumed to be sequentially complete. A subset $\mathcal{B}$ of $L(X)$ is called equicontinuous if and only if for each continuous semi-norm q on X, there exists a continuous semi-norm p on X such that

$q(Bx) \leq p(x)$ for all $B \in \mathcal{B}$ and all $x \in X$. A Boolean algebra of projections in X is a family $\mathcal{B}$ of commuting indempotents in $L(X)$ which is partially ordered with respect to range inclusion and which is a Boolean algebra with respect to the lattice operations defined by setting $Q \vee R = Q + R - QR$ and $Q \wedge R = QR$ for all Q,R in $\mathcal{B}$. It is assumed that the unit element of $\mathcal{B}$ is the identity operator I.

An equicontinuous Boolean algebra of projections in X will be called strongly equicontinuous if and only if $Q_n \longrightarrow O$ in $L(X)$ whenever the sequence $\{Q_n\} \subseteq \mathcal{B}$ is pairwise disjoint.

We may now state our principal result, which is a reflexivity theorem for the closed algebras generated by strongly equicontinuous Boolean algebras of projections and which is (essentially) due to Bade [1] for the case that X is Banach.

THEOREM 1.1. *If* $\mathcal{B}$ *is a strongly equicontinuous Boolean algebra of projections in* X, *then the closed algebra generated by* $\mathcal{B}$ *in* $L(X)$ *consists precisely of those continuous linear operators on* X *which leave invariant each* $\mathcal{B}$*-invariant (closed) subspace of* X.

The proof of the preceding theorem contains several ingredients which are of interest in their own right. It is necessary to introduce some further terminology. The equicontinuous Boolean algebra $\mathcal{B}$ is called Bade σ-complete if and only if $\mathcal{B}$ is σ-complete as an abstract Boolean algebra and whenever $\{Q_n\}$ is a decreasing sequence of elements of $\mathcal{B}$ for which $inf_n Q_n = O$ it follows that $Q_n \longrightarrow O$. Similarly, the equicontinuous Boolean algebra $\mathcal{B}$ is called Bade-complete if and only if $\mathcal{B}$ is complete as an abstract Boolean algebra and whenever $\{Q_\tau\}$ is a downwards filtering system of elements of $\mathcal{B}$ for which $inf_\tau\, Q_\tau = O$, it follows that the net $\{Q_\tau\}$ converges to O. We remark that each Bade complete Boolean algebra of projections is Bade σ-complete and that each Bade σ-complete Boolean algebra of projections is strongly equicontinuous. The precise relation between strong equicontinuity and Bade-completeness now follows.

THEOREM 1.2 *If* $\mathcal{B}$ *is an equicontinuous Boolean algebra of projections in* X, *then the following statements are equivalent.*

(i) $\mathcal{B}$ *is strongly equicontinuous.*

(ii) *The closure of* $\mathcal{B}$ *in* $L(X)$ *is a Bade-complete Boolean algebra of projections in* X.

If $\mathcal{B}$ *is a strongly equicontinuous Boolean algebra of projections in* X, *then each projection in the closed algebra generated by* $\mathcal{B}$ *lies in the closure of* $\mathcal{B}$.

If $\mathcal{B}$ *is a Bade-complete Boolean algebra of projections in* X, *then* $\mathcal{B}$ *is a topologically complete subset of* $L(X)$.

We remark that the implication (i) $\Rightarrow$ (ii) of the preceding theorem is due to Walsh [12] for Boolean algebras of projections which are Bade σ-complete. The practical effect of the theorem is that only Boolean algebras of projections with very strong completeness properties need be considered.

If $x \in X$, and if $\mathcal{B}$ is an equicontinuous Boolean algebra of projections in X, then $\mathcal{B}[x]$, the cyclic subspace generated by x, is the smallest closed subspace of X which contains x and is $\mathcal{B}$-invariant. The second ingredient in the reflexivity theorem is an analysis of the structure of cyclic subspaces, in which closed densely-defined operators with very special properties enter quite naturally. The essential structure is given by the following result.

THEOREM 1.3 *Let* $\mathcal{B}$ *be a Bade-complete Boolean algebra of projections in* X *and let* $x \in X$. *If* $y \in \mathcal{B}[x]$, *then there exists a closed, densely defined linear operator* T *with domain* $D(T)$ *such that*

(i) $x \in D(T)$ *and* $y = Tx$.

(ii) *There exists an increasing sequence* $\{Q_n\} \subseteq \mathcal{B}$ *such that* $\sup_n Q_n = I$ *and such that* TQ_n *is in the closed algebra generated by* $\mathcal{B}$ *for each* $n = 1,2,\ldots$ *Moreover* $z \in D(T)$ *if and only if* $\lim_n TQ_n z$ *exists in* X, *in which case* $Tz = \lim_n TQ_n z$.

(iii) T *leaves invariant each closed* $\mathcal{B}$-*invariant subspace of* X.

It is quite simple to see that the reflexivity theorem is a direct consequence of theorem 1.3 for the very special case that X has a cyclic vector, i.e. when there exists an element $x \in X$ with the property that $\mathcal{B}[x] = X$. If X does not have a cyclic vector, then a more delicate argument is required, based once again on the structure theorem 1.3 combined with some of the original ideas of Bade.

2. SPECTRAL MEASURES

In this section we will indicate how the general theory of vector measures can be effectively applied to prove the basic structure theorem 1.3 in an efficient and transparent way. A by-product of this aim is a number of new results concerning the structure of spectral measures in locally convex spaces.

We recall first some terminology from the general theorem of vector-valued measures [8]. An X-valued vector measure is a σ-additive map $m : \mathcal{M} \longrightarrow X$ whose domain is a σ-algebra $\mathcal{M}$ of subsets of some set Ω.

If X' is the dual space of X, then, for each $x' \in X'$, the complex-valued measure $E \longrightarrow \langle m(E), x' \rangle$, $E \in M$ will be denoted by $x' \circ m$ and its variation by $|x' \circ m|$. A complex-valued, M-measurable function f is said to be m-integrable if and only if f is integrable with respect to each scalar measure $x' \circ m$, $x' \in X'$ and if, for each $E \in M$, there exists an element, denoted $\int_E fdm$, of X such that

$$\langle \int_E fdm, x' \rangle = \int_E fd(x' \circ m)$$

for each $x' \in X'$. If f is m-integrable, the measure $E \longrightarrow \int_E fdm$, $E \in M$ is denoted by fm.

If q is a continuous semi-norm on X and if f is m-integrable, we define

$$q(m)(f) = \sup\{|x' \circ fm|(\Omega): \ x' \in U_q^{\circ}\}$$

where $U_q^{\circ} \subseteq X'$ is the polar of the q-unit ball $U_q \subseteq X$.

If $L(m)$ denotes the space of m-integrable functions, then $L(m)$ equipped with the family of semi-norms $f \longrightarrow q(m)(f)$ is a locally convex space. The corresponding Hausdorff quotient space will be denoted by $L^1(m)$. In general $L^1(m)$ is not topologically complete. Measures m for which $L^1(m)$ is topologically complete are called closed. Such measures were introduced and characterised by Kluvánek (see [18]).

A spectral measure on X is a countably additive map $P: M \longrightarrow L(X)$, whose domain M is a σ-algebra of subsets of some set Ω, and which is multiplicative and satisfies $P(\Omega) = I$. The spectral measure P is called equicontinuous if the range of P is an equicontinuous subset of $L(X)$. The range of an equicontinuous spectral measure is a Bade σ-complete Boolean algebra B of projections in X. Conversely, each Bade

σ-complete Boolean algebra $\mathcal{B}$ of projections in X is the range of an equicontinuous spectral measure defined on the Baire subsets of the Stone space of $\mathcal{B}$.

Let P: $\mathcal{M} \longrightarrow L(X)$ be an equicontinuous spectral measure on X. If $x \in X$, then Px will denote the X-valued vector measure $E \longrightarrow P(E)x$, $E \in \mathcal{M}$. Let now f by <u>any</u> complex-valued $\mathcal{M}$-measurable function. We define the linear map $\int_\Omega fdP$ as follows. The domain $D\left(\int_\Omega fdP\right)$ of $\int_\Omega fdP$ is the linear subspace consisting of elements $x \in X$ for which f is $x' \circ Px$ integrable for each $x' \in X'$ and for which there exists a uniquely determined element of X, denoted $\left(\int_\Omega fdP\right)(x)$ such that

$$\langle \left(\int_\Omega fdP\right)x, x'\rangle = \int_\Omega fd(x' \circ Px)$$

The mapping $\int_\Omega fdP$ is a closed, densely defined linear map in X which commutes with the spectral measure P in the sense $D\left(\int_\Omega fdP\right)$ is P-invariant and

$$\left(\int_\Omega fdP\right)P(E)x = P(E)\left(\int_\Omega fdP\right)x, \; x \in D\left(\int_\Omega fdP\right)$$

If $\int_\Omega fdP$ is an element of $L(X)$, then it may be shown that f is P-integrable and the integral of f with respect to P is precisely $\int_\Omega fdP$, so there is no possibility of confusion of notation. Further if $x \in X$, then f is Px-integrable if and only if $x \in D\left(\int_\Omega fdP\right)$, and in this case, we have $\int_\Omega fdPx = \left(\int_\Omega fdP\right)x$.

Let P be an equicontinuous spectral measure in X, let q be a continuous semi-norm on X and let $x \in X$. From the fact that P is equicontinuous and multiplicative, it follows that there exists a continuous semi-norm p on X such that

(1) $$q\left(\left(\int_\Omega fdP\right)x\right) \le q(P)(f) \le 4p\left(\left(\int_\Omega fdP\right)x\right)$$

for all f integrable with respect to P. Similarly, there exists a continuous semi-norm p on X such that

(2) $$q\left(\int fdPx\right) \le q(Px)(f) \le 4p\left(\int fdPx\right)$$

for all f integrable with respect to the vector measure Px. The estimate (1) asserts that the integration map

$$f \longrightarrow \int_\Omega fdP, \qquad f \in L^1(P)$$

is a topological isomorphism. Similarly, the estimate (2) asserts that the integration map

$$f \longrightarrow \int_\Omega fdPx, \qquad f \in L^1(Px)$$

is also a topological isomorphism.

We have remarked that, for arbitrary vector measures m, the locally convex space $L^1(m)$ is not in general topologically complete. The extent to which this pathology is not shared by equicontinuous spectral measures is indicated by the following result, which is the key to our present approach.

THEOREM 2.1 *If P is an equicontinuous spectral measure in X and if the range of P is a Bade complete Boolean algebra of projections in X then the locally convex spaces $L^1(P)$, $L^1(Px)$, $x \in X$, are topologically complete.*

We remark that, while the proof of the theorem is not completely trivial, it can be based quite effectively on criteria given by Kluvánek for topological completeness of spaces of type $L^1(m)$, [7], [8].

The following representation theorems are now immediate consequences of the topological completeness of the spaces $L^1(P)$, $L^1(Px)$. We remark that it may be shown that $L^1(P)$ is an algebra.

THEOREM 2.2 *Let* $\mathcal{B}$ *be a Bade complete Boolean algebra of projections in* X, *displayed as the range of an equicontinuous spectral measure* P *defined on the Borel subsets of the Stone space* Ω *of* $\mathcal{B}$.

(i) *The integration map*

$$f \longrightarrow \int_\Omega f dP, \quad f \in L^1(P)$$

is a topological (and algebra) isomorphism of $L^1(P)$ *onto the closed algebra generated by* $\mathcal{B}$.

(ii) *The integration map*

$$f \longrightarrow \int_\Omega f dPx, \quad f \in L^1(Px)$$

is a topological isomorphism of $L^1(Px)$ *onto the cyclic subspace* $\mathcal{B}[x]$, *for each* $x \in X$.

(iii) *For each* $x \in X$

$$\mathcal{B}[x] = \left\{ \left(\int_\Omega f dP \right)(x) : \; f \text{ is Borel and } x \in D\left(\int_\Omega f dP \right) \right\}.$$

Part (i) of the preceding theorem is due to Ricker [10]. It implies that the closed algebra generated by $\mathcal{B}$ consists entirely of scalar type spectral operators. The representation theorem for cyclic subspaces given by part (ii) is new, even for the case that X is Banach and easily yields part (iii). For the case that X is Banach, part (iii) was given by Bade [2]. However, the Banach space methods used by Bade fail, even for the case that X is metrizable, as has been pointed out by Walsh [12], who also obtained (ii) for the case of metrizable X. The present approach, based on the theory of closed vector measures, is quite different and yields considerable simplification of technique. Finally, it is easily seen that part (iii) yields Theorem 1.3 of the preceding section.

We will conclude by stating a result which underlines the very special nature of closed algebras generated by Bade complete Boolean algebras of projections. Once again suppose that $\mathcal{B}$ is a Bade complete Boolean algebra of projections in X, displayed as the range of an equicontinuous spectral measure P on the Stone space Ω of $\mathcal{B}$. Let $\mathcal{S}$ be the subalgebra of $L^1(P)$ determined by the P-integrable subsets of Ω. $\mathcal{S}$ is itself a Boolean algebra and the restriction of the integration map to $\mathcal{S}$ is a Boolean algebra isomorphism of $\mathcal{S}$ onto $\mathcal{B}$. Since $L^1(P)$ is an order complete (complex) vector lattice, we obtain the following result.

<u>THEOREM 2.3</u> *Let* $\mathcal{B}$ *be a Bade complete Boolean algebra of projections in* X *and let* P *be the associated equicontinuous spectral measure on the Stone space* Ω *of* $\mathcal{B}$. *The integration map*

$$f \longrightarrow \int_{\Omega} f dP, \qquad f \in L^1(P)$$

induces on the closed algebra generated by $\mathcal{B}$ *the structure of an order complete, topologically complete complex vector lattice whose topology is defined by order continuous lattice semi-norms and which contains* $\mathcal{B}$ *as a closed, order complete, complemented sublattice.*

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