Radiation fields on asymptotically Minkowski spacetimes

Dean Baskin

Texas A&M University
joint with Jeremy Marzuola, András Vasy, and Jared Wunsch

March 24, 2018

The nicest wave equations

Consider the wave equation on $\mathbb{R} \times \mathbb{R}^n$:

$$\Box u = D_t^2 u - \Delta u = 0$$
$$(u, \partial_t u)|_{t=0} = (\phi, \psi) \in C_c^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n)$$

Or on $\mathbb{R} \times (0, \infty)$:

$$\Box u = D_t^2 u - D_x^2 u = 0$$
$$(u, \partial_t u)|_{t=0} = (\phi, \psi) \in C_c^\infty(\mathbb{R}_+) \times C_c^\infty(\mathbb{R}_+)$$
$$u(t, 0) = 0$$

The radiation field

Let (r, θ) be polar coordinates on $\mathbb{R}^n$ and set $s = t - r$ $(s = t - x)$.

Define Friedlander's *forward radiation field of u*:

$$\mathcal{R}_+[u] = \lim_{r \to \infty} r^{(n-1)/2} u(s + r, r\theta)$$

FACT: This limit exists. (In fact, $r^{(n-1)/2} u(s + r, r\theta)$ is smooth in $1/r$ as $1/r \to 0$.)

Why care?

- Provides a measure of decay of solutions of the wave equation
- Dynamical definition of Radon transform
- Translation representation of the wave group:

$$\begin{array}{ccc} \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) & \xrightarrow{U(t)} & \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \\ \Big\downarrow \partial_s \mathcal{R}_+ & & \Big\downarrow \partial_s \mathcal{R}_+ \\ L^2(\mathbb{R} \times \mathbb{S}^{n-1}) & \xrightarrow{T_t} & L^2(\mathbb{R} \times \mathbb{S}^{n-1}) \end{array}$$

Asymptotics of the radiation field

If $\Box u = 0$ on $\mathbb{R} \times \mathbb{R}^n$ has smooth, compactly supported data, then as $s \to \infty$,

$$\mathcal{R}_+[u](s,\theta) \sim \begin{cases} 1 + \mathcal{O}(\langle s \rangle^{-\infty}) & n = 1 \\ \mathcal{O}(\langle s \rangle^{-\infty}) & n \geq 3 \text{ odd} \\ \sum_{j=0}^{\infty} s^{-\frac{n-1}{2}-j} a_j(\theta) & n \text{ even} \end{cases}$$

On $\mathbb{R} \times \mathbb{R}_+$:

$$\mathcal{R}_+[u](s) = \mathcal{O}(\langle s \rangle^{-\infty}).$$

Where do these exponents come from? What is their significance?

Radial compactification of Minkowski space

Take the radial compactification of $\mathbb{R} \times \mathbb{R}^n$ by setting

$$t = \frac{1}{\rho}\cos\theta, \quad r = \frac{1}{\rho}\sin\theta$$

The Minkowski metric $-dt^2 + dr^2 + r^2 d\omega^2$ becomes:

$$-\frac{d\rho^2}{\rho^4}\cos 2\theta - 2\sin 2\theta \frac{d\rho}{\rho^2}\frac{d\theta}{\rho} + \frac{d\theta^2}{\rho^2}\cos 2\theta + \sin^2\theta \frac{d\omega^2}{\rho^2}$$

With $v = \cos 2\theta$, we get:

$$-v\frac{d\rho^2}{\rho^4} + \frac{d\rho}{\rho^2}\frac{dv}{\rho} + \frac{v}{4(1-v^2)}\frac{dv^2}{\rho^2} + \frac{1-v}{2}\frac{d\omega^2}{\rho^2}$$

Features of the compactification

The Minkowski metric induces conformal metrics on $C_\pm$ and C_0:

- $C_\pm$ inherit metrics conformal to the standard metric on n-dimensional hyperbolic space.
- C_0 inherits a metric conformal to the standard Lorentzian metric on the n-dimensional de Sitter space.

All light light rays hit $S_\pm$ in the past/future.

The radiation field blow-up

To separate the light rays, we blow up $S_{\pm}$, giving two new boundary hypersurfaces corresponding to "null infinity": $\mathcal{I}^+$ and $\mathcal{I}^-$.

The lapse parameter $s = t - r$ is a coordinate along $\mathcal{I}^+$.

The forward radiation field is the restriction of $\rho^{-(n-1)/2} u$ to $\mathcal{I}^+$.

Asymptotics of the radiation field

Theorem (DB–Vasy–Wunsch)

Suppose u is the forward solution of $\Box u = f \in C_c^\infty$ on a non-trapping asymptotically Minkowski space. The forward radiation field of u is well-defined and satisfies

$$\mathcal{R}_+[u](s,y) \sim \sum_j \sum_{k=0}^{m_j} s^{-\imath\sigma_j} (\log s)^k a_{jk}(y)$$

Asymptotics of the radiation field

Theorem (DB–Vasy–Wunsch)

Suppose u is the forward solution of $\Box u = f \in C_c^\infty$ on a non-trapping asymptotically Minkowski space. The forward radiation field of u is well-defined and satisfies

$$\mathcal{R}_+[u](s, y) \sim \sum_j \sum_{k=0}^{m_j} s^{-\imath\sigma_j} (\log s)^k a_{jk}(y)$$

Theorem (DB–Marzuola)

The same is true on $\mathbb{R} \times C(Y)$ and the exponents can be explicitly identified.

Remarks

1. This is really a joint asymptotic expansion (u is polyhomogeneous...).

Remarks

1. This is really a joint asymptotic expansion (u is polyhomogeneous...).
2. The complex numbers σ_j "are" the resonances of the asymptotically hyperbolic polar cap (and $m_j + 1$ is the multiplicity of σ_j).

Remarks

1. This is really a joint asymptotic expansion (u is polyhomogeneous...).
2. The complex numbers σ_j "are" the resonances of the asymptotically hyperbolic polar cap (and $m_j + 1$ is the multiplicity of σ_j).
3. The non-trapping hypothesis implies that $\operatorname{Im} \sigma_j \to -\infty$ as $j \to \infty$.

Remarks

1. This is really a joint asymptotic expansion (u is polyhomogeneous...).
2. The complex numbers σ_j "are" the resonances of the asymptotically hyperbolic polar cap (and $m_j + 1$ is the multiplicity of σ_j).
3. The non-trapping hypothesis implies that $\operatorname{Im} \sigma_j \to -\infty$ as $j \to \infty$.
4. The exponents seen depend only on the induced geometry at infinity. (Warning: caveat on next slide!)

Remarks

1. This is really a joint asymptotic expansion (u is polyhomogeneous...).
2. The complex numbers σ_j "are" the resonances of the asymptotically hyperbolic polar cap (and $m_j + 1$ is the multiplicity of σ_j).
3. The non-trapping hypothesis implies that $\operatorname{Im} \sigma_j \to -\infty$ as $j \to \infty$.
4. The exponents seen depend only on the induced geometry at infinity. (Warning: caveat on next slide!)
5. "Long-range" metrics need more log terms.

"Price's Law"

For spacetime perturbations of $\mathbb{R} \times \mathbb{R}^3$ that don't change the geometry at infinity, we don't recover the rapid decay; instead we have

$$\mathcal{R}_+[u](s,\theta) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{j} s^{-2} a_{jk}(\theta)$$

(This corresponds to t^{-3} decay for the solution u.)

Ideas of the proof

What you might want to do to get an expansion in ρ:

1. Apply Mellin transform in ρ to convert to a problem on the boundary.
2. Find a Fredholm problem, shift the contour, apply inverse Mellin transform.

And for an expansion in s? There are a few approaches.

I: Set-up

We start with a solution u of $\Box u = f$ and rescale and conjugate it as

$$Lw = g,$$

where

$$L = \rho^{-(n-1)/2}\rho^{-2}\Box\rho^{(n-1)/2}$$

and

$$w = \rho^{(n-1)/2}u, \quad g = \rho^{-(n-1)/2}\rho^{-2}f.$$

On Minkowski space:

$$\begin{aligned} L &= -v(\rho\partial_\rho)^2 + 4(1-v^2)\rho\partial_\rho\partial_v + 4(1-v^2)v\partial_v^2 - \frac{2}{1-v}\Delta_\omega \\ &\quad + (1-n-\mathcal{O}(v))\rho\partial_\rho + (4+\mathcal{O}(v))\partial_v \\ &= 4\partial_v\left(\rho\partial_\rho + v\partial_v\right) + (\text{nicer}) \end{aligned}$$

Why? (Two/three reasons)

II: Why can't I just take the Mellin transform?

It would be nice to just take the Mellin transform now to turn the problem into one on the boundary, but you want to know that the Mellin transform ends up in reasonable spaces.

Forward solution is zero in the past (near S_-), so want to propagate this regularity foward. This works well until S_+ (with a caveat on cones), where you reach radial points.

As András mentioned, you can only propagate regularity up to a threshold value, so we can't necessarily get conormality with respect to the boundary. We can, however, obtain regularity under vector fields tangent to S_+ as well!

III: Finding a Fredholm problem

We apply the Mellin transform to obtain a family of differential operators P_σ on the boundary;

$$P_\sigma \tilde{w}_\sigma = R_\sigma \tilde{w}_\sigma + \tilde{g}_\sigma$$

On Minkowski space, $R_\sigma = 0$ and (in C_+)

$$v^{\frac{n+3}{4} + \frac{\imath\sigma}{2}} P_\sigma v^{-\frac{n+1}{4} - \frac{\imath\sigma}{2}} = -\Delta_{\mathbb{H}^n} - \left(\sigma^2 + \frac{(n-1)^2}{4} \right).$$

Analogous propagation estimates on the boundary shows that P_σ is Fredholm as a map

$$P_\sigma : \mathcal{X}^s \to \mathcal{Y}^{s-1},$$

where $\mathcal{X}^s$ and $\mathcal{Y}^s$ are as in an earlier talk. They are essentially variable order Sobolev spaces that get worse as $\operatorname{Im}\sigma \to -\infty$.
The non-trapping hypothesis is used to show that P_σ^{-1} is a meromorphic operator with only finitely many poles in any horizontal strip in $\mathbb{C}$.

IV: Global asymptotic expansion

We have a family of equations of the form

$$P_\sigma \tilde{w}_\sigma = \tilde{g}_\sigma (+ R_\sigma \tilde{w}_\sigma),$$

where we know that $\tilde{w}_\sigma$ is analytic in a half space $\operatorname{Im}\sigma \geq \varsigma_0$, and $\tilde{g}_\sigma$ is entire.
We invert P_σ to see that $\tilde{w}_\sigma$ is mermorphic in a larger space, and continue.
Inverting the Mellin transform turns poles of $\tilde{w}_\sigma$ into terms in an asymptotic expansion in ρ. The coefficients in this expansion are getting *worse* (at $v = 0$) as $\operatorname{Im}\sigma \to -\infty$, though!

V: Full expansion

To turn the expansion into a full expansion on the manifold with corners, show that w has improved asymptotics under applications of

$$(R + \imath j) \dots (R + \imath)R,$$

where $R = vD_v + \rho D_\rho$. (This is somewhat more complicated in the long-range case.)
This is essentially a combinatorial argument; you get to play a game commuting R through the operator.

Thank you

Kerr metric near null infinity

Near null infinity, we can write the Kerr metric in the desired form using

$$\rho = 1/t, \quad v_0 = 2(1 - r/t), \quad v = v_0 - v_0^2/4,$$

but it's ugly. The important terms are the coefficients of $d\rho/\rho^4$ and $d\rho\, dv/\rho^3$.

The coefficient of $d\rho\, dv/\rho^3$ is $1 +$ (decaying). The coefficient of $d\rho^2/\rho^4$ is:

$$-v + 4M\rho + \mathcal{O}(\rho^2, \rho v, v^2).$$