NOTES ON EXPLODED MANIFOLDS AND A TROPICAL GLUING FORMULA FOR GROMOV-WITTEN INVARIANTS

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ABSTRACT. Notes for a short lecture series, covering exploded manifolds, the moduli stack of curves in exploded manifolds, and a tropical gluing formula for Gromov-Witten invariants which gives a degeneration formula for Gromov-Witten invariants in normal crossing degenerations. I gave the original lecture series in April 2016 at the Simons Center for Geometry and Physics at Stonybrook. Video of the lectures is available on the SCGP website.

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1. Introduction

To write a degeneration or gluing formula for Gromov-Witten invariants in a normal crossing degeneration, it is essential to keep track of information on two different scales: a small scale involving differential geometry of manifolds with products of cylindrical ends, and a large scale involving piecewise integral affine geometry. On this large scale, holomorphic curves look like tropical curves, and the gluing formula for Gromov-Witten invariants is a sum over tropical curves. This large and small scale information appears simultaneously in the degenerating family of targets, the domain of holomorphic curves, and the moduli stack of holomorphic curves, and it becomes convenient to work in a category that keeps track of this large and small scale information systematically. In this series of talks, I will explain how the category of exploded manifolds does this. The original degenerating family of targets may be replaced by a smooth family of exploded manifolds. Holomorphic curves in this family of exploded manifolds are holomorphic maps of curves within the category of exploded manifolds. The moduli stack of holomorphic curves in this family is a nice stack over the category of exploded manifolds, and is naturally smooth (modulo the usual transversality considerations.) Gromov-Witten invariants of appropriate exploded manifolds are well defined, and do not change in families. By the end of this lecture series I will explain how the gluing formula for Gromov-Witten invariants arises, and why a special cohomology theory must be used.

Without using an appropriate category, such as exploded manifolds or log schemes, an explanation of Gromov-Witten invariants relative normal crossing divisors or the associated gluing formula will seen surprising, unnatural, and unmotivated.[1] By contrast, once log schemes or exploded manifolds are understood, Gromov-Witten invariants relative normal crossing divisors define themselves naturally with no creative leaps required.[2] Similarly, the associated tropical gluing formula involving these relative Gromov-Witten invariants is extremely natural and unsurprising using exploded manifolds. Thus, the definition of Gromov-Witten invariants relative to normal crossing divisors, and the associated degeneration formulas may be succinctly summarized by asserting that Gromov-Witten invariants work in the category of exploded manifolds as nicely, and as naturally as any optimist could hope— more naturally than in the category of smooth manifolds.

Accordingly, these notes start with explaining exploded manifolds before moving on to the desired relative Gromov-Witten invariants or gluing formula. Exploded manifolds will most likely seem strange to the reader, but all their strangest features are forced by the geometry of holomorphic curves in normal crossing degenerations. The strong connection between exploded manifolds and log schemes was not deliberate, but arose because of these geometric constraints on what properties a category dealing with holomorphic curves in normal crossing degenerations should have.

These notes are meant to accompany my series of 3 lectures recorded in April 2016 at the Simons Center for Geometry and Physics at Stonybrook. The videos are available on the SCGP website. Video of an introductory talk I refer to in these

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[1] Some motivation is given in the introduction to [10] or slides from talks linked to from my website, http://maths-people.anu.edu.au/∼parkerb. The gluing formula is also described in an easy case without special language in [7]. The actual lecture series started with an unrecorded talk similar to the one at this link.

[2] This is not to say that defining these Gromov Witten invariants is easy. In both the log and exploded settings, it requires hard, technical work. For log Gromov-Witten invariants, see [4, 3, 1]. A separate, but related approach to defining GW invariants relative a version of normal crossing divisors in the symplectic setting is found in [5]. For the relationship between these approaches and the approach using exploded manifolds, see [11, 13].

recordings is not available, but slides from analogous talks are available at this link or this link. Please do not hesitate to contact me if you have any questions or comments on this work.

2. Exploded manifolds

We begin with the definition of an exploded manifold, however we will be able to extract very little from it until after making some more definitions and explaining some examples.

Definition 2.1. *An exploded manifold is an abstract exploded space locally isomorphic to* $\mathbb{R}^n \times \mathbf{T}^m_P$.

Obviously, we need to understand what an abstract exploded space is, and what these $\mathbb{R}^n \times \mathbf{T}^m_P$ are. All that can be extracted from the above definition at this point is that exploded manifolds with have coordinate charts isomorphic to $\mathbb{R}^n \times \mathbf{T}^m_P$. The type of these coordinate charts depends on nonnegative integers n and m and a m-dimensional integral affine polytope P (defined later). We shall see that the (real) dimension of such a coordinate chart is $n + 2m$.

Definition 2.2. *An abstract exploded space is a set* $\mathbf{B}$ *with topology induced from a surjective map to a Hausdorff topological space* $\mathbf{B} \longrightarrow \lceil \mathbf{B} \rceil$*, and a sheaf,* $\mathcal{E}^\times$*, of* $\mathbb{C}^*\mathfrak{t}^{\mathbb{R}}$*-valued functions, containing constants, and closed under multiplication and taking inverses, where* $\mathbb{C}^*\mathfrak{t}^{\mathbb{R}}$ *is our notation for the group* $(\mathbb{C}^*, \times) \times (\mathbb{R}, +)$.

With this definition, it should be clear that a morphism of abstract exploded spaces is a continuous map $\phi : \mathbf{A} \longrightarrow \mathbf{B}$ so that $\phi^*(\mathcal{E}^\times(\mathbf{B})) \subset \mathcal{E}^\times(\mathbf{A})$. We call functions in $\mathcal{E}^\times$ exploded functions.

We choose the strange notation $\mathbb{C}^*\mathfrak{t}^{\mathbb{R}}$ for $(\mathbb{C}^*, \times) \times (\mathbb{R}, +)$ because it is the group of units for a semiring $\mathbb{C}\mathfrak{t}^{\mathbb{R}}$. In some sense, I consider the algebraic version of exploded manifolds to be nice algebraic spaces over the exploded semiring, $\mathbb{C}\mathfrak{t}^{\mathbb{R}}$. As a set, $\mathbb{C}\mathfrak{t}^{\mathbb{R}}$ is $\mathbb{C} \times \mathbb{R}$, with the element (c, a) written $c\mathfrak{t}^a$. The laws for multiplication and addition are then

$$c_1\mathfrak{t}^{a_1} \times c_2\mathfrak{t}^{a_2} := (c_1c_2)\mathfrak{t}^{a_1+a_2}$$

and

$$c_1\mathfrak{t}^{a_1} + c_2\mathfrak{t}^{a_2} := \begin{cases} c_1\mathfrak{t}^{a_1} & \text{if } a_1 < a_2 \\ (c_1 + c_2)\mathfrak{t}^{a_1} & \text{if } a_1 = a_2 \\ c_2\mathfrak{t}^{a_2} & \text{if } a_2 < a_1 \end{cases}$$

The same construction works with any ring replacing $\mathbb{C}$; for example $0\mathfrak{t}^{\mathbb{R}}$ gives the tropical semiring. There is an associated tropical part homomorphism from $\mathbb{C}^*\mathfrak{t}^{\mathbb{R}}$ to the tropical numbers which will be very important to us.

$$\underline{c\mathfrak{t}^a} := a$$

Equally important, is the following smooth part homomorphism which is only defined on $\mathbb{C}\mathfrak{t}^{[0,\infty)}$, the set of elements $c\mathfrak{t}^a$ where $a \geq 0$.

$$\lceil c\mathfrak{t}^a \rceil := \begin{cases} 0 \text{ if } a > 0 \\ c \text{ if } a = 0 \end{cases}$$

The addition law on $\mathbb{C}^*\mathfrak{t}^{\mathbb{R}}$ and the smooth part homomorphism are both heuristically based on the interpretation that $\mathfrak{t}$ is infinitesimal. For an exploded manifold, the surjective map to a Hausdorff topological space, $\mathbf{B} \longrightarrow \lceil \mathbf{B} \rceil$, is closely related to the above smooth part homomorphism, and $\lceil \mathbf{B} \rceil$ is called the smooth part of $\mathbf{B}$. Corresponding to the tropical part homomorphism, there shall be a surjective tropical part map $\mathbf{B} \longrightarrow \underline{\mathbf{B}}$, where $\underline{\mathbf{B}}$ is some complex of integral affine polytopes called the tropical part of $\mathbf{B}$. The tropical part of a coordinate chart $\mathbb{R}^n \times \mathbf{T}^m_P$ is

P, so coordinate charts on exploded manifolds are classified by dimension and their tropical part.

We now give some important examples of exploded manifolds, building towards understanding a general coordinate chart $\mathbb{R}^n \times \mathbf{T}^m_P$.

Example 2.3. $\mathbf{T} := \mathbf{T}^1_{\mathbb{R}}$

As a topological space $\mathbf{T} = \mathbb{C}^*\mathfrak{t}^{\mathbb{R}}$ with the trivial indiscrete topology. Let $z : \mathbf{T} \longrightarrow \mathbb{C}^*\mathfrak{t}^{\mathbb{R}}$ be the corresponding coordinate. The exploded functions on $\mathbf{T}$ are the monomials:

$$\mathcal{E}^\times(\mathbf{T}) := \{cz^n \text{ so that } n \in \mathbb{Z} \text{ and } c \in \mathbb{C}^*\mathfrak{t}^{\mathbb{R}}\}$$

Although a rather trivial example, $\mathbf{T}$ is important: for any abstract exploded space $\mathbf{B}$, the sheaf of exploded functions is the sheaf of morphisms to $\mathbf{T}$.

Example 2.4. $\mathbf{T}^1_{[0,\infty)} := \operatorname{Expl}(\mathbb{C}, 0)$

As a set, $\mathbf{T}^1_{[0,\infty)} := \mathbb{C}^*\mathfrak{t}^{[0,\infty)}$, with topology induced from the smooth part homomorphism $\mathbb{C}^*\mathfrak{t}^{[0,\infty)} \longrightarrow \mathbb{C} := \lceil \mathbf{T}^1_{[0,\infty)} \rceil$. Let $z : \mathbf{T}^1_{[0,\infty)} \longrightarrow \mathbb{C}^*\mathfrak{t}^{[0,\infty)} \subset \mathbb{C}^*\mathfrak{t}^{\mathbb{R}}$ be the coordinate on $\mathbf{T}^1_{[0,\infty)}$ corresponding to our identification, and let $\lceil z \rceil : \mathbf{T}^1_{[0,\infty)} \longrightarrow \mathbb{C}$ be the composition of z with the smooth part homomorphism. The exploded functions on $\mathbf{T}^1_{[0,\infty)}$ are all functions in the form

$$h(\lceil z \rceil)\mathfrak{t}^a z^n$$

where h is a smooth[3], $\mathbb{C}^*$-valued function, $a \in \mathbb{R}$ is locally constant, and $n \in \mathbb{Z}$.

Later on, we shall discuss the explosion functor, which produces a holomorphic exploded manifold from a complex manifold with normal crossing divisors or more generally, a complex, log smooth, log scheme. The explosion of $\mathbb{C}$ with the divisor 0 is $\mathbf{T}^1_{[0,\infty)}$. For now, we can think of this explosion as 'exploding' the divisor 0 and replacing it with $\mathbb{C}^*\mathfrak{t}^{(0,\infty)}$. From one perspective, this replaces the divisor with a collection of cylinders $\mathbb{C}^*$ indexed by $(0, \infty)$.

Remark 2.5. *[Relationship of exploded manifolds to log schemes] For exploded manifolds $\mathbf{B}$, it turns out that the sheaf of morphisms to $\mathbf{T}^1_{[0,\infty)}$ is equal the the sheaf of exploded functions with tropical part in $[0, \infty) \subset \mathbb{R}$. We can compose such exploded functions with the smooth part homomorphism to obtain $\mathbb{C}$-valued functions. It turns out that such $\mathbb{C}$-valued functions come from functions on $\lceil \mathbf{B} \rceil$. So maps from $\mathbf{B}$ to $\mathbf{T}^1_{[0,\infty)}$ form a sheaf of monoids on $\lceil \mathbf{B} \rceil$ with a homomorphism to the sheaf of $\mathbb{C}$-valued functions on $\lceil \mathbf{B} \rceil$. Such structure should be familiar to log geometers as a log structure. Exploded manifolds with tropical part consisting of polytopes not containing any entire lines may be defined using this log structure together with the natural map to a point $p^\dagger$ with log structure give by the monoid $\mathbb{C}^*\mathfrak{t}^{[0,\infty)}$ and the smooth part homomorphism $\mathbb{C}^*\mathfrak{t}^{[0,\infty)} \longrightarrow \mathbb{C}$. From this perspective, the explosion of a complex log scheme M is $M \times p^\dagger$ with the natural projection to $p^\dagger$. For further details on the relationship between exploded manifolds and log schemes, see* [11].

After this warmup, we are ready to define $\mathbf{T}^m_P$. Let $P \subset \mathbb{R}^m$ be an integral affine polytope, i.e. a subset of $\mathbb{R}^m$ with nonempty interior, cut out by finitely many integral affine inequalities in the form

$$x \cdot \alpha + a \geq 0 \text{ or } x \cdot \alpha + a > 0$$

[3] We can put a different regularity on $\mathbf{T}^1_{[0,\infty)}$, such as holomorphic, or continuous, by changing the regularity required of h. Holomorphic exploded manifolds are abstract exploded spaces locally isomorphic to *an open subset of* $\mathbb{C}^n \times \mathbf{T}_P$ with its sheaf of holomorphic exploded functions. Later on, we shall need to use a regularity slightly weaker, but pretty much as good as smooth called $C^{\infty,\underline{1}}$. See Remark 2.14.

where $\alpha \in \mathbb{Z}^m$ and $a \in \mathbb{R}$.

Example 2.6. $\mathbf{T}_P^m := \mathbf{T}_P$

As a set, $\mathbf{T}_P^m$ is the subset of $\left(\mathbb{C}^*\mathfrak{t}^{\mathbb{R}}\right)^m$ with tropical part in P.

$$\mathbf{T}_P^m := \left\{(c_1\mathfrak{t}^{a_1}, \dots, c_m\mathfrak{t}^{a_n}) \in \left(\mathbb{C}^*\mathfrak{t}^{\mathbb{R}}\right)^m \text{ so that } (a_1, \dots, a_m) \in P\right\}$$

Accordingly, we get coordinate functions $z_i : \mathbf{T}_P^m \longrightarrow \mathbf{T}$ so that $(\underline{z_1}, \dots, \underline{z_m})$ defines a surjective map, $\mathbf{T}_P \longrightarrow \underline{\mathbf{T}_P} := P$, which is our tropical part map. Given an integral affine map $P \longrightarrow [0,\infty)$ in the form $x \mapsto a + x \cdot \alpha$, there is a corresponding monomial on $\mathbf{T}_P^m$

$$\mathfrak{t}^a z^\alpha : \mathbf{T}_P^m \longrightarrow \mathbb{C}^*\mathfrak{t}^{[0,\infty)} \qquad \text{where } z^\alpha := \prod_{i=1}^m z_i^{\alpha_i}$$

We can compose such a monomial with the smooth part homomorphism to define a $\mathbb{C}$-valued function we shall call a smooth monomial.

$$\zeta := \lceil \mathfrak{t}^a z^\alpha \rceil : \mathbf{T}_P \longrightarrow \mathbb{C}$$

The definition of the smooth part homomorphism implies that such a monomial is nonzero exactly where $\underline{\mathfrak{t}^a z^\alpha} = 0$. Such smooth monomials form a finitely generated monoid; choose some basis $\zeta_1, \dots, \zeta_n$. Then the smooth part of $\mathbf{T}_P$ is equal to the image of $(\zeta_1, \dots, \zeta_n)$ in $\mathbb{C}^n$, and the smooth part map $\mathbf{T}_P \longrightarrow \lceil \mathbf{T}_P \rceil$ is $(\zeta_1, \dots, \zeta_n)$. The induced topology on $\mathbf{T}_P$ is the coarsest topology so that smooth monomials are continuous.

With the above understood, we can now define the sheaf of exploded functions $\mathcal{E}^\times(\mathbf{T}_P)$. The exploded functions are all functions in the form

$$h(\zeta_1, \dots, \zeta_n)\mathfrak{t}^a z^\alpha$$

where h is smooth and $\mathbb{C}^*$-valued, and $a \in \mathbb{R}$ and $\alpha \in \mathbb{Z}^m$ are locally constant. Again, we can change the regularity of our exploded manifold by changing the regularity required of h.

Once $\mathbf{T}_P$ is defined, there are no surprises in defining a general coordinate chart $\mathbb{R}^n \times \mathbf{T}_P$. As a topological space, this is just the product of $\mathbf{T}_P$ with $\mathbb{R}^n$, its smooth part is just the product of $\lceil \mathbf{T}_P \rceil$ with $\mathbb{R}^n$, and its tropical part is P. The exploded functions on $\mathbb{R}^n \times \mathbf{T}_P$ are as above, but now h may also depend on $\mathbb{R}^n$.

Remark 2.7. *The reader should now be able to verify the following facts.*

1. *A map from any open subset U of a coordinate chart to $\mathbf{T}_P^m$ is equivalent to m exploded functions $f_i \in \mathcal{E}^\times(U)$ so that $(\underline{f_1}, \dots, \underline{f_m}) \in P$.*
2. *A map $\mathbf{T}_P \longrightarrow \mathbf{T}_Q$ induces an integral affine map $P \longrightarrow Q$.*
3. *Given any integral affine map $P \longrightarrow Q$ there exists a map $\mathbf{T}_P \longrightarrow \mathbf{T}_Q$ inducing it.*
4. *$\mathbf{T}_P$ is isomorphic to $\mathbf{T}_Q$ if and only if P is isomorphic to Q as an integral affine polytope.*
5. *Any map $\mathbf{T}_P \longrightarrow \mathbf{T}_Q$ induces a map $\lceil \mathbf{T}_P \rceil \longrightarrow \lceil \mathbf{T}_Q \rceil$ compatible with the smooth structures on $\lceil \mathbf{T}_P \rceil$ and $\lceil \mathbf{T}_Q \rceil$.*
6. *For any exploded manifold $\mathbf{B}$, there exist functorial smooth part and tropical part maps $\mathbf{B} \longrightarrow \lceil \mathbf{B} \rceil$ and $\mathbf{B} \longrightarrow \underline{\mathbf{B}}$ that are as we have described on coordinate charts. Any morphism $\phi : \mathbf{A} \longrightarrow \mathbf{B}$ has a smooth part $\lceil \phi \rceil$ and*

tropical part $\underline{\phi}$ so that the following diagram commutes.

$$\begin{array}{ccc}
\lceil \mathbf{A} \rceil & \xrightarrow{\lceil \phi \rceil} & \lceil \mathbf{B} \rceil \\
\uparrow & & \uparrow \\
\mathbf{A} & \xrightarrow{\phi} & \mathbf{B} \\
\downarrow & & \downarrow \\
\underline{\mathbf{A}} & \xrightarrow{\underline{\phi}} & \underline{\mathbf{B}}
\end{array}$$

The tropical part can be complicated by how the tropical parts of different coordinate charts glue together. See section 4 of [10] *for more details.*

Example 2.8. $\mathbf{T}_{[0,l]}$

As a set $\mathbf{T}_{[0,l]} = \mathbb{C}^* \mathfrak{t}^{[0,l]}$. A basis for the smooth monomials on $\mathbf{T}_{[0,l]}$ is $\zeta_1 := \lceil z \rceil$ and $\zeta_2 = \lceil \mathfrak{t}^l z^{-1} \rceil$. The smooth part of $\mathbf{T}_{[0,l]}$ is

$$\lceil \mathbf{T}_{[0,l]} \rceil = \{\zeta_1 \zeta_2 = 0\} \subset \mathbb{C}^2 ,$$

two complex planes joined at the origin— the local model around a node of a holomorphic curve (in the smooth category.) Any holomorphic curve $\mathbf{C}$ in the category of exploded manifolds has a smooth part which is a nodal curve $\lceil \mathbf{C} \rceil$. The nodes of $\lceil \mathbf{C} \rceil$ correspond to strata of $\mathbf{C}$ we shall call internal edges. The local model around an internal edge is $\mathbf{T}_{[0,l]}$. Over the node in $\lceil \mathbf{C} \rceil$ is an entire stratum of $\mathbf{C}$ isomorphic to $\mathbf{T}_{(0,l)}$. Because $\mathbf{T}_{(0,l)}$ has no nonconstant smooth monomials, the restriction of exploded functions to an edge are just the monomials cz^n, so maps of $\mathbf{T}_{[0,1]}$ are very rigid restricted to $\mathbf{T}_{(0,l)}$. A key observation which shall later lead to our gluing formula for GW invariants is that choosing a map from $\mathbf{T}_{[0,l]}$ is equivalent to choosing a maps from $\mathbf{T}_{[0,l)}$ and $\mathbf{T}_{(0,l]}$ which agree on $\mathbf{T}_{(0,l)}$.

An instructive way to construct $\mathbf{T}_{[0,l]}$ is as the subset of $\mathbf{T}^2_{[0,\infty)^2}$ where $z_1 z_2 = 1\mathfrak{t}^l$, so ζ_i is the restriction of $\lceil z_i \rceil$. This subset is a fiber of the map

$$z_1 z_2 : \mathbf{T}_{[0,\infty)^2} \longrightarrow \mathbf{T}_{[0,\infty)} .$$

The smooth part of this map is the local model for node formation.

$$\zeta_1 \zeta_2 : \mathbb{C}^2 \longrightarrow \mathbb{C}$$

Over points $\epsilon \mathfrak{t}^0 \in \mathbf{T}_{[0,\infty)}$ we get smooth manifolds $\zeta_1 \zeta_2 = \epsilon$, but over points $c\mathfrak{t}^l$ where $l > 0$, we get exploded manifolds isomorphic to $\mathbf{T}_{[0,l]}$. Under the smooth part map, all these different exploded manifolds are sent to $\{\zeta_1 \zeta_2 = 0\}$. The extra information which is lost in this smooth part map is parametrized by $\mathbb{C}^* \mathfrak{t}^{(0,\infty)}$. This extra information may be thought of as gluing information.

2.1. **The explosion functor.** Note that $\zeta_1 \zeta_2 : \mathbb{C}^2 \longrightarrow \mathbb{C}$ may be thought of as a normal crossing degeneration. The explosion functor applied to such normal crossing degenerations (when they are proper) give smooth families of exploded manifolds. Given a complex manifold M with normal crossing divisors, $\operatorname{Expl} M$ is a holomorphic exploded manifold with smooth part equal to M. The explosion functor replaces coordinate charts that are open subsets $U \subset (\mathbb{C}^n, \{\zeta_1 \cdots \zeta_n = 0\})$ with the corresponding open subsets of $\mathbf{T}^n_{[0,\infty)^n}$ with smooth part $U \subset \mathbb{C}^n = \lceil \mathbf{T}_{[0,\infty)^n} \rceil$, so we replace coordinates ζ_i (whose vanishing locus give the divisor) with exploded coordinates z_i with smooth part $\lceil z_i \rceil = \zeta_i$. Given any holomorphic map $h : M \longrightarrow N$ sending the interior of M to the interior of N and each stratum[4]

[4]The strata of $(\mathbb{C}^n, \{\zeta_1 \dots, \zeta_n = 0\})$ are the sets where $\zeta_i = 0$ for $i \in I$ and $\zeta_j \neq 0$ for $j \notin I$. The strata of M are connected and locally isomorphic to these strata.

of M into some stratum of N, there is a unique explosion of h so that the following diagram commutes.

$$\begin{array}{ccc} \operatorname{Expl} M & \xrightarrow{\operatorname{Expl} h} & \operatorname{Expl} N \\ \downarrow & & \downarrow \\ \lceil \operatorname{Expl} M \rceil = M & \xrightarrow{h} & \lceil \operatorname{Expl} N \rceil = N \end{array}$$

Deligne-Mumford space has a natural structure of a complex orbifold with normal crossing divisors the boundary divisors. It turns out that $\operatorname{Expl} \bar{M}_{g,n}$ represents the moduli stack of stable exploded curves with genus g and n punctures, and the explosion of the forgetful map $\bar{M}_{g,n+1} \longrightarrow \bar{M}_{g,n}$ is the universal curve over this moduli stack. As discussed in Remark 2.5, the explosion functor also applies to nice complex log schemes (including log smooth log schemes.) If X is such a log scheme, we can also apply the explosion functor to the moduli stack of curves in X. As discussed in [11], the result of this is the moduli stack of holomorphic curves in $\operatorname{Expl} X$.

One purpose of these lectures is to argue that a good way to define Gromov-Witten invariants of M relative to its normal crossing divisor is to use holomorphic curves in $\operatorname{Expl} M$. We shall see that the moduli stack of such curves is naturally compact, and (modulo the usual transversality issues) has the structure of an orbifold in the category of exploded manifolds. A virtual fundamental class and Gromov-Witten invariants can be defined that are invariant in families of exploded manifolds. In particular, by using the explosion of a normal crossing or log smooth degeneration, we get a degeneration formula for Gromov-Witten invariants by calculating Gromov-Witten invariants of an exploded manifold with smooth part the singular fiber.

The explosion functor works for complex manifolds with normal crossing divisors, or complex log schemes. There is no completely functorial analogue of the explosion functor for symplectic manifolds with normal crossing divisors, however, by making a contractible choice of extra structure (such as that discussed in [6]) we can construct an explosion of such a symplectic manifold with normal crossing divisors. Gromov-Witten invariants of this exploded manifold are invariants of the original symplectic manifold with normal crossing divisors, so are suitable for defining Gromov Witten invariants relative normal crossing divisors in the symplectic setting. The degeneration formula also works for the symplectic analogue of normal crossing or log smooth degenerations.

2.2. **Tangent space.** Our defining sheaf of exploded functions $\mathcal{E}^\times(\mathbf{B})$ only has the operation of multiplication, so is not suitable for defining tangent vectors as derivations. Let $\mathcal{E}(\mathbf{B})$ be the sheaf of $\mathbb{C}\mathfrak{t}^{\mathbb{R}}$-valued functions generated from $\mathcal{E}^\times(\mathbf{B})$ by allowing addition. We can then define vectorfields v as derivations on $\mathcal{E}(\mathbf{B})$ satisfying the following:

$$v(cx + dy) = cvx + dvy \text{ for } x, y \in \mathcal{E}(\mathbf{B}) \text{ and } c, d \in \mathbb{C}\mathfrak{t}^{\mathbb{R}}$$

$$v(xy) = (vx)y + xvy$$

$$\underline{v(1\mathfrak{t}^0)} = \underline{0\mathfrak{t}^0}$$

and if x is $\mathbb{R}\mathfrak{t}^{\mathbb{R}}$ valued, then so is vx. These conditions ensure that $\underline{vx} = \underline{x}$ and if $x \in \mathcal{E}^\times$, then vx is a smooth $\mathbb{C}$-valued function times x.

With these definitions, in standard coordinates $z_1, \dots, z_m$ on $\mathbf{T}^m_P$, the tangent vectorfields are real smooth functions times the real and imaginary parts of $z_i \frac{\partial}{\partial z_i}$. These are sections of a vector bundle $T\mathbf{T}^m_P$ isomorphic to $\mathbb{R}^{2m} \times \mathbf{T}^m_P$. Vector fields and the tangent functor behave largely as they do for smooth manifolds:

- The relationship between 1-parameter groups of isomorphisms and flows of vectorfields is the same as usual.
- The cotangent bundle and tensors are defined as usual. Construction of tensors using partitions of unity also work as usual (assuming $\lceil \mathbf{B} \rceil$ is second countable.)
- Transversality, intersection, and fiber products of transverse maps work as usual.[5]

The first surprise about $T\mathbf{B}$ is the existence of integral vectors ${}^{\mathbb{Z}}T_p\mathbf{B} \subset T_p\mathbf{B}$. A vector v is integral if $vx/x \in \mathbb{Z}$ for all exploded functions $x \in \mathcal{E}^{\times}(\mathbf{B})$. An example of a nonzero integral vector is the real part of $z\frac{\partial}{\partial z}$ at any point in $\mathbf{T}_{[0,\infty)}$ with tropical part in $(0,\infty) \subset [0,\infty)$. An arbitrary exploded function $x = h(\lceil z \rceil)\mathfrak{t}^a z^n$, so $vx/x = n$. Over the interior of strata with m-dimensional tropical part, ${}^{\mathbb{Z}}T_p\mathbf{B}$ is $\mathbb{Z}^m$-lattice inside $T_p\mathbf{B}$. We can identify this lattice with the lattice of integral vectors at the image of p in $\underline{\mathbf{B}}$. The obvious identification is functorial.

2.3. **Almost complex structures and holomorphic curves.**

Definition 2.9. *An almost complex structure J on $\mathbf{B}$ is an endomorphism of $T\mathbf{B}$ (represented as a section of $T^*\mathbf{B} \otimes T\mathbf{B}$) so that $J^2 = -\operatorname{id}$, and so that for any integral vector v and exploded function x,*

$$(Jv)x = i(vx)$$

For example, this extra condition says that on $\mathbf{T}_{[0,\infty)}$, J of the real part of $z\frac{\partial}{\partial z}$ is the imaginary part of $z\frac{\partial}{\partial z}$ plus a vectorfield that vanishes where $\lceil z \rceil = 0$.

Before we define what a holomorphic curve is, we need the appropriate analogue of 'compact' for exploded manifolds:

Definition 2.10. $\mathbf{B}$ *is complete if $\lceil \mathbf{B} \rceil$ is compact, and every polytope in $\underline{\mathbf{B}}$ is complete (i.e. $\mathbf{B}$ is locally isomorphic to $\mathbb{R}^n \times \mathbf{T}_P$ where $P \subset \mathbb{R}^m$ is closed).*

For example, $\mathbf{T}_{(0,1)}$ is not complete, but $\mathbf{T}$ is. The explosion of any compact manifold with normal crossing divisors is also complete. There is a similar notion of a complete map, which is the analogue of a proper map. See Definition 3.5 of [10].

Definition 2.11. *A holomorphic curve is a complete, 2-dimensional exploded manifold $\mathbf{C}$ with an almost complex structure j.*

A curve in $(\mathbf{B}, J)$ is a holomorphic curve $(\mathbf{C}, j)$ with a map

$$f : \mathbf{C} \longrightarrow \mathbf{B}\ .$$

This is a holomorphic curve if $df \circ j = J \circ df$.

Apart from the exceptional example of $\mathbf{C} = \mathbf{T}$, each holomorphic curve $(\mathbf{C}, j)$ is locally isomorphic to one of three models:

- an open subset of $\mathbb{C}$— here our curve behaves just like a usual smooth holomorphic curve, the tropical part of such a connected open subset is just a point;
- an open subset of $\mathbf{T}_{[0,\infty)}$— the smooth part of our curve here is an open subset of $\mathbb{C}$ with the special point 0, the tropical part is $[0,\infty)$, we call the strata of $\mathbf{C}$ over $(0,\infty)$ an external edge or end of $\mathbf{C}$;

[5] It is important to note that transversality and intersection products are determined using $T\mathbf{B}$ not with a notion of the tangent space of the smooth part $\lceil \mathbf{B} \rceil$. This is important, as an attempt to use fiber products using $\lceil \mathbf{B} \rceil$ will not give a correct gluing formula. A consequence of us needing to do fiber products on the level of exploded manifolds is that we will need to use an interesting cohomology theory, called refined cohomology.

- an open subset of $\mathbf{T}_{[0,l]}$— the smooth part of our curve here is isomorphic to an open subset of $\{\zeta_1\zeta_2 = 0\} \subset \mathbb{C}^2$, the model for a node. The strata over this node, $\mathbf{T}_{(0,1)}$ is called an internal edge of $\mathbf{C}$.

The smooth part $\lceil \mathbf{C} \rceil$ of our curve is some compact nodal curve with a finite collection of special points. The tropical part $\underline{\mathbf{C}}$ is a complete integral affine graph with a vertex for every component of $\lceil \mathbf{C} \rceil$, an internal edge isomorphic to $[0, l]$ for each node, and a external edge isomorphic to $[0, \infty)$ for each special point. The extra information in $\mathbf{C}$, not seen in the nodal curve $\lceil \mathbf{C} \rceil$, is a $\mathbb{C}^*\mathfrak{t}^{(0,\infty)}$-worth of gluing information for each node, discussed in Example 2.8.

Because $f : \mathbf{C} \longrightarrow \mathbf{B}$ has more information than $\lceil f \rceil : \lceil \mathbf{C} \rceil \longrightarrow \lceil \mathbf{B} \rceil$, sometimes $\lceil f \rceil$ has more automorphisms than f.[6] Call f *stable* if both f and $\lceil f \rceil$ only have a finite number of automorphisms.

2.4. **Families of exploded manifolds and families of curves.**

Definition 2.12. *A family of exploded manifolds is a complete map $\pi : \hat{\mathbf{B}} \longrightarrow \mathbf{B}_0$ so that for all $p \in \hat{\mathbf{B}}$,*

$$T_p\pi : T_p\hat{\mathbf{B}} \longrightarrow T_{\pi(p)}\mathbf{B}_0 \text{ is surjective,}$$
$$\text{and } T_p\pi\left({}^{\mathbb{Z}}T_p\hat{\mathbf{B}}\right) = {}^{\mathbb{Z}}T_{\pi(p)}\mathbf{B}_0 \ .$$

For example, exploding any proper normal crossing degeneration gives a family of exploded manifolds.

As usual, there is a notion of vertical (co)tangent bundle on $\hat{\mathbf{B}} \longrightarrow \mathbf{B}_0$. A family of tensors such as almost complex structures is a section of the appropriate vertical tensor bundle. Using fiber products, we can perform base changes of families as usual, and as usual, any family of tensors pulls back under such base changes. As well as holomorphic curves in a single exploded manifold $\mathbf{B}$, we are interested in holomorphic curves in a family of exploded manifolds $\hat{\mathbf{B}} \longrightarrow \mathbf{B}_0$ with a family of almost complex structures J.

Definition 2.13. *A family of (stable, holomorphic) curves $\hat{f}$ in a family of targets $(\hat{B}, J)$ is a family $\mathbf{C}(\hat{f}) \longrightarrow \mathbf{F}(\hat{f})$ with a family of almost complex structures j, and a map $\hat{f}$ so that the following diagram commutes*

$$\begin{array}{ccc} (\mathbf{C}(\hat{f}), j) & \xrightarrow{\hat{f}} & (\hat{\mathbf{B}}, J) \\ \downarrow & & \downarrow \\ \mathbf{F}(\hat{f}) & \longrightarrow & \mathbf{B}_0 \end{array}$$

and so that $\hat{f}$ restricted to any fiber is a (stable, holomorphic) curve in the corresponding fiber of $\hat{\mathbf{B}} \longrightarrow \mathbf{B}_0$.

Remark 2.14. *The natural regularity for families of curves, and indeed individual curves[7] is just a tiny bit weaker than smooth. There is a notion of $C^{\infty,\underline{1}}$ regularity, defined in section 7 of [10], which for all practical purposes is as good as smooth. We can still take derivatives to all orders with $C^{\infty,\underline{1}}$ maps, and $C^{\infty,\underline{1}}$ exploded manifolds form a category that works just as well as smooth exploded manifolds (except it takes*

[6]The cases in which $\lceil f \rceil$ may have more automorphisms are those in which gluing information at nodes is not preserved. The only case in which f has more automorphisms than $\lceil f \rceil$ is when $\mathbf{C} = \mathbf{T}$ and f is not injective.

[7]We can achieve more regularity for families of curves if J is integrable, and it may be true that 'smooth' is the correct regularity for families of curves when J satisfies an extra condition called being $\bar{\partial}$-log compatible. I am unaware of a proof or a counter example.

longer to define...). The almost complex structure j and all maps in Definitions 2.11 and 2.13 should be regarded as $C^{\infty,\underline{1}}$, although for proving compactness of the moduli stack of J-holomorphic curves in [14], *we use stronger regularity on $(\hat{\mathbf{B}}, J) \longrightarrow \mathbf{B}_0$.*

On $\mathbf{T}_{[0,\infty)}$, a $C^{\infty,\underline{1}}$ function is a continuous function which is smooth on $\mathbb{C}\setminus 0 \subset \mathbf{T}_{[0,\infty)}$, and which converges as $\lceil z\rceil \to 0$ along with all derivatives faster than $|\lceil z\rceil|^{\delta}$ for every $\delta < 1$. Remembering that our vector fields on $\mathbf{T}_{[0,\infty)}$ are generated by the real and imaginary parts of $z\frac{\partial}{\partial z}$, we can consider $\mathbb{C}\setminus 0$ as having a cylindrical end at 0, and a $C^{\infty,\underline{1}}$ function is one for which all derivatives converge exponentially with all weights < 1. In this lecture series, we shall not describe $C^{\infty,\underline{1}}$ regularity on more general coordinate charts, but note that $C^{\infty,\underline{1}}$ is the same as smooth on $\mathbb{R}$ and $\mathbf{T}$, and all other coordinate charts can be obtained as fiber products from $\mathbb{R}$, $\mathbf{T}$, and $\mathbf{T}_{[0,\infty)}$, and these fiber products are also valid in the category of $C^{\infty,\underline{1}}$ exploded manifolds.

Definition 2.15. *A morphism of $C^{\infty,\underline{1}}$ families of curves $\hat{f} \longrightarrow \hat{g}$ is a commutative diagram of $C^{\infty,\underline{1}}$ maps*

$$\begin{array}{ccccc} \mathbf{F}(\hat{f}) & \longleftarrow & (\mathbf{C}(\hat{f}), j) & \xrightarrow{\hat{f}} & \hat{\mathbf{B}} \\ \downarrow & & \downarrow & \nearrow_{\hat{g}} & \\ \mathbf{F}(\hat{f}) & \longleftarrow & (\mathbf{C}(\hat{f}), j) & & \end{array}$$

so that the left hand square is a pullback diagram.

2.5. **The moduli stack of curves.** Use the notation $\mathcal{M}^{st}(\hat{\mathbf{B}})$ for the moduli stack of stable curves in $\hat{\mathbf{B}}$. This is the category of $C^{\infty,\underline{1}}$ families of stable curves in $\hat{\mathbf{B}}$ along with the functor $\mathbf{F}$ which assigns the exploded manifold $\mathbf{F}(\hat{f})$ to the family of curves $\hat{f}$. Use the notation $\mathcal{M}(\hat{\mathbf{B}}) \subset \mathcal{M}^{st}(\hat{\mathbf{B}})$ for the substack of holomorphic curves.

The reader unfamiliar with stacks should think of $\mathcal{M}^{st}$ as something so that a map $\mathbf{F} \longrightarrow \mathcal{M}^{st}$ corresponds to a family of curves parametrized by $\mathbf{F}$. In the case of curves mapping to a point, $\mathcal{M}^{st}(\cdot) = \mathcal{M}(\cdot)$, and both are represented by (the stack of maps into) $\coprod_{g,n} \operatorname{Expl} \bar{M}_{g,n}$.

$\mathcal{M}^{st}$ comes with a natural topology. Say that a subcategory $\mathcal{U}$ of $\mathcal{M}^{st}$ is a substack if $\hat{f}$ is in $\mathcal{U}$ if and only if all the individual curves in $\hat{f}$ are isomorphic to a curve in $\mathcal{U}$. Define $\mathcal{U}$ to be open if for all $\hat{f}$ in $\mathcal{M}^{st}$, the subset of $\mathbf{F}(\hat{f})$ parametrizing curves in $\mathcal{U}$ is open. For example $\mathcal{M} \subset \mathcal{M}^{st}$ is closed, and the moduli stack of curves contained in an open subset of $\mathbf{B}$, or consisting of curves f so that $\underline{f}$ has no internal edges, is open. A slightly more interesting example of a closed substack is the substack of $\mathcal{M}^{st}(\mathbf{B})$ consisting of curves f with at least one component of $\lceil \mathbf{C}(f)\rceil$ unstable and representing a trivial homology class in $\lceil \mathbf{B}\rceil$ either by itself or with other components. It's good to know that there is an open neighborhood of $\mathcal{M}$ excluding those annoying little buggers.

Interesting phenomena occur in the natural topology on the moduli stack of (not necessarily stable) $C^{\infty,\underline{1}}$ curves in $\mathbf{B}$. The unstable curves are a closed substack, but every neighborhood of an unstable curve includes its stabilization (if the stabilization exists).

2.6. **Regularity of the moduli stack of holomorphic curves.**

There is a reasonably nice (but infinite-dimensional) bundle $\mathcal{Y}$ over $\mathcal{M}^{st}(\hat{\mathbf{B}})$ discussed in Section 2.2 of [12]. There is section $\bar{\partial} : \mathcal{M}^{st} \longrightarrow \mathcal{Y}$ which on a curve f is $\bar{\partial} f := (df + J \circ df \circ j)/2$, so $\mathcal{M} \subset \mathcal{M}^{st}$ is the intersection of $\bar{\partial}$ with 0. It is proved in [12] that if $\bar\partial$ is transverse[8] to 0 at a holomorphic curve f, then there exists an open neighborhood $\mathcal{U} \subset \mathcal{M}^{st}$ of f and a $C^{\infty,\underline{1}}$ family of curves $\hat f$ with group of automorphisms G so that $\mathcal{M} \cap \mathcal{U}$ is represented by $\hat f/G$. In other words, the moduli stack of curves is an orbifold close to where $\bar\partial$ is transverse to 0.

In general, $\bar\partial$ is not transverse to 0. In [12], it is shown that on a neighborhood $\mathcal{U}$ of f, there exist suitably nice[9] finite-dimensional vector sub-bundles V of $\mathcal{Y}$ so that $\bar\partial$ is transverse to V. Theorem 6.6 of [12] then states that in this situation, $\{\bar\partial \subset V\} \subset \mathcal{U}$ is an orbifold locally represented by $\hat f/G$ for some family of curves $\hat f$. Then $V(\hat f)$ is a finite-dimensional, G-equivariant vector bundle over $\mathbf{F}(\hat f)$ with a natural section

$$\bar\partial \hat f : \mathbf{F}(\hat f) \longrightarrow V(\hat f)$$

and the moduli stack of holomorphic curves is locally represented by the quotient by G of the intersection of $\bar\partial \hat f$ with 0. The data $(\mathcal{U}, V, \hat f/G)$ with the section $\bar\partial$ defines a nice kind of Kuranishi chart for $\mathcal{M} \subset \mathcal{M}^{st}$ which is embedded in $\mathcal{M}^{st}$. We can also choose V to be a complex vector bundle, and $\bar\partial$ to be transverse to 0 in a stronger sense[10] so that the homotopy of $D\bar\partial$ to a $\mathbb{C}$-linear operator defines a canonical orientation of $\mathbf{F}(\hat f)$ (relative to $\mathbf{B}_0$ in the case of curves in a family of targets $\hat{\mathbf{B}} \longrightarrow \mathbf{B}_0$.)

Remark 2.16. *The statement of regularity above includes all necessary gluing statements, as a neighborhood of a nodal curve includes all relevant gluings. It also includes the gluing statements required for relating the Gromov-Witten invariants of a fiber of* $\hat{\mathbf{B}} \longrightarrow \mathbf{B}_0$ *with singular smooth part to the Gromov-Witten invariants of nearby fibers.*

Under the assumption that $\mathcal{M} \subset \mathcal{M}^{st}$ obeys suitable compactness conditions, Theorem 7.3 of [12] constructs a compatible collection[11] of such Kuranishi charts covering $\mathcal{M}$, and a virtual fundamental class is constructed from this data in [15].

2.7. Taming forms and compactness of the moduli stack of holomorphic curves.

The smooth part $\lceil \mathbf{B} \rceil$ of $\mathbf{B}$ also has a (co)tangent space, with a canonical complex structure normal to strata.[12]

A taming form ω on $\mathbf{B}$ is a closed 2-form on $\lceil \mathbf{B} \rceil$, symplectic on strata, and positive on holomorphic planes normal to strata.[13]

An almost complex structure on $\mathbf{B}$ does not necessarily induce an almost complex structure on $\lceil \mathbf{B} \rceil$. A nice kind of almost complex structure that does induce an almost complex structure on $\lceil \mathbf{B} \rceil$ is called a $\bar\partial$-log compatible complex structure. $\bar\partial$-log compatible complex structures are defined in section 3 of [14]. Roughly speaking J is $\bar\partial$-log compatible if for all exploded functions $x \in \mathcal{E}^\times \mathbf{B}$,

$$f^{-1}(df - i df \circ J)$$

---

[8]See [12], sections 2.7 and 2.8 for a discussion of the tangent space of $\mathcal{M}^{st}$ and the linearization of $\bar\partial$.

[9]Definition 2.24 of [12]

[10]Definition 2.26 of [12]

[11]We are omitting many details. Roughly speaking, we construct a collection of embedded Kuranishi charts, compatible in the sense that on their common domain of definition, one includes into the other. One property of our Kuranishi charts we have not mentioned is that they can be shrunk appropriately. This property is essential for making global constructions of sections of sheaves over Kuranishi charts, and is related to the Hausdorff condition one could desire for making such global constructions.

[12]See section 2 of [14]. Unlike $T\mathbf{B}$, the tangent space of $\lceil \mathbf{B} \rceil$ is not always a vector bundle, and is not particularly nice.

[13]For a more precise statement, see section 2 of [14].

is a $\mathbb{C}$-valued 1-form on $\mathbf{B}$ pulled back from $\lceil \mathbf{B} \rceil$. The standard complex structure on $\mathbf{T}_P$ is $\bar\partial$-log compatible, and Lemma 3.10 of [14] gives that the space of $\bar\partial$-log compatible almost complex structures on $\mathbf{B}$ tamed by a given taming form is nonempty and connected. ($\bar\partial$-log compatible almost complex structures on smooth manifolds are just the same as smooth almost complex structures.)

The following theorem follows from Theorem 6.1 and Lemma 4.2 of [14] along with Lemma 2.8 of [12] to confirm that the topology used in [14] is the natural topology.

Theorem 2.17. *Suppose that J is a $\bar\partial$-log compatible almost complex structure tamed by ω on $\mathbf{B}$, a complete exploded manifold, and suppose further that there exists an affine immersion $\underline{\mathbf{B}} \longrightarrow [0,\infty)^N$. Then the moduli stack of holomorphic curves in $\mathcal{M}^{st}(\mathbf{B})$ with genus g, n ends, and ω-energy at most E is compact.*[14]

If there only exists an immersion $\underline{\mathbf{B}} \longrightarrow \mathbb{R}$, then $\mathcal{M}$ is compact restricted to every connected component of $\mathcal{M}^{st}(\mathbf{B})$.

The analogous result holds for holomorphic curves in a family of exploded manifolds $\hat{\mathbf{B}} \longrightarrow \mathbf{B}_0$, but now $\mathcal{M} \longrightarrow \mathbf{B}_0$ is proper when restricted to curves with bounded energy, number of ends and genus or when restricted to connected components of $\mathcal{M}^{st}(\hat{\mathbf{B}})$ respectively.

2.8. **DeRham cohomology theories.**

We shall need to discuss some features of three different DeRham cohomology theories[15] for exploded manifolds: ordinary cohomology H^*, refined cohomology ${}^rH^*$ and refined cohomology generated by functions, ${}^r_{fg}H^*$. All three are identical to usual DeRham cohomology when applied to smooth manifolds, and have integration and Stokes' theorem working as usual. These 3 cohomology theories are related as follows:

$$H^*(\mathbf{B}) \hookrightarrow {}^rH^*(\mathbf{B}) \longleftarrow {}^r_{fg}H^*(\mathbf{B})$$

Ordinary cohomology $H^*(\mathbf{B})$ is finite dimensional if $\mathbf{B}$ is compact, invariant[16] in connected families of exploded manifolds, and for M a complex manifold with normal crossing divisors, $H^*(\operatorname{Expl} M) = H^*(M, \mathbb{R})$. The problem with ordinary cohomology is that it does not have pushforwards compatible with fiber products of exploded manifolds, so it is inappropriate for our gluing formula.

Refined cohomology, ${}^rH^*$ is the minimal extension of H^* that does possess pushforwards compatible with fiber products. It is also invariant in connected families of exploded manifolds and is usual cohomology on smooth manifolds, but when M is a complex manifold with normal crossing divisors, ${}^rH^*(\operatorname{Expl} M)$ is in general infinite dimensional.

Refined cohomology generated by functions ${}^r_{fg}H^*$ also possesses pushforwards compatible with fiber products, but it is only invariant in families of exploded manifolds parametrized by $\mathbb{R}$, not general families. This non-invariance is actually a convenient feature, because it allows multiple different gluing formulas for the same Gromov-Witten invariant. The main advantage of ${}^r_{fg}H^*$ is its compatablity with tropical completion, discussed in section 7 of [15]. For a given point $p \in \underline{\mathbf{B}}$, the set of points in $\mathbf{B}$ with tropical part p has the structure of a manifold $\mathbf{B}|_p$ which has a canonical completion $\mathbf{B}\check{|}_p$ to a complete exploded manifold (when $\mathbf{B}$ starts off

[14] The reader may wonder why we use 'compact' here instead 'complete'. The virtual moduli space is complete, but $\mathcal{M}$ may not be complete when $\bar\partial$ is not transverse to 0. The non-transverse intersection with 0 of a section of a vector bundle over a complete exploded manifold may not be complete, but it is always compact.

[15] H^* and ${}^rH^*$ are defined in [8]. ${}^r_{fg}H^*$ is defined in [15].

[16] More accurately, H^* of the fibers of a family of exploded manifolds forms a kind of flat bundle over the family with interesting monodromy. See section 11 of [8].

compact). Any differential form θ on $\mathbf{B}$ restricts to give a differential form on $\mathbf{B}\rfloor_p$ which then extends canonically to a differential form $\theta\check{\rfloor}_p$ on $\mathbf{B}\check{\rfloor}_p$. The map $\theta \mapsto \theta\check{\rfloor}_p$ gives a well defined map

$$ {}^{r}_{fg}H^*(\mathbf{B}) \longrightarrow {}^{r}H^*(\mathbf{B}\check{\rfloor}_p) \ . $$

2.9. **Gromov-Witten invariants.**

A virtual fundamental class $[\mathcal{M}]$ for the moduli stack of holomorphic curves is constructed in [12]. We can integrate forms from H^*, ${}^rH^*$ or ${}^r_{fg}H^*$ over $[\mathcal{M}]$ to define numerical Gromov-Witten invariants. In particular, so long as $\mathbf{B}$ is complete, given any evaluation map

$$ ev : \mathcal{M}^{st}(\mathbf{B}) \longrightarrow \mathbf{X} $$

where $\mathbf{X}$ is an oriented exploded orbifold or manifold so that ev is proper restricted to $\mathcal{M}_{g,n,E}$, (the moduli stack of holomorphic curves with genus g, n ends, and ω-energy E), we can pull back and integrate $\theta \in {}^rH^*_c(\mathbf{X})$.[17] The integral

$$ \int_{[\mathcal{M}_{g,n,E}]} ev^*\theta $$

is well defined independent of all choices, and invariant in families $\hat{\mathbf{B}} \longrightarrow \mathbf{B}_0$ for which the evaluation map makes sense. We can also push forward the virtual fundamental class to a class $\eta_{g,n,E} \in {}^r_{fg}H^*(\mathbf{X})$ so that

$$ \int_{[\mathcal{M}_{g,n,E}]} ev^*\theta = \int_{\mathbf{X}} \eta_{g,n,E} \wedge \theta \ . $$

This class $\eta_{g,n,E} \in {}^r_{fg}H^*(\mathbf{X})$ is independent of all choices and invariant in families parametrized by $\mathbb{R}$. Its image in ${}^rH^*(\mathbf{X})$ is invariant in all connected families in the following sense. Given a family of evaluation maps to a family of targets, relatively oriented over $\mathbf{B}_0$,

$$ \begin{array}{ccc} \mathcal{M}^{st}(\hat{\mathbf{B}}) & \xrightarrow{ev} & \hat{\mathbf{X}} \\ \downarrow & & \downarrow \\ \mathbf{B}_0 & \xrightarrow{\text{id}} & \mathbf{B}_0 \end{array} $$

so long as ev is proper restricted to $\mathcal{M}^{st}_{g,n,E}(\hat{\mathbf{B}})$, we can again push forward the virtual fundamental class to define $\eta_{g,n,E} \in {}^r_{fg}H^*(\hat{\mathbf{X}})$.[18] The precise statement of invariance is that this class $\eta_{g,n,E}$ is compatible with base changes. This follows from the observation that embedded Kuranishi structures pull back under base changes, and Theorem 5.22 of [12].

For stating our gluing theorem, it is convenient to package the different $\eta_{g,n,E}$ into a generating function

$$ \eta := \sum_{g,n,E} \hbar^{2g+n-2} q^E \eta_{g,n,E} $$

Remark 2.18. *It is also possible to encapsulate the pushforward of tautological classes in η so that it also records descendant Gromov-Witten invariants. The same invariance properties still hold, and the gluing formula still works. See* [7] *for an example of this.*

[17] We can also use differential forms representing classes in H^* or ${}^r_{fg}H^*$, as these differential forms automatically represent classes in ${}^rH^*$.

[18] For tagging the moduli space with ω-energy E, we are assuming the the cohomology class represented by the taming form ω does not change in our family $\hat{\mathbf{B}} \longrightarrow \mathbf{B}_0$. An analogous statement holds if this is not the case.

2.10. **Tropical gluing formula.**

Let γ be a tropical curve in $\underline{\mathbf{B}}$, and let $\mathcal{M}^{st}_\gamma(\mathbf{B})$ denote the moduli stack of stable curves in $\mathbf{B}$ with a chosen isomorphism of their tropical part with γ. This is an $|\mathrm{Aut}\,\gamma|$-fold cover of the substack of $\mathcal{M}^{st}(\mathbf{B})$ consisting of curves with tropical part isomorphic to γ. The contribution of $\mathcal{M}^{st}_\gamma$ to η is well defined:

$$\text{for } p \in \underline{\mathbf{X}}, \quad \eta\rceil_p = \sum_{\gamma \in \underline{ev}^{-1}(p)} \eta_\gamma / \left|\mathrm{Aut}\,\gamma\right|$$

and $\mathcal{M}^{st}_\gamma$ is a fiber product of moduli stacks which we shall explain below.

$$\begin{array}{ccc} \mathcal{M}^{st}_\gamma(\mathbf{B}) & \longrightarrow & \prod_{e\in \mathrm{ied}(\gamma)} \mathbf{B}|_e/\mathbb{C}^* \\ \downarrow \text{cut} & & \downarrow \Delta \\ \prod_v \mathcal{M}^{st}_{\gamma_v}(\mathbf{B}\rceil_v) & \longrightarrow & \prod_{e\in \mathrm{ied}(\gamma)} (\mathbf{B}|_e/\mathbb{C}^*)^2 \end{array} \tag{1}$$

For any vertex v of γ, applying tropical completion at v to a curve in $\mathcal{M}^{st}_\gamma(\mathbf{B})$ gives a curve in $\mathbf{B}\rceil_v$. The subset of the domain $\mathbf{C}$ over v is a punctured Riemann surface $\mathbf{C}|_v$, which maps to $\mathbf{B}|_v$. This map has a canonical completion to a curve $\mathbf{C}\rceil_v \longrightarrow \mathbf{B}\rceil_v$, with tropical part parametrized by γ_v, the tropical curve obtained from γ by cutting at all internal edges, and extending the cut edges of the component containing v to be semi-infinite. The map cut : $\mathcal{M}^{st}_\gamma(\mathbf{B}) \longrightarrow \prod \mathcal{M}^{st}_{\gamma_v}(\mathbf{B}\rceil_v)$ is the corresponding map of stacks.

In the above diagram, $\mathrm{ied}(\gamma)$ indicates the set of internal edges of γ. There is a copy of $\mathbb{C}^*$ in the domain of a curve f over each point in an edge of γ. Over every point p in $\underline{\mathbf{B}}$, the manifold $\mathbf{B}|_p$, is a $(\mathbb{C}^*)^n$ bundle when p is in a n-dimensional stratum of $\mathbf{B}$. Pick a point in the tropical part of each edge e of γ, and let $\mathbf{B}|_e \subset \mathbf{B}$ be the subset of $\mathbf{B}$ with tropical part equal to this point. The restriction of f determines a map $\mathbb{C}^* \longrightarrow \mathbf{B}|_e$ that is equivariant with respect to a $\mathbb{C}^*$ action on $\mathbf{B}|_e$ determined by the edge e of γ. In particular, the restriction of f is equivariant with respect to the action with weight $(a_1, \dots, a_n)$ when the derivative of γ on e is $(a_1, \dots, a_n)$. The top line of Diagram 1 is the associated evaluation map.

$$\mathcal{M}^{st}_\gamma(\mathbf{B}) \longrightarrow \prod_{e\in \mathrm{ied}(\gamma)} \mathbf{B}|_e/\mathbb{C}^* \tag{2}$$

There is a similar evaluation map for each of the edges of γ_v corresponding to an internal edge of γ. There are two such edges in $\coprod_v \gamma_v$, which is why the bottom line of Diagram 1 has target $\prod(\mathbf{B}|_e/\mathbb{C}^*)^2$.

In each case $\mathbf{B}|_e$ is the quotient of a manifold X_e by a trivial group action. Use m_e to denote the number so that derivative of γ at the edge e is m_e times a primitive integral vector. If m_e is zero, the $\mathbb{C}^*$ action on $\mathbf{B}|_e$ is trivial, and we set $X_e := \mathbf{B}|_e$. Otherwise, $\mathbf{B}|_e/\mathbb{C}^*$ is the quotient of a manifold X_e by the trivial $\mathbb{Z}_{m_e}$ action.

Remark 2.19. *Each of the stacks in Diagram 1 is actually a stack over the category of manifolds. The associated moduli stacks of holomorphic curves are not compact. We really need to work with appropriate completions of these stacks in order to prove our gluing theorem.*

$\mathcal{M}^{st}_{\gamma_v}(\mathbf{B}\rceil_v)$ has a natural closure $\mathcal{M}^{st}_{[\gamma_v]}(\mathbf{B}\rceil_v)$ in the moduli stack of curves with labeled ends. X_e sits inside a related exploded manifold $\mathrm{End}(\mathbf{B}\rceil_v)$ with a similar evaluation map constructed in section 3.1 of [9].

$$\begin{array}{ccc}
\mathcal{M}^{st}_{\gamma_v}(\mathbf{B}\check{\rceil}_v) & \longrightarrow & X_e \\
\big\downarrow & & \big\downarrow \\
\mathcal{M}^{st}_{[\gamma_v]}(\mathbf{B}) & \longrightarrow & \mathrm{End}(\mathbf{B}\check{\rceil}_v)
\end{array}$$

The corresponding evaluation map at all k edges of γ_v gives a map to a component $\mathrm{End}_{\gamma_v}(\mathbf{B}\check{\rceil}_v)$ of $(\mathrm{End}(\mathbf{B}\check{\rceil}_v))^k$. There is a similar evaluation map from $\mathcal{M}^{st}(\mathbf{B})$, which records the location of all external edges. As we don't have a labeling of external edges, this evaluation map takes values in $\coprod_n(\mathrm{End}\,\mathbf{B})^n/S_n$. We now have the elements of the key diagram for our gluing formula.

$$\begin{array}{ccc}
\mathcal{M}^{st}(\mathbf{B}) & \xrightarrow{\ ev\ } & \coprod_n(\mathrm{End}\,\mathbf{B})^n/S_n \\
\big\uparrow & & \big\uparrow i_\gamma \\
\mathcal{M}^{st}_\gamma & \longrightarrow & \prod_{\mathrm{ed}(\gamma)} X_e \\
\big\downarrow \mathrm{cut} & & \big\downarrow \Delta \\
\prod_v \mathcal{M}^{st}_{[\gamma_v]}(\mathbf{B}\check{\rceil}_v) & \xrightarrow{\prod_v ev_{\gamma_v}} & \prod_v \mathrm{End}_{\gamma_v}(\mathbf{B}\check{\rceil}_v)
\end{array}$$

In the above, $\mathrm{ed}(\gamma)$ indicates all edges of γ, and i_γ forgets X_e for internal edges and otherwise uses the inclusion of $X_e \subset \mathrm{End}\,\mathbf{B}$ for external edges.

The gluing formula reads[19]

$$\eta\check{\rceil}_p = \sum_{\gamma\in\underline{ev}^{-1}(p)} \frac{k_\gamma}{|\mathrm{Aut}\,\gamma|}(i_\gamma)_!\Delta^*\bigwedge \eta_{\gamma_v}$$

where $\eta_{\gamma_v} \in {}^{r}_{fg}H^*(\mathrm{End}_{\gamma_v}(\mathbf{B}))$ is the Gromov-Witten invariant associated with ev_{γ_v}, and

$$k_\gamma = \prod_{\mathrm{ied}(\gamma)} m_e$$

is an extra combinatorial factor introduced because the bottom square of the above diagram is no longer a fiber product diagram after we forgot the trivial group action on X_e present in Diagram 1.

2.11. **Refinements.**

In this section, we shall discuss the need for refined cohomology, and the operation of refinement, which is important for any reader who wants to understand the tropical gluing formula in terms of manifolds with normal crossing divisors, or log schemes instead of exploded manifolds.

Definition 2.20. *A refinement* $\mathbf{B}' \longrightarrow \mathbf{B}$ *is a complete, bijective submersion.*

As is explained in section 10 of [10], refinements of $\mathbf{B}$ are equivalent to subdivisions of the tropical part $\underline{\mathbf{B}}$ of $\mathbf{B}$. An instructive example is a refinement of $\mathbf{T}^n$ determined by subdividing $\mathbb{R}^n = \underline{\mathbf{T}^n}$ into a toric fan of some toric manifold M.

[19]We have cheated a little bit. To know that the pushforward $(i_\gamma)_!$ gives a well defined cohomology class, we need an extension to a map from a compact exploded manifold containing X_e. There are some complications involving tropical completions. Further details are given in section 4.3 of [9], and I am in the process of writing up a more clearly stated, and more powerful version of this gluing formula.

This refinement is $\operatorname{Expl} M$, the explosion of M relative to its toric boundary divisors. A further subdivision[20] of this toric fan would result in a further refinement of $\operatorname{Expl} M$, which in turn would be the explosion of some toric blowup of M.

Refinements do not affect Gromov Witten invariants, suitably interpreted. (The virtual fundamental class of) the moduli stack of holomorphic curves in $\mathbf{B}'$ is a refinement of (the virtual fundamental class of) the moduli stack of holomorphic curves in $\mathbf{B}$.[21]

As can be seen from the definition, refinements are very close to being isomorphisms. One manifestation of this is that the following diagram is a fiber product diagram.

$$\begin{array}{ccc} \mathbf{B}' & \longrightarrow & \mathbf{B}' \\ \downarrow & & \downarrow \\ \mathbf{B}' & \longrightarrow & \mathbf{B} \end{array}$$

This implies that any cohomology theory with pushforwards compatible with fiber product diagrams and containing H^* must also contain $H^*(\mathbf{B}')$ for any refinement $\mathbf{B}'$. Refined cohomology ${}^rH^*(\mathbf{B})$ is the minimal such cohomology theory. Refinements of $\operatorname{Expl} M$ correspond to boundary blowups M' of M locally modeled on toric blowups. Refined cohomology (defined differently, but equivalently in [8]) of $\operatorname{Expl} M$ is the direct limit of the cohomology of all such M'.

We can use refinements to identify the key elements of the tropical gluing formula in more familiar terms. First, let us describe the evaluation map to $\operatorname{End} \mathbf{B}$. Suppose $\mathbf{B}$ is the explosion of a complex manifold M relative to a simple normal crossing divisor $\bigcup_{i=1}^k D_i$ with irreducible components. Suppose for simplicity that the intersection of any number of these D_i is connected. (This can be achieved by suitably blowing up M, corresponding to a refinement of $\mathbf{B}$.) The tropical part of $\mathbf{B}$ may be identified with the vectors $(v_1, \dots, v_k) \in [0, \infty)^k$ so that if $v_i > 0$ for some set of indices $i \in I$, then $\bigcap_{i \in I} D_i \neq \emptyset$. The connected components of $\operatorname{End} \mathbf{B}$ are then indexed by nonnegative integral vectors in $\mathbb{N}^k$ satisfying the same condition. We can consider these integral vectors as either specifying the derivative at an end of a tropical curve, or the degree of contact with each D_i. The zero vector corresponds to no contact with any D_i. The corresponding component of $\operatorname{End} \mathbf{B}$ is $\mathbf{B} = \operatorname{Expl} M$. The vector $(n, 0, \dots, 0)$ corresponds to order n contact with D_1, and no contact with any other D_i. The corresponding component of $\operatorname{End} \mathbf{B}$ is $\operatorname{Expl} D_1$, where we use the divisor given by intersection with the other D_i. Similarly, the component of $\operatorname{End} \mathbf{B}$ corresponding to order n contact with D_i and no contact with other divisors is $\operatorname{Expl} D_i$. No other component of $\operatorname{End} \mathbf{B}$ will be the explosion of anything, but a refinement of it will be; simply blow up M (refine $\mathbf{B}$ to $\mathbf{B}'$) until the specified order of contact is with only one component D of the divisor. Then $\operatorname{End} \mathbf{B}'$ is a refinement of $\operatorname{End} \mathbf{B}$, and the refinement of our component is $\operatorname{Expl} D$. As $\mathcal{M}(\mathbf{B}')$ is also a refinement of $\mathcal{M}(\mathbf{B})$, we can use the map $\mathcal{M}(\mathbf{B}') \longrightarrow \operatorname{Expl} D \subset \operatorname{End} \mathbf{B}'$ to define GW invariants in place of $\mathcal{M}(\mathbf{B}) \longrightarrow \operatorname{End} \mathbf{B}$.

The exploded manifolds $\mathbf{B}\check{\rceil}_v$ may also be described with the help of refinements. Suppose that $\mathbf{B}$ is a fiber of the explosion of a simple normal crossing degeneration with singular fiber $\cup_i N_i$. When $\mathbf{B}$ is a fiber with nontrivial tropical part, its smooth part is equal to this singular fiber. The tropical part of $\mathbf{B}$ has a vertex p_i for each N_i,

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[20] A conical subdivision of the toric fan is the explosion of the toric manifold with that subdivided fan. A more general subdivision would result in an exploded manifold $\mathbf{B}'$ which is not the explosion of a log scheme or complex manifold with normal crossing divisors. The smooth part $\lceil \mathbf{B}' \rceil$ of $\mathbf{B}'$ would have extra components. Special cases of these extra components have been variously called holomorphic buildings, rubber components and expansions.

[21] As well as my proof of this using an older construction of the virtual fundamental class in [9], Abramovich and Wise have proved this in the log setting in [2].

and a simplex with corners p_i for $i \in I$ for each connected component of $\bigcap_{i\in I} N_i$. If v is one of these p_i, then $\mathbf{B}\check{\rceil}_v$ is $\operatorname{Expl} N_i$. When v is in a higher dimensional stratum of $\underline{\mathbf{B}}$, we can refine $\mathbf{B}$ to $\mathbf{B}'$, subdividing $\underline{\mathbf{B}}$ so that v becomes a zero-dimensional stratum.[22] So long as every stratum in $\underline{\mathbf{B}'}$ with v as a corner has v as a standard corner,[23] the stratum of $\lceil \mathbf{B} \rceil$ corresponding to v is a manifold N with normal crossing divisors. Then $\mathbf{B}'\check{\rceil}_v = \operatorname{Expl} N$, and we can use $\mathbf{B}'\check{\rceil}_v$ in place of $\mathbf{B}\check{\rceil}_v$ for computing Gromov-Witten invariants because $\mathbf{B}'\check{\rceil}_v$ is a refinement of $\mathbf{B}\check{\rceil}_v$.

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[22]We can understand refinements of $\mathbf{B}$ in terms of refinements of $\operatorname{Expl} M$, where M is the total space of the normal crossing degeneration. $\underline{\operatorname{Expl} M}$ is a cone over $\underline{\mathbf{B}}$. Our refinement is equivalent to a subdivision of $\underline{\mathbf{B}}$; if the cone over this subdivision has rational slopes, it is a subdivision defining a refinement of $\underline{\operatorname{Expl} M}$. Beware that refining the total space over a family will generally not produce a family because it will no longer satisfy the requirement on integral vectors. This is dealt with by a base change.

[23]A standard corner of an integral affine polytope is locally isomorphic to an open neighborhood of 0 in $[0, \infty)^n$.