NOTE ON (SUPER) HEAVY SUBSETS IN SYMPLECTIC MANIFOLDS

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Abstract. In the first part of this paper, we use symplectic cohomology to construct partial symplectic quasi-states on symplectic manifolds with boundaries of contact type. We apply this partial symplectic quasi-states to the study of symplectic cohomology itself. We prove that a variant of symplectic cohomology $\widetilde{SH}(M)$ vanishes if M is displaceable in its symplectic completion $(\widehat{M}, \widehat{\omega})$. In the second part of this paper, we use finite coverings of closed symplectic manifolds to give a criteria of (super)heavy sets.

1. Introduction

Entov and Polterovich introduced the notion of partial symplectic quasi-states and derived remarkable applications in symplectic geometry, see [4, 5]. In this note, we present some observations on partial symplectic quasi-states and their consequences. Firstly, we recall the definition of a partial symplectic quasi-state on a closed symplectic manifold. Let (M, ω) be a closed symplectic manifold and let $C(M)$ be the space of continuous functions on M. A partial symplectic quasi-states on (M, ω) is a functional

$$\zeta : C(M) \longrightarrow \mathbb{R}$$

which enjoys the following properties ([4, 5]).

(1) (Lipschitz continuity) $|\zeta(F) - \zeta(G)| \leq |F - G|_{C^0}$.
(2) (Semi-homogenuity) $\zeta(\lambda F) = \lambda \zeta(F)$ $(\forall \lambda \in \mathbb{R}_{\geq 0})$.
(3) (Monotinicity) $F \leq G \Longrightarrow \zeta(F) \leq \zeta(G)$.
(4) (Additivity with respect to constants and Normalization) $\zeta(F + a) = \zeta(F) + a$ holds for any $a \in \mathbb{R}$. In particular, $\zeta(1) = 1$.
(5) (Partial additivity) Let F_1 and F_2 be smooth functions on M. If F_1 and F_2 Poisson commute, i.e., $\{F_1, F_2\} = 0$,

$$\zeta(F_1 + F_2) \leq \zeta(F_1) + \zeta(F_2).$$

If, in addition, Supp F_2 is Hamiltonianly displaceable,

$$\zeta(F_1 + F_2) = \zeta(F_1).$$

(6) (Invariance) $\zeta(F) = \zeta(F \circ \varphi)$ holds for any Hamiltonian diffeomorphism φ.

Here we say a subset $A \subset M$ Hamiltonianly displaceable, if there is a Hamiltonian diffeomorphism φ of (M, ω) such that $\varphi(A) \cap \overline{A} = \emptyset$. We also note that these properties imply that $\zeta(F) = 0$ if Supp F is Hamiltonianly displaceable. Once a

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partial symplectic quasi-states ζ is defined, we have the notion of heavy sets and superheavy sets ([4, 5]), see Definition 2.6.

Entov and Polterovich used quantum cohomology ring and spectral invariants of Floer homology to construct partial symplectic quasi-state on closed symplectic manifolds ([4]). And in [5], they studied properties of heavy sets and superheavy sets. In the first part of this paper, we consider symplectic manifolds with boundaries of contact type and their symplectic cohomology. We use symplectic cohomology to construct partial symplectic quasi-states on such symplectic manifolds. There are desirable similarities between partial symplectic quasi-states in Part I of this paper and partial symplectic quasi-states on closed manifolds. We applied heaviness to the study of symplectic cohomology itself. In particular, we proved that if a symplectic manifold with a boundary of contact type (M, ω) is Hamiltonianly displaceable in its symplectic completion $(\widehat{M}, \widehat{\omega})$, then a variant of symplectic cohomology $\widetilde{SH}(M)$, which will be defined in Definition 2.4, vanishes. This is a generalization of Ritter's theorem [13] which states that if a Liouville subdomain (V, λ) in a Liouville domain (W, γ) is Hamiltonian displaceable in W, then symplectic cohomology $SH(V)$ vanishes.

In the second part of this paper, we consider a finite covering of a closed symplectic manifold and give a criteria of (super)heaviness by using this covering. We also present a few examples of superheavy sets detected by this covering trick.

Part 1. **Criterion using symplectic cohomology**

2. Constructions and Main results

We first define symplectic manifolds with boundaries of contact type and their symplectic completions.

Definition 2.1. *The boundary ∂M of a symplectic manifold (M, ω) is called contact type if there is a vector field X which satisfies the following properties.*

- *X is defined on a neighborhood of ∂M*
- $\mathcal{L}_X \omega = \omega$
- *X is outward pointing on ∂M*

Such a vector field X is called Liouville vector field.

As in [17], we define the symplectic completion $(\widehat{M}, \widehat{\omega})$ of (M, ω) by using above Liouville vector field X.

$$\widehat{M} = M \cup_{\partial M} [1, \infty) \times \partial M$$

$$\widehat{\omega}(x) = \begin{cases} \omega(x) & (x \in M) \\ d(r\alpha) & (x = (r, y) \in [1, \infty) \times \partial M) \end{cases}$$

Here α is a 1-form on ∂M defined by $\alpha = \iota_X \omega|_{\partial M}$. We consider a family of pairs of a periodic Hamiltonian function $H \in (S^1 \times \widehat{M})$ and a periodic almost complex structure $J \in C^\infty(S^1, \mathrm{End}(T\widehat{M}))$.

$$\mathcal{H} = \left\{ (H, J) \;\middle|\; \begin{matrix} H(t, (r, y)) = \alpha r + \beta \ (\alpha < 0), \ J(t, (r, y)) : \text{contact type} \\ \text{for any } (r, y) \in [R, \infty) \times \partial M \ (\exists R \geq 1) \end{matrix} \right\}$$

$$\mathcal{H}_{reg} = \{(H, J) \in \mathcal{H} | H : \text{non-degenerate}\}$$

For any relatively compact subset $U \subset \widehat{M}$, we define subsets of $\mathcal{H}$ as follows.

$$\mathcal{H}(U) = \{(H,J) \in \mathcal{H} \mid \inf_{(t,x)\in S^1\times U} H(t,x) > 0\}$$
$$\mathcal{H}_{reg}(U) = \mathcal{H}_{reg} \cap \mathcal{H}(U)$$

We consider the following forgetful map.

$$\mathcal{H}(U) \longrightarrow C^\infty(S^1 \times M)$$
$$(H,J) \longmapsto H$$

Later, we often denote the image of this forgetful map by $\mathcal{H}(U)$, too.

For $(H,J) \in \mathcal{H}$, we consider its contractible periodic orbits and its Novikov covering.

$$P(H) = \{x : S^1 \to M \mid \dot{x}(t) = X_{H_t}(x(t)),\ x : \text{contractible}\}$$
$$\widetilde{P(H)} = \{(x,u) \in C^\infty(D^2, M) \times P(H) \mid \partial u = x\}/\sim$$
$$(x,u) \sim (y,w) \Leftrightarrow \begin{cases} x = y \\ \widehat{\omega}(u\sharp\overline{w}) = 0 \\ c_1(u\sharp\overline{w}) = 0 \end{cases}$$

The action functional A_H is defined for $\widetilde{P(H)}$ by

$$A_H([(x,u)]) = -\int_{D^2} u^*\omega + \int_0^1 H(t,x(t))dt$$

The action spectrum and Hofer norm are defined as follows.

$$\mathrm{Spec}(H) = \{A_H([(x,u)]) \mid [(x,u)] \in \widetilde{P(H)}\}$$
$$||H|| = \int_0^1 \max_x H(t,x) - \min_x H(t,x)dt$$

For $(H,J) \in \mathcal{H}_{reg}$, we consider Floer chain complex and its boundary operator ∂ as follows. (In the situation that $\widehat{M}$ is the completion of M with boundary of contact type, an argument using maximum principle enables us to construct Floer chain complex in a similar way to the case of closed symplectic manifolds, [6] in monotone case, [7] in general.) The underlying module of Floer complex is defined by

$$CF(H,J) = \{ \sum_{\tilde{\ell}\in\widetilde{P(H)}, a_l\in\mathbb{Q}} a_l \cdot \tilde{\ell} \parallel \sharp\{\tilde{\ell} \mid a_l \neq 0, A_H(\tilde{\ell}) < C\} < \infty\ (\forall C \in \mathbb{R})\}.$$

Roughly speaking, the boundary operator ∂ of the complex $CF(H,J)$ is defined by counting *isolated* Floer connecting orbits, i.e., the solutions of the following equation:

$$\frac{\partial v}{\partial s} + J(v)(\frac{\partial v}{\partial t} - X_{H_t}(u)) = 0 \tag{2.1}$$

for $v : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ so that $\lim_{s\to\pm\infty} u(s,\cdot) = \ell_\pm$, where $\tilde{\ell_\pm} = (\ell_\pm, u_\pm) \in \widetilde{P(H)}$ and $(\ell_+, u_+) \sim (\ell_+, v\sharp u_-)$.

Let $n(\tilde{\ell}_-, \tilde{\ell}_+) \in \mathbb{Q}$ be the weighted sign count of Floer connecting orbits from $\tilde{\ell_-}$ to $\tilde{\ell_+}$. Then the boundary operator is given by

$$\partial(\tilde{\ell}_-) = \sum_{\tilde{\ell}_+ \in \widetilde{P(H)}} n(\tilde{\ell}_-, \tilde{\ell}_+) \cdot \tilde{\ell}_+.$$

We denote the homology of $(CF(H,J), \partial)$ by $HF(H,J)$. For any $a \in \mathbb{R}$ or $a < b$, we have a subcomplex and their quotient defined as follows.

$$CF^{<a}(H,J) = \{ \sum_{\tilde{\ell} \in \widetilde{P(H)}, A_H(\tilde{\ell}) < a} a_{\tilde{\ell}} \cdot \tilde{\ell} \in CF(H,J) \}$$

$$CF^{[a,b)}(H,J) = CF^{<b}(H,J)/CF^{<a}(H,J)$$

We denote their homologies by $HF^{<a}(H,J)$ and $HF^{[a,b)}(H,J)$.

Definition 2.2. *We define a partial order on $\mathcal{H}$ as follows.*

$$(H, J_1) \leq (K, J_2) \Longleftrightarrow H \geq K$$

Definition 2.3. *For any relatively compact subset $U \subset \widehat{M}$ and $-\infty \leq a < b \leq \infty$, we define a symplectic cohomology of M with respect to U by*

$$SH^{[a,b)}(\widehat{M} : U) = \varinjlim_{(H,J) \in \mathcal{H}_{reg}(U)} HF^{[a,b)}(H,J).$$

We write

$$SH^{[a,b)}(M) = SH^{[a,b)}(\widehat{M} : M).$$

Remark 2.1. *$SH^{[-\infty,\infty)}(M)$ is isomorphic to the ordinary symplectic cohomology $SH(M)$.*

Next we consider the following type of symplectic cohomology which is appropriate for the construction of partial symplectic quasi-state.

Definition 2.4. *For $-\infty < b \leq \infty$ and $U \subset M$, we define*

$$\widetilde{SH}^{<b}(M : U) = \varprojlim_{a \to -\infty} SH^{[a,b)}(\widehat{M} : U),$$

$$\widetilde{SH}(M : U) = \varinjlim_{b \to \infty} \varprojlim_{a \to -\infty} SH^{[a,b)}(\widehat{M} : U).$$

We write

$$\widetilde{SH}(M) = \widetilde{SH}(M : M) = \varinjlim_{b \to \infty} \varprojlim_{a \to -\infty} SH^{[a,b)}(\widehat{M} : M).$$

We consider the following type of the spaces of continuous functions as in [10].

$$C_c(M) = \{ f : M \to \mathbb{R} \mid \mathrm{Supp}\ f \ \subset \mathrm{Int}(M) \}$$

$$C_{cc}(M) = \{ f : M \to \mathbb{R} \mid \exists C_f \in \mathbb{R} \ \text{ s.t. } \ f - C_f \in C_c(M) \}$$

We also use a similar class of functions on $S^1 \times M$. For any continuous function $F \in C_{cc}(S^1 \times M)$, we define the following subspaces of $\mathcal{H}$.

$$\mathcal{H}(F) = \{ (H,J) \in \mathcal{H} \mid H|_{S^1 \times M} > F \}$$

$$\mathcal{H}_{reg}(F) = \mathcal{H}(F) \cap \mathcal{H}_{reg}$$

We consider the following forgetful map.

$$\mathcal{H}(F) \longrightarrow C^\infty(S^1 \times \widehat{M})$$
$$(H, J) \longmapsto H$$

Later, we often denote the image of this forgetful map by $\mathcal{H}(F)$, too.

We also define symplectic cohomology of M with respect to F as follows.

Definition 2.5. *For continuous function $F \in C_{cc}(S^1 \times M)$ and $a < b$, we define the following cohomologies.*

$$SH^{[a,b)}(M; F) = \varinjlim_{(H,J)\in\mathcal{H}_{reg}(F)} HF^{[a,b)}(H, J)$$
$$\widetilde{SH}^{<b}(M; F) = \varprojlim_{a\to-\infty} SH^{[a,b)}(M : F)$$

Remark 2.2. *We can easily check that there is a canonical isomorphism*

$$\widetilde{SH}(M; F) := \varinjlim_{b\to\infty} \widetilde{SH}^{<b}(M; F) \cong \widetilde{SH}(M).$$

The filtrations are different, in general. In particular, it is the case when $F \notin \mathcal{H}(M)$.

For closed symplectic manifolds, spectral numbers are defined as Floer theoretical min-max argument. We define the spectral number of $F \in C_{cc}(S^1 \times M)$ associated with $e \in \widetilde{SH}(M)\backslash\{0\}$.

$$\rho_e(F) = \inf\{b \in \mathbb{R} \mid e \in \mathrm{Im}(\widetilde{SH}^{<b}(M : F) \to \widetilde{SH}(M))\}$$

Remark 2.3. *The construction of $\widetilde{SH}(M)$ implies that*

$$\rho_e(F) > -\infty$$

holds for any $e \in \widetilde{SH}(M) \setminus \{0\}$ and any such F.

First, we define a product structure on $\widetilde{SH}(M : U)$. The product structure on symplectic cohomology was studied in [1] on cotangent bundles. The construction works in more general setting, see, e.g., [13]. We can define product structure on our symplectic cohomology. We take $(H, J_1), (K, J_2) \in \mathcal{H}_{reg}(U)$ such that $(H\sharp K, J_3) \in \mathcal{H}_{reg}(U)$. Here we set

$$(H\sharp K)(t, x) = H(t, x) + K(t, (\phi_f^t)^{-1}(x)).$$

Then we have the ordinary pair of pants product of Floer chain complex as follows.

$$CF^{<a}(H, J_1) \times CF^{<b}(K, J_2) \longrightarrow CF^{<a+b}(H\sharp K, J_3)$$

For any pairs $a < b$, $c < d$, above chain map induces the following map.

$$HF^{[a,b)}(H, J_1) \times HF^{[c,d)}(K, J_2) \longrightarrow HF^{[b+d-\min\{b-a,d-c\},b+d)}(H\sharp K, J_3)$$

By taking direct limit of such (H, J_1) and (K, J_2), we have a product structure

$$\widetilde{SH}(M : U) \times \widetilde{SH}(M : U) \longrightarrow \widetilde{SH}(M : U)$$

Remark 2.4. *$\widetilde{SH}(M:U)$ has the unit element 1_U. In fact, 1_U is the image of the fundamental class $[M, \partial M]$ under the composition of the following maps.*

$$H_*(M, \partial M) \to SH(M) \to \widetilde{SH}(M:U)$$

¿From this observation, we can see that if $U \subset W \subset M$ and $\widetilde{SH}(M:W) = 0$ holds, then $\widetilde{SH}(M:U) = 0$ holds.

Remark 2.5. *Similar to the case of closed sympectic manifolds, ρ_e induces a map $\rho_e : \widetilde{Ham}^c(M,\omega) \longrightarrow \mathbb{R}$ where $\widetilde{Ham}^c(M,\omega)$ is the universal cover of $Ham^c(M,\omega)$.*

Theorem 2.1. *Suppose that $\widetilde{SH}(M) \neq 0$. Let $e \in \widetilde{SH}(M)$, $e \neq 0$, be an idempotent. Then*

$$\zeta_e(f) = \lim_{k \in \mathbb{N}, k \to +\infty} \frac{\rho_e(kf)}{k}$$

is well defined for $f \in C_{cc}(M)$ and

$$\zeta_e : C_{cc}(M) \longrightarrow \mathbb{R}$$

becomes a partial symplectic quasi-state. In other words, ζ_e satisfies the following properties.

- *(Lipschitz continuity) $|\zeta_e(F) - \zeta_e(G)| \leq |F - G|_{C^0}$.*
- *(Semi-homogenuity) $\zeta_e(\lambda F) = \lambda \zeta_e(F)$ $(\forall \lambda \in \mathbb{R}_{\geq 0})$.*
- *(Monotonicity) $F \leq G \Longrightarrow \zeta_e(F) \leq \zeta_e(G)$.*
- *(Additivity with respect to constants and Normalization) $\zeta_e(F+a) = \zeta_e(F) + a$ holds for any $a \in \mathbb{R}$. In particular, $\zeta_e(1) = 1$.*
- *(Partial additivity) Let F_1 and F_2 be smooth functions in $C_{cc}(M)$. If $\{F_1, F_2\} = 0$, then*

  $$\zeta_e(F_1 + F_2) \leq \zeta_e(F_1) + \zeta_e(F_2).$$

  If, in addition, Supp F_2 *is Hamiltonianly displaceable,*

  $$\zeta_e(F_1 + F_2) = \zeta_e(F_1).$$

  and Supp F_2 *is $Ham^c(M,\omega)$-displaceable, then $\zeta_e(F_1 + F_2) = \zeta_e(F_1)$ holds.*
- *(Invariance) $\zeta_e(F) = \zeta_e(F \circ \phi)$ $\quad (\forall \phi \in Ham^c(M,\omega))$*

Remark 2.6. *If $\widetilde{SH}(M) \neq 0$ holds, we can construct partial symplectic quasi-states by using the unit element $1_M \in \widetilde{SH}(M)$.*

As in the closed case ([5]), we define heavy set and superheavy set as follows. From now on, we assume that $e \in \widetilde{SH}(M) \backslash \{0\}$ is an idempotent.

Definition 2.6. *A closed subset $A \subset M$ is called e-heavy if*

$$\zeta_e(H) \geq \inf_A H \quad (\forall H \in C_{cc}(M))$$

holds and is called e-superheavy if

$$\zeta_e(H) \leq \sup_A H \quad (\forall H \in C_{cc}(M))$$

holds.

Remark 2.7. *A closed subset $A \subset M$ is e-heavy if and only if $\zeta_e(f) \geq 0$ holds for any $f \in C_{cc}(M)$ such that $f|_A = 0$. A closed subset $A \subset M$ is e-superheavy if and only if $\zeta_e(f) \leq 0$ holds for any $f \in C_{cc}(M)$ such that $f|_A = 0$. See Remark 5.1 (4) for the case of closed symplectic manifolds.*

Definition 2.7. *A subset $A \subset M$ is called Hamiltonianly displaceable if there is $\varphi \in Ham(M, \omega)$ such that $\varphi(A) \cap \overline{A} = \emptyset$ holds. Otherwise, A is called Hamiltonian non-displaceable.*

When A is compact and Hamiltonianly displaceable, we can take φ generated by a compactly supported Hamiltonian function such that φ displaces A.

Next two propositions are important for the proof of theorem.

Proposition 2.1. *A closed subset $A \subset M$ is e-heavy if and only if e does not vanish under the map*

$$\widetilde{SH}(M) \longrightarrow \widetilde{SH}(M : A).$$

Proposition 2.2. *If a closed subset $A \subset M$ is e-heavy, A is non-displaceable in M.*

The proof of Proposition 2.2 goes in the the same way as [5].

Using these two propositions, we prove the following vanishing theorem.

Theorem 2.2. *If $\overline{U}$ is displaceable in $(\widehat{M}, \widehat{\omega})$, then $\widetilde{SH}(M : U) = 0$ holds. In particular, if M is displaceable in $(\widehat{M}, \widehat{\omega})$, then $\widetilde{SH}(M) = 0$ holds.*

Remark 2.8. *Let (V, λ) be a Liouville domain and Let ι be an exact embedding of V into another Liouville domain (W, γ) of the same dimension. Then we can prove that*

$$SH(V) \cong \widetilde{SH}(V) \cong \widetilde{SH}(W : \iota(V)).$$

The first isomorphism follows from the fact that $SH(V) \cong SH^{[a,\infty)}(V)$ is satisfied for any $a < 0$, since V is a Liouville domain. The second isomorphism follows from the following arguments ([16]). We fix $a \in \mathbb{R}$. It suffices to prove that there is a canonical isomorphism between $SH^{[a,\infty)}(V)$ and $SH^{[a,\infty)}(W : \iota(V))$. Without loss of generality, we can assume that $\iota^*\gamma = \lambda$. In the completion $\widehat{V}$, a neighborhood of ∂V can be identified with the following domain

$$((1-\epsilon, 1+\epsilon) \times \partial V, r\lambda|_{\partial V}).$$

Here r is the coordinate of $(1-\epsilon, 1+\epsilon)$. If we choose ϵ small enough, a neighborhood of $\iota(\partial V)$ in $\widehat{W}$ can be identified with the following domain

$$((1-\epsilon, 1+\epsilon) \times \iota(\partial V), r\gamma|_{\iota(\partial V)}).$$

So these two neighborhoods can be identified canonically. Note that we can take outward collar neighborhoods V and $\iota(V)$ to $\widehat{V}$ and $\widehat{W}$, respectively:

$$V_\epsilon = V \cup [1, 1+\epsilon] \times \partial V \subset \widehat{V},$$

$$\iota(V)_\epsilon = \iota(V) \cup [1, 1+\epsilon] \times \iota(\partial V) \subset \widehat{W}.$$

Thus ι can be extended to V_ϵ canonically. We construct $H_V \in \mathcal{H}(V)$ and $H_W \in \mathcal{H}(W)$ as follows.

- *$H_V(t, x) = H_W(t, \iota(x))$ holds on $x \in V_\epsilon$*
- *$H_V(t, (r, y)) = \alpha(r)$ holds on $(r, y) \in [1, 1+\epsilon] \times \partial V$*
- *There are no periodic orbits on $[1, 1+\epsilon] \times \partial V$*

Then, the periodic orbits $P(H_V)$ and $P(H_W)$ can be divided into the following four types.

(1) *$x(t) \in P(H_V)$ contained in V.*

(2) $x(t) \in P(H_V)$ *contained in* $\widehat{V} \setminus V_\epsilon$.
(3) $x(t) \in P(H_W)$ *contained in* $\iota(V)$.
(4) $x(t) \in P(H_W)$ *contained in* $\widehat{W} \setminus \iota(V_\epsilon)$.

Periodic orbits (1) *and* (3) *are identified canonically. We can choose* H_V *and* H_W *so that the action functional* A_{H_V} *(resp.* A_{H_W}*) take values smaller than a fixed constant* $a \in \mathbb{R}$ *for periodic orbits in* (2) *(resp.* (4)*). (Here we also used the assumption that* V *and* W *are Liouville domains and* V *is an exactly embedded subdomain.) So* $CF^{[a,\infty)}(H_V)$ *is generated by periodic orbits in* (1) *and* $CF^{[a,\infty)}(H_W)$ *is generated by periodic orbits in* (3). *This implies that* $CF^{[a,\infty)}(H_V)$ *and* $CF^{[a,\infty)}(H_V)$ *can be identified as vector spaces. We choose almost complex structures* $J_{\widehat{V}}$ *on* $\widehat{V}$ *and* $J_{\widehat{W}}$ *on* $\widehat{W}$ *as follows.*

- $J_{\widehat{V}}|_{V_\epsilon} = \iota^*(J_{\widehat{W}})|_{V_\epsilon}$
- $J_{\widehat{V}}$ *is contact type on* $[1, 1+\epsilon] \times \partial V$

Then the image of any Floer connecting orbit (2.1) *between periodic orbits in* (1) *(resp.* (3)*) is contained in* V *(resp.* $\iota(V)$*) (see* [1]*). So,* $CF^{[a,\infty)}(H_V, J_{\widehat{V}})$ *and* $CF^{[a,\infty)}(H_W, J_{\widehat{W}})$ *are also canonically identified as chain complexes. By taking direct limit with respect to such* H_V *and* H_W*, there is the following canonical isomorphisms.*

$$SH^{[a,\infty)}(V) = \varinjlim_{H_V} HF^{[a,\infty)}(H_V, J_{\widehat{V}})$$
$$\cong \varinjlim_{H_W} HF^{[a,\infty)}(H_W, J_{\widehat{W}}) = SH^{[a,\infty)}(W : \iota(V))$$

3. Fundamental properties of ρ_e and ζ_e

In this section, we prove Theorem 2.1. We have the following subadditivity.

Lemma 3.1. *For any two periodic Hamiltonian functions* F, G*, following inequality holds.*

$$\rho_e(F) + \rho_e(G) \geq \rho_e(F \sharp G)$$

Proof. We take $(H, J) \in \mathcal{H}(f)$, $(K, J) \in \mathcal{H}(g)$ so that $(H \sharp K, J) \in \mathcal{H}(F \sharp G)$ holds. Then we have a map

$$HF^{<a}(H, J) \times HF^{<b}(K, J) \longrightarrow HF^{<a+b}(H \sharp K, J)$$

for any $a, b \in \mathbb{R}$. By taking direct limit of such (H, J) and (K, J), we get a map

$$SH^{<a}(M : F) \times SH^{<b}(M : G) \longrightarrow SH^{<a+b}(M : F \sharp G)$$

This implies that subadditivity is satisfied. □

We also have the following lemma.

Lemma 3.2. *Let* $F, G \in C_c^\infty(S^1 \times M)$ *be two Hamiltonian functions. Suppose that the Hamiltonian diffeomorphism* ϕ_G *generated by* G *satisfies*

$$\phi_G(\text{Supp } F) \cap \text{Supp } F = \emptyset$$

holds. Then $\rho_e(G \sharp F) = \rho_e(G)$ *holds.*

Proof. Let $G' \in \mathcal{H}_{reg}$ be a Hamiltonian function such that

- $G' \in \mathcal{H}_{reg}(G)$
- $G' \sharp F \in \mathcal{H}(G \sharp F)$

- $\phi_{G'}(\text{Supp } F) \cap \text{Supp } F = \emptyset$.

From these assumptions, we have that $G' \sharp tF \in \mathcal{H}_{reg}(G \sharp tF)$ for any t. We take G' so that the restriction of G' to M is a small perturbation of that of G. Let

$$\{e_{b,G\sharp F} \in SH^{[b,\infty)}(M; G\sharp F)\}_{b\in\mathbb{R}} \in \widetilde{SH}(M; G\sharp F)$$
$$\{e_{b,G} \in SH^{[b,\infty)}(M; G)\}_{b\in\mathbb{R}} \in \widetilde{SH}(M; G)$$

be representatives of $e \in \widetilde{SH}(M)$, cf. Remark 2.2. We choose a negative real number $a < 0$ so that $|a| > ||F||$ holds. We assume that G' is sufficiently small outside M so that we can choose

$$e_0 \in HF^{[4a,\infty)}(G', J)$$
$$e_1 \in HF^{[4a,\infty)}(G' \sharp F, J)$$

which are representatives of $e_{4a,G}$ and $e_{4a,G\sharp F}$. ¿From now on, we assume that $a < 0$ also satisfies the following condition

$$a < \min\{\rho_e(G\sharp F), \rho_e(G)\}. \tag{3.1}$$

We consider the following two constants for $4a \le d \le a$.

$$\tau^d(0) = \inf\{c \in \mathbb{R} \mid e_0^d \in \text{Im}(HF^{[d,c)}(G') \to HF^{[d,\infty)}(G'))\}$$
$$\tau^d(1) = \inf\{c \in \mathbb{R} \mid e_1^d \in \text{Im}(HF^{[d,c)}(G'\sharp F) \to HF^{[d,\infty)}(G'\sharp F))\}$$

Here e_t^d is the image of e_t $(t = 0, 1)$. Then the condition (3.1) on $a \in \mathbb{R}$ implies that $\tau^d(t) > a$ holds and it does not depend on the choice of d if $4a \le d \le a$ is satisfied. We denote them by $\tau(0)$ and $\tau(1)$. For any $d \ge 3a$, we have the following one parameter family of maps induced by continuation homomorphisms $(t \in [0, 1])$.

$$\kappa_t^d : HF^{[4a,\infty)}(G', J) \longrightarrow HF^{[d,\infty)}(G' \sharp tF, J)$$

Next, we compare $\tau(0)$ and $\tau(1)$. For any $d \in [3a, 2a]$ and $t \in [0, 1]$, we consider the following quantity

$$\tau^d(t) = \inf\{c \in \mathbb{R} \mid \kappa_t^d(e_0) \in \text{Im}(HF^{[d,c)}(G'\sharp tF) \to HF^{[d,\infty)}(G'\sharp tF)\}.$$

Then $\tau^d(t) \ge 2a$ holds and $\tau^d(t)$ does not depend on the choice of $3a \le d \le 2a$. We denote them by $\tau(t)$. Note that above two definitions of $\tau(0)$ and $\tau(1)$ coincide, because $\kappa_0^d(e_0) = e_0^d$ and $\kappa_1^d(e_1) = e_1^d$ hold for $3a \le d \le 2a$. We have the following commutative diagram for any $s, t \in [0, 1]$.

$$\begin{array}{ccc} HF^{[4a.\infty)}(G') & \xrightarrow{\kappa_s^{3a}} & HF^{[3a,\infty)}(G'\sharp sF) \\ \| & & \downarrow \\ HF^{[4a.\infty)}(G') & \xrightarrow{\kappa_t^{2a}} & HF^{[2a,\infty)}(G'\sharp tF) \end{array}$$

Applying the standard argument on filtrations to this diagram, we find that

$$\tau(t) \le \tau(s) + |s - t| \cdot ||F|| \text{ for } s, t \in [0, 1].$$

In particular, we obtain the following Lipschitz continuity.

$$|\tau(s) - \tau(t)| \le |s - t| \cdot ||F||.$$

The assumption $\phi_{G'}(\text{Supp } F) \cap \text{Supp } F = \emptyset$ implies that

$$\text{Spec}(G'\sharp tF) = \text{Spec}(G'),$$

(Ostrover's argument, see, e.g., [12]). Since $G'\sharp tF \in \mathcal{H}_{reg}(G\sharp tF)$, we obtain $\tau(t) \in \mathrm{Spec}(G'\sharp tF) = \mathrm{Spec}(G')$ (the spectrality for non-degenerate Hamiltonians). Because $\mathrm{Spec}(G')$ is nowhere dense, the continuity of $\tau(t)$ implies that $\tau(t)$ is a constant function. In particular, $\tau(0) = \tau(1)$ holds. If we make the Hamiltonian function G' smaller and smaller, $\tau(0)$ converges to $\rho_e(G)$ and $\tau(1)$ converges to $\rho_e(G\sharp F)$. So $\rho_e(G) = \rho_e(G\sharp F)$ holds.

□

The ingredients of the proof of the fact that $\zeta_e(f)$ is well-defined are as follows.

- subadditivity
- Lemma 3.2
- Fragmentation Lemma [3]
- Let $F \in C_c^\infty(S^1 \times M)$ and set $\overline{F} = -F(t, \phi_F^t(x))$. Then, if $\mathrm{Supp}(F)$ is contained in U, which is Hamiltonianly displaceable, $\rho_e(F) + \rho_e(\overline{F}) \leq C_U$ for some constant C_U which depends only on U.

The rest of the proof of well-definedness is algebraic and this part is the same as in the closed case ([4]). Lipschitz continuity, semi-homogenuity, monotonicity, additivity with respect to constants and normalization, partial additivity, and invariance also goes in the same way as in the closed case.

4. Proof of Theorem 2.2

In this section, we prove Theorem 2.2. We first prove Proposition 2.1.

Proof of Proposition 2.1. Assume that $e \in \widetilde{SH}(M)\backslash\{0\}$ be an idempotent and $A \subset M$ is a subset such that e does not vanish under the map

$$\iota : \widetilde{SH}(M) \longrightarrow \widetilde{SH}(M : A)$$

The condition that $\iota(e) \neq 0$ implies that

$$c(e : A) = \inf\{a \in \mathbb{R} \mid \iota(e) \in \mathrm{Im}(\widetilde{SH}^{<a}(M : A) \to \widetilde{SH}(M : A))\} > -\infty,$$

see Remark 2.3. Since e is an idempotent, we have that $c(e : A) = c(e^{*k} : A) \leq kc(e : A)$ for any $k \in \mathbb{N}$. Hence we find that $c(e : A) \geq 0$. Let $f \in C_{cc}(M)$ be any function such that $f|A = 0$. It suffices to prove that $\zeta_e(f) \geq 0$ holds. In this case, since $\mathcal{H}(f) \subset \mathcal{H}(A)$, we have the following map for any $a \in \mathbb{R}$.

$$\tau : \widetilde{SH}^{<a}(M : f) \longrightarrow \widetilde{SH}^{<a}(M : A).$$

So the fact that $c(e : A) \geq 0$ implies that $\rho_e(f) \geq 0$. Obviously, $kf \in C_{cc}(M)$, thus we have $\rho_e(kf) \geq 0$. This implies that

$$\zeta_e(f) = \lim_{k\to\infty} \frac{\rho_e(kf)}{k} \geq 0$$

and $A \subset M$ is e-heavy.

Next, we assume that $\iota(e) = 0$. We show that $A \subset M$ is not heavy. We fix a constant $c < 0$ and let $\{e_a \in SH^{[a,\infty)}(M)\}_{a\in\mathbb{R}}$ be the representative of $e \in \widetilde{SH}(M)$. We choose $\overline{e}_c \in HF^{[c,\infty)}(H)$ $(H \in \mathcal{H}(M))$, which represents e_c. The assumption that $\iota(e) = 0$ implies that there exists $H_A \in \mathcal{H}(A)$ such that

- $H(t, x) \geq H_A(t, x)$.

- $\overline{e}_c$ vanishes under the following map

$$HF^{[c,\infty)}(H) \longrightarrow HF^{[c,\infty)}(H_A).$$

We choose $f \in C_{cc}(M)$ which satisfies the following conditions.

- $f|_A = 0$
- $f(x) \leq H_A(t,x)$ holds on $(t,x) \in S^1 \times M$

Then $H_A \in \mathcal{H}(f)$ holds and $\overline{e}_c \in HF^{[c,\infty)}(H)$ vanishes under the composition of the following maps.

$$HF^{[c,\infty)}(H) \longrightarrow HF^{[c,\infty)}(H_A) \longrightarrow SH^{[c,\infty)}(M;f)$$

This implies that $\rho_e(f) \leq c < 0$ holds. In particular, subadditivity implies that $\zeta_e(f) \leq c$ holds and $A \subset M$ is not e-heavy. □

Next we prove Theorem 2.2.

Proof. We assume that $\overline{U} \subset M$ is displaceable in $(\widehat{M}, \widehat{\omega})$. For $R \geq 1$, we define the following subset of $\widehat{M}$

$$\begin{cases} M_R = M \cup_{\partial M} [1,R) \times \partial M \\ \omega_R = \widehat{\omega}|_{M_R} \end{cases}$$

For such (M_R, ω_R)we have the following natural isomorphism.

$$\widetilde{SH}(M:U) \cong \widetilde{SH}(M_R:U)$$

We assume that $\overline{U}$ is displaceable in M_R. We divide the proof into the following two parts.

(1) Assume that $\widetilde{SH}(M_R) = 0$ holds. Then the unit element $1_{M_R} \in \widetilde{SH}(M_R)$ is zero. There is a canonical map

$$\widetilde{SH}(M_R) \longrightarrow \widetilde{SH}(M_R:U) \cong \widetilde{SH}(M:U)$$

and this maps the unit 1_{M_R} to the unit $1_U \in \widetilde{SH}(M:U)$. So 1_U is zero and $\widetilde{SH}(M:U) = 0$ holds.

(2) Assume that $\widetilde{SH}(M_R) \neq 0$ holds. In this case we can use $e = 1_{M_R}$ to construct partial symplectic quasi-state ζ_e. Because $\overline{U}$ is displaceable in M_R, Proposition 2.2 implies that $\overline{U}$ is not e-heavy. Then Proposition 2.1 implies that $1_{\overline{U}} = 0$. So $1_U = 0$ and $\widetilde{SH}(M:U) = 0$ holds.

□

Appendix to Part 1.

This appendix is an extract from the master's thesis of the second author [16]. A similar result for wrapped Floer homology was obtained in [15]. In this appendix, we explain that if (M,ω) is a weakly exact symplectic manifold (i.e. $\omega|_{\pi_2(M)} = 0$) with a contact type boundary, we can prove a stronger result than Theorem 2.2. We explain the following theorem.

Theorem 4.1. *Let (M,ω) be a compact weakly exact symplectic manifold with a contact type boundary and let $U \subset M$ be a subset of M such that $\overline{U}$ is Hamiltonian displaceable in $\widehat{M}$. Let $e(\overline{U})$ be its displacement energy. Then for any $-\infty < a < b \leq \infty$ and $c > e(\overline{U})$, the canonical map*

$$SH^{[a,b)}(M:U) \longrightarrow SH^{[a+c,b+c)}(M:U)$$

is zero.

Theorem 2.2 for weakly exact symplectic manifolds is a corollary of above theorem. We explain how to prove Theorem 4.1. A detailed arguments in a similar situation in wrapped Floer homology can be found in [15]. Let $K \in C_c^\infty(S^1 \times \widehat{M})$ be a Hamiltonian function which satisfies the following properties.

- $\phi_K(\overline{U}) \cap \overline{U} = \phi$
- $||K|| < c$

Let W be a open neighborhood of $\overline{U}$ such that $\phi_K(W) \cap W = \phi$ holds. We fix $R \geq 1$ so that

$$M_R = M \cup_{\partial M} [1,R] \times \partial M \supset (W \cup \text{Supp } K)$$

We take a (non-smooth) Hamiltonian function h as follows.

- $h|_U > 0$
- $h|_{M_R \setminus W} \equiv C$
- $h((r,y)) = \alpha r + \beta, \quad (r,y) \in [R,\infty) \times \partial M, \alpha < 0$
- $|\alpha|$ is sufficiently small with respect to $|C|$

For such h, we can perturb h to $H \in C^\infty(S^1 \times \widehat{M})$ so that

- H is smooth
- H is non-degenerate
- $\text{Spec}(H \sharp K) \subset [-\infty, a)$

hold. Then we have the following commutative diagram. We omit almost complex structures.

$$\begin{array}{ccc} HF^{[a,b)}(H) & \longrightarrow & HF^{[a+||K||,b+||K||)}(H) \\ \downarrow & & \uparrow \\ HF^{[a+||K||_+,b+||J||_+)}(H\sharp K) & = & HF^{[a+||K||_+,b+||J||_+)}(H\sharp K) \end{array}$$

where $||K||_+$ is the positive part of Hofer norm

$$||K||_+ = \int_0^1 \max_x K(t,x)dt$$

By the conditions of H and , we have

$$HF^{[a+||K||_+,b+||J||_+)}(H\sharp K) = 0$$

and above commutative diagram implies that

$$HF^{[a,b)}(H) \longrightarrow HF^{[a+||K||,b+||K||)}(H)$$

is zero. By taking direct limit of such H, we can see that

$$SH^{[a,b)}(M:U) \longrightarrow SH^{[a+||K||,b+||K||)}(M:U)$$

is zero. Because we assumed $||K|| < c$, we can see that Theorem 4.1 holds.

Part 2. Covering tricks

In this part, we consider covering tricks. We consider a finite covering of a closed symplectic manifold and give a sufficent condition for (super)heaviness by using suitable covering spaces.

5. A brief review of (super)heaviness on closed symplectic manifolds

In this section, we briefly review the construction of symplectic quasi-states on closed symplectic manifolds and the definition of heaviness and superheaviness ([4, 5]). Let (M, ω) be a closed symplectic manifold and let $QH(M, \omega)$ be the quantum cohomology ring which is defined as follows. $H^*(M : \mathbb{Q})$ is the singular cohomology group of M and Λ is the Novikov ring of (M, ω)

$$QH(M, \omega) = H^*(M : \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda$$

The quantum cohomology has a product structure which is known as quantum cup product([14, 11])

$$* : QH(M, \omega) \times QH(M, \omega) \longrightarrow QH(M, \omega)$$

and there is a ring isomorphism between quantum cohomology ring and Floer homology.

$$\Psi : QH(M, \omega) \longrightarrow HF(H, J)$$

For any pair of a Hamiltonian function and an almost complex structure (H, J) and for any element $e \in QH(M, \omega)$, the spectral invariant is defined as follows.

$$\rho_e(H) = \inf\{a \in \mathbb{R} \mid \Psi(e) \in \mathrm{Im}(HF^{<a}(H, J) \to HF(H, J))\}$$

$\rho_e(H)$ does not depend on the choice of J and it is Lipschitz continuous with respect to the Hofer norm of H. So we can extend $\rho_e(H)$ for any continuous function H.

Let $e \in QH(M, \omega)$ be an idempotent (i.e. $e * e = e$ holds). Then, symplectic quasi-states ζ_e is defined as follows([4, 5]). Let H be an autonomous Hamiltonian function on M.

$$\zeta_e(H) = \lim_{k \in \mathbb{N}, k \to +\infty} \frac{\rho_e(kH)}{k}$$

Then, heavy sets and superheavy sets are defined as follows.

Definition 5.1 (heavy set, superheavy set [4, 5]). *We fix an idempotent e.*

(1) *A closed subset $X \subset M$ is called e-heavy if*

$$\zeta_e(H) \geq \inf_X H \quad (\forall H \in C(M))$$

holds.

(2) *A closed subset $X \subset M$ is called e-superheavy if*

$$\zeta_e(H) \leq \sup_X H \quad (\forall H \in C(M))$$

holds.

Remark 5.1. (1) *ζ_e satisfies Lipschitz continuity, semi-homogenuity, monotonicity, additivity with respect to constants and normalization, partial additivity, invariance as in Theorem 2.1.*

(2) *Let $X \subset M$ be a e-heavy set. Then X can not be displaced by any Hamiltonian diffeomorphism. In other words, $X \cap \phi(X) \neq \emptyset$ holds for any Hamiltonian diffeomorphism ϕ.*

(3) *Let $X \subset M$ be a e-superheavy set. Then X can not be displaced by any symplecitc isotopy. In other words, $X \cap \phi(X) \neq \emptyset$ holds for any symplectomorphism ϕ if ϕ is isotopic to the identity via symplectomorphisms.*

(4) *A closed subset $X \subset M$ is e-heavy (resp. e-superheavy) if and only if $\zeta_e(f) = 0$ for any continuous function f such that $f|_X = 0$ and $f \leq 0$ (resp. $f \geq 0$), see Proposition 4.1 in* [5]. *The latter condition is equivalent to the condition that $\zeta_e(f) \geq 0$ (resp. $\zeta_e(f) \leq 0$) for any continuous function f such that $f|_X = 0$.*

6. Covering tricks

Let (M, ω) be a closed symplectic manifold and let

$$\pi : \widetilde{M} \longrightarrow M$$

be a k-fold covering ($k < +\infty$). $\widetilde{M}$ has a symplectic form $\widetilde{\omega} = \pi^*\omega$. Let (H, J) be a pair of a Hamiltonian function and an almost complex structure on M, then we denote its pull-back to $(\widetilde{M}, \widetilde{\omega})$ by $(\widetilde{H}, \widetilde{J})$.

$$\widetilde{H}(t, x) = H(t, \pi(x))$$
$$\widetilde{J}(t, x) = J(t, \pi(x))$$

Next, we take the average of $CF(\widetilde{H}, \widetilde{J})$ as follows.

$$av : CF(\widetilde{H}, \widetilde{J}) \longrightarrow CF(\widetilde{H}, \widetilde{J})$$
$$[x, u] \longmapsto \frac{1}{k} \sum_{\pi(u_i)=\pi(u), u_i \neq u_j} [x_i, u_i]$$

We denote the image of av by $CF(\widetilde{H}, \widetilde{J})^{av}$ and its homology by $HF(\widetilde{H}, \widetilde{J})^{av}$. Note that av preserves the filtration by the action functional.

Let π_*^{HF} be the projection of Floer chain complexes as follows.

$$\pi_*^{HF} : CF(\widetilde{H}, \widetilde{J}) \longrightarrow CF(H, J)$$
$$[x, u] \longmapsto [\pi(x), \pi(u)]$$

Then we have the following lemma.

Lemma 6.1. *The restriction of π_*^{HF} to $CF(\widetilde{H}, \widetilde{J})^{av}$ induces an isomorphism between $HF(\widetilde{H}, \widetilde{J})^{av}$ and $HF(H, J)$.*

Remark 6.1. *The arguments in the proof of Lemma 6.1 implies that av is a chain map and $CF(\widetilde{H}, \widetilde{J})^{av}$ is a subcomplex. So its homology is well defined. Lemma 6.1 also implies that the natural map*

$$HF(\widetilde{H}, \widetilde{J})^{av} \longrightarrow HF(\widetilde{H}, \widetilde{J})$$

is injective.

Proof. For any $l = [x, u] \in \widetilde{P(H)}$, there is k discs $u_1, \cdots, u_k$ of $\widetilde{M}$ which are the lifts of u.

$$u_i : D^2 \longrightarrow \widetilde{M}$$
$$\pi(u_i) = u$$

We denote the boundary of u_i by x_i and $[x_i, u_i] \in \widetilde{P(\widetilde{H})}$ by l^i. Then $\widetilde{l} = \frac{1}{k}\sum_{1\leq i\leq k} l^i$ generates $CH(\widetilde{H}, \widetilde{J})^{av}$. So it suffices to compare the matrix coefficients $n(l_-, l_+)$ and $n(\widetilde{l_-}, \widetilde{l_+})$ for every $l_-, l_+ \subset \widetilde{P(H)}$.

Let $v : \mathbb{R} \times S^1 \longrightarrow M$ be a connecting orbit from l_- to l_+. Because v converges to contractible periodic orbits in the ends, there are k cylinders $v_1, \cdots, v_k$ which are the lifts of v. These lifts determine a permutation τ of $\{1, \cdots, k\}$ so that v_i is a connecting orbit from l^i_- to $l^{\tau(i)}_+$. This implies that $n(l_-, l_+)$ and $n(\widetilde{l_-}, \widetilde{l_+})$ coincide. So, we can identify $CF(\widetilde{H}, \widetilde{J})^{av}$ and $CH(H, J)$. □

On the other hand, we define $QH(\widetilde{M}, \widetilde{\omega})^{av}$ using the transfer map for the finite covering space. Namely, for any singular simplex $\sigma : \Delta \to M$, there is k singular simplices $\{\sigma_i : \Delta \to \widetilde{M}\}_{i=1,\ldots,k}$ such that $\pi \circ \sigma_i = \sigma$. Then, we set $tf(\sigma) = \frac{1}{k}\sum \sigma_i$, which is a chain homomorphism and induces

$$tf_* : H_*(M : \mathbb{Q}) \longrightarrow H_*(\widetilde{M} : \mathbb{Q}).$$

We set

$$av = tf_* \circ \pi_* : H_*(\widetilde{M} : \mathbb{Q}) \to H_*(\widetilde{M} : \mathbb{Q}).$$

By identifying cohomology and homology via Poincaré duality, we can also define av on singular cohomology and also on quantum cohomology.

$$av : QH(\widetilde{M}, \widetilde{\omega}) \longrightarrow QH(\widetilde{M}, \widetilde{\omega})$$

$QH(\widetilde{M}, \widetilde{\omega})^{av}$ is the image of av. As in the case of Floer homology, the projection

$$\pi^{QH}_* : QH(\widetilde{M}, \widetilde{\omega}) \longrightarrow QH(M, \omega)$$

induces an isomorphism between $QH(\widetilde{M}, \widetilde{\omega})^{av}$ and $QH(M, \omega)$. For any $e \in QH(M, \omega)$, we denote the corresponding element in $QH(\widetilde{M}, \widetilde{\omega})^{av}$ by $\widetilde{e}$.

Proposition 6.1. (1) *There is a following commutative diagram.*

$$\begin{array}{ccc} QH(\widetilde{M}, \widetilde{\omega}) & \xrightarrow{\cong} & HF(\widetilde{H}, \widetilde{J}) \\ {\scriptstyle av}\big\downarrow & & {\scriptstyle av}\big\downarrow \\ QH(\widetilde{M}, \widetilde{\omega})^{av} & \xrightarrow{\cong} & HF(\widetilde{H}, \widetilde{J})^{av} \\ {\scriptstyle \pi^{QH^{av}}_*}\big\downarrow & & {\scriptstyle \pi^{HF^{av}}_*}\big\downarrow \\ QH(M, \omega) & \xrightarrow{\cong} & HF(H, J) \end{array}$$

Here We denote $\pi^{QH}_*|_{QH(\widetilde{M}, \widetilde{\omega})^{av}}$ *by* $\pi^{QH^{av}}_*$ *and* $\pi^{HF}_*|_{HF(\widetilde{H}, \widetilde{J})^{av}}$ *by* $\pi^{HF^{av}}_*$.

(2) $\pi^{QH^{av}}_*$ *is a ring isomorphism. In particular, e is an idempotent if and only if $\widetilde{e}$ is an idempotent.*

Proof. For a closed symplectic manifold, the quantum cohomology and Hamiltonian Floer homology are isomorphic. The isomorphism is constructed roughly as follows. Pick a function $\rho \in C^\infty(\mathbb{R})$ such that $\rho(s) = 0$ for $s << 0$ and $\rho(s) = 1$ $s >> 0$. Then consider a family of Hamiltonian functions $K_s = \rho(s) \cdot H$ and $L_s = (1 - \rho(s)) \cdot H$. The isomorphism is constructed using solutions of s-dependent analog of the equation (2.1) for Floer connecting orbits using $\{K_s\}$ and $\{L_s\}$. The asymptotic condition for solution v when using $\{K_s\}$ is $\lim_{s\to-\infty} v(s, \dot{)}$ lies in a given chain in

M and $\lim_{s\to+\infty} v(s,\cdot) = \ell$ for $\ell \in P(H)$. In the case using $\{L_s\}$, these asymptotic conditions are switched. The projection π maps solutions in $\widetilde{M}$ to those in M and any solution in M is lifted to k solutions in $\widetilde{M}$. Therefore Claim (1) is a direct consequence of the construction of the isomorphism between quantum cohomology and Floer homology.

Claim (2) follows from the following observation. The quantum cup product $*$ is defined by using three points genus zero Gromov-Witten invariants. Note that every genus zero stable map to M has k lifts to $\widetilde{M}$. Then, we can see that $\pi_*^{QH^{av}}$ is a ring isomorphism. □

Lemma 6.2. (1) *For any periodic Hamiltonian function H and any idempotent $e \in QH(M,\omega)$,*

$$\rho_e(H) = \rho_{\widetilde{e}}(\widetilde{H})$$

holds.

(2) *For any autonomous Hamiltonian function H and any idempotent $e \in QH(M,\omega)$,*

$$\zeta_e(H) = \zeta_{\widetilde{e}}(\widetilde{H})$$

holds.

Proof. (2) follows from (1). So it suffices to prove (1). Because we can identify $CH(H,J)$ and $CH(\widetilde{H},\widetilde{J})^{av}$,

$$e \in \mathrm{Im}(HF^{<a}(H,J) \longrightarrow HF(H,J))$$

implies

$$\widetilde{e} \in \mathrm{Im}(HF^{<a}(\widetilde{H},\widetilde{J}) \longrightarrow HF^(\widetilde{H},\widetilde{J})).$$

This implies $\rho_e(H) \geq \rho_{\widetilde{e}}(\widetilde{H})$ holds.

Let $\widetilde{c} \in CF(\widetilde{H},\widetilde{J})^{av}$ be a cycle which satisfies $[\widetilde{c}] = \widetilde{e}$ and let $\widetilde{d'} \in CF(\widetilde{H},\widetilde{J})$ be a chain which is not necessarily av-invariant. Then $c = \pi_*^{HF}(\widetilde{c}) \in CF(H,J)$ is a cycle which represents e. We take the average of $\widetilde{d'}$ and we denote it by $\widetilde{d} = av(\widetilde{d'})$. Then we have the following equalities.

$$\begin{aligned} c + \partial(\pi_*^{HF}(\widetilde{d})) &= \pi_*^{HF}(\widetilde{c} + \partial\widetilde{d}) = \pi_*^{HF}(\widetilde{c} + \partial(av(\widetilde{d'}))) \\ &= \pi_*^{HF}(\widetilde{c} + av(\partial\widetilde{d'})) = \pi_*^{HF}(av(\widetilde{c} + \partial\widetilde{d'})) \end{aligned}$$

So $\widetilde{c} + \partial\widetilde{d'} \in CF^{<a}(\widetilde{H},\widetilde{J})$ implies $c + \partial(\pi_*^{HF}(\widetilde{d})) \in CF^{<a}(H,J)$ holds. This implies $\rho_e(H) \leq \rho_{\widetilde{e}}(\widetilde{H})$ holds. □

The lemma above is the key of the following criterion for (super)heaviness.

Proposition 6.2. (1) *A closed subset $X \subset M$ is e-heavy if and only if $\pi^{-1}(X)$ is $\widetilde{e}$-heavy.*

(2) *A closed subset $X \subset M$ is e-superheavy if and only if $\pi^{-1}(X)$ is $\widetilde{e}$-superheavy.*

Proof. We prove the claim (1). The proof of (2) is similar. Let $X \subset M$ be a e-heavy subset. Let f be a continuous function on $\widetilde{M}$ such that $f \leq 0$ and $f|_{\pi^{-1}(X)} = 0$. Let $\underline{f}$ be a continuous function on M which is defined as follows.

$$\underline{f}(x) = \min_{z\in\widetilde{M},\pi(z)=x} f(z)$$

Note that $\underline{f} \leq 0$, $\underline{f}|_X = 0$ and $f \geq \pi^*(\underline{f})$. So we have the following inequality.

$$\zeta_{\widetilde{e}}(f) \geq \zeta_{\widetilde{e}}(\pi^*\underline{f}) = \zeta_e(\underline{f}) = 0$$

This implies that $\zeta_{\widetilde{e}}(f) = 0$, hence $\pi^{-1}(X)$ is $\widetilde{e}$-heavy.

Conversly, assume that $\pi^{-1}(X)$ is $\widetilde{e}$-heavy and let f be a continuous function on M such that $f \leq 0$ and $f|_X = 0$. Then we have $\pi^* f \leq 0$ and $\pi^* f|_{\pi^{-1}(X)} = 0$, which imply that

$$\zeta_e(f) = \zeta_{\widetilde{e}}(\pi^* f) = 0.$$

Hence we find that X is e-heavy. □

Let $X \subset M$ be a closed subset and let U_i be a connected component of $\pi^{-1}(M \backslash X)$.

$$\pi^{-1}(M \backslash X) = \bigsqcup_i U_i$$

Then we can prove the following theorem.

Theorem 6.1. *Assume that each U_i is Hamiltonian displaceable. Then X is e-superheavy.*

Proof. It suffices to prove that $\pi^{-1}(X)$ is $\widetilde{e}$-superheavy. Let f be a continuous function on $\widetilde{M}$ such that $f|_{\pi^{-1}(X)} = 0$ holds. We approximate this function by smooth function g on $\widetilde{M}$ as follows. We fix $\epsilon > 0$.

$$|f - g| < \epsilon$$

$$g = \sum g_i$$

$$\text{Supp } g_i \subset U_i$$

Note that g_i Poisson commute, the partial additivity of $\zeta_{\widetilde{e}}$ implies that

$$\zeta_{\widetilde{e}}(g) = \sum \zeta_{\widetilde{e}}(g_i) = 0.$$

Lipschitz continuity of $\zeta_{\widetilde{e}}$ implies that $|\zeta_{\widetilde{e}}(f)| < \epsilon$ holds. So $\zeta_{\widetilde{e}}(f) = 0$ holds and $\pi^{-1}(X)$ is $\widetilde{e}$-superheavy. □

7. Examples

In this section, we construct a few examples of superheavy sets by using the covering trick. We consider the following situation. Let (X, ω) be a closed symplectic manifold such that the fundamental group $\Gamma = \pi_1(X)$ is residually finite group. Moreover, any element of Γ acts on the universal covering of X by a Hamiltonian diffeomorphism.

Definition 7.1 (residually finite group)**.** *A group Γ is residually finite if and only if the intersection of all normal subgroup of finite index is trivial. In other words, the following holds.*

$$\bigcap_{N \triangleleft \Gamma, |\Gamma/N| < \infty} N = \{e\}$$

Let $\pi : \widetilde{X} \to X$ be a universal covering and write $\widetilde{\omega} = \pi^* \omega$. Assume that $\widetilde{D} \subset X$ is a relatively compact open set which satisfies the following conditions.

- the restriction of π to $\widetilde{D}$ is injective.
- $\widetilde{D}$ is Hamiltonianly displaceable in $(\widetilde{X}, \widetilde{\omega})$

We write $D = \pi(\widetilde{D})$. Multiplying a suitable cut-off function to a displacing Hamiltonian for $\widetilde{D}$, if necessary, we can pick a compactly supported 1-periodic Hamiltonian function $H \in C_c^\infty([0,1] \times \widetilde{X})$ such that ϕ_H satisfies $\phi_H(\widetilde{D}) \cap \widetilde{D} = \emptyset$ and let $K \subset \widetilde{X}$ be a compact set such that Supp $H \subset K$ holds.

Since Γ is residually finite, we can take a normal subgroup $\Gamma' \subset \Gamma$ of finite index such that

$$(\widetilde{X}, \widetilde{\omega}) \xrightarrow{\pi'} (X' = \widetilde{X}/\Gamma', \omega') \xrightarrow{\pi''} (X, \omega)$$

- π'' is a finite covering.
- $\pi'|_K : K \longrightarrow X'$ is injective.

Namely, we choose elements $\gamma_1, \cdots, \gamma_l \in \Gamma$ which satisfies the following condition. For any $x, y \in K$ with $\pi(x) = \pi(y)$, there exists j, $1 \leq j \leq l$ such that $\gamma_i x = y$.

Since Γ is residually finite,we can pick normal subgroups $\{N_i \lhd \Gamma\}_{1 \leq i \leq l}$ of finite index such that $\gamma_i \notin N_i$ Then we set $\Gamma' = \bigcap_{1 \leq i \leq l} N_i$, which enjoys the properties mentioned above.

Set $D' = \pi'(\widetilde{D})$. Then we have that $\phi_H(D') \cap D' = \emptyset$. We find that each connected component of $\pi''^{-1}(D) \subset X'$ is Hamiltonianly displaceable in (X', ω'). Then Theorem 7.1 implies that $X \backslash D$ is superheavy with respect to any idempotent $e \in QH(X, \omega)$.

Example 7.1. *Let M be a closed Kähler manifold of constant negative holomorphic sectional curvature. Then there is a cocompact discrete subgroup $\Gamma \subset PU(1, n : \mathbb{C})$ such that M is Kähler isometric to $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, where $\mathbb{H}^n_{\mathbb{C}}$ is the complex hyperbolic space. It is known that Γ is residually finite. Let $\widetilde{D} \subset \mathbb{H}^n_{\mathbb{C}}/\Gamma$ be the interior of a fundamental domain for Γ-action on $\mathbb{H}^n_{\mathbb{C}}$. We denote by D the image of $\widetilde{D}$ by the projection $\mathbb{H}^n_{\mathbb{C}} \to M$. We pick $g \in \Gamma$ such that the closure of $g(\widetilde{D})$ is disjoint from the closure of $\widetilde{D}$. Since g belongs to the identity component of the group of Kähler isometres, $\widetilde{D}$ is Hamiltonianly displaceable in $\mathbb{H}^n_{\mathbb{C}}$. Then we can apply the above arguments to D. Hence $M \setminus D$ is superheavy.*

The same argument works for symplectic tori $\mathbb{R}^{2n}/\Gamma$, where Γ is a lattice of rank $2n$. See also, the work of Kawasaki [9].

Example 7.2. *First of all, we recall the construction by Fine and Panov* [8]. *Let $SO(2n, 1)$ be the set of orientation preserving linear automorphism which preserves the following Minkowski metric.*

$$x_0^2 - x_1^2 - \cdots - x_{2n}^2$$

Then $Z_{2n} = SO(2n, 1)/U(n)$ is realized as a coadjoint orbit of $SO(2n, 1)$, so Z_{2n} has a symplectic form ([8]*). Z_{2n} fibers over the real hyperbolic space $\mathbb{H}^{2n}$.*

$$\mathbb{H}^{2n} \cong SO(2n, 1)/SO(2n)$$

Z_{2n} is the twistor space of $\mathbb{H}^{2n}$. Let $\Gamma \subset SO(2n, 1)$ be a fundamental group of a closed hyperbolic manifold X. Then Γ action on Z_{2n} preserves the symplectic form on Z^{2n}. So the symplectic form on Z_{2n} induces a symplectic form on Z_{2n}/Γ. Z_{2n}/Γ is the twistor space of $X = \mathbb{H}^{2n}/\Gamma$.

$$\begin{array}{ccc} Z_{2n} & \longrightarrow & Z_{2n}/\Gamma \\ \big\downarrow & & \eta \big\downarrow \\ \mathbb{H}^{2n} & \xrightarrow{\pi} & X = \mathbb{H}^{2n}/\Gamma \end{array}$$

Let $\widetilde{D} \subset \mathbb{H}^{2n}$ *be a fundamental domain for the* Γ*-action on* $\mathbb{H}^{2n}$. *We write* $D = \pi(\widetilde{D})$. *Pick an element* $g \in SO(2n,1)$*, which displaces* $\widetilde{D}$ *from itself in* $\mathbb{H}^{2n}$. *Since* g *induces an action on* Z_{2n}/Γ *preserving the symplectic form and* Z_{2n} *is simply connected,* $\widetilde{D}$ *is Hamiltonianly displaceable in* Z_{2n}. *Then the above arguments imply that* $(Z_{2n}/\Gamma) \setminus \eta^{-1}(D) \subset Z_{2n}/\Gamma$ *is superheavy.*

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