Spectral theory for linear operators on L^1 or $C(K)$ spaces

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Abstract

It is known that on a Hilbert space, the sum of a real scalar-type operator and a commuting well-bounded operator is well-bounded. The corresponding property has been shown to be fail on L^p spaces, for $1 < p \neq 2 < \infty$. We show that it does hold however on every Banach space X such that X or X^* is a Grothendieck space. This class notably includes L^1 and $C(K)$ spaces.
MSC (2000): 47B40, 46B03, 46B20 (primary) 46B10, 46B22, 46B26 (secondary). *Received: 13 June 2006 / Accepted: 19 December 2006.*

1 Introduction

This paper is concerned with scalar-type spectral and well-bounded operators on a Banach space X.

The theory of scalar-type spectral operators was initiated by N. Dunford (see his survey [6]) in order to generalize the theory of self-adjoint operators to operators on general Banach spaces. These operators are those which admit an integral representation with respect to a countably additive spectral measure (the precise definition is given in section 2) and therefore a functional calculus for bounded measurable functions on their spectrum. In particular, the spectral expansion of such an operator converges unconditionally. An old result of J. Wermer ([19]) asserts that finitely many commuting scalar-type spectral operators on a Hilbert space can be simultaneously transformed into normal operators, by passing to an equivalent inner product. An application of this is that the sum and product of two commuting scalar-type spectral operators on a Hilbert space are also scalar-type spectral. This result has been extended by many authors, beginning with C.A. McCarthy ([14], [15]) in the 1960s who considered operators acting on L^p spaces for $1 < p < \infty$.

More recently T.A. Gillespie ([8]) showed that, on a Banach lattice, the Boolean algebra generated by two commuting bounded Boolean algebras is itself bounded. As a consequence he obtains that the sum and product of two commuting scalar-type spectral operators on a weakly complete Banach lattice (and also on a wide class of subspaces of Banach lattices) are scalar-type spectral. It has been long known however that counterexamples to this result exist even within the class of super-reflexive Banach spaces. Indeed, a counterexample can be given on the von Neumann-Schattten classes C_p, for $1 < p \neq 2 < \infty$.

One may wonder under what conditions the sum of two commuting scalar-type spectral operators would have the weaker property of being well-bounded. Well-bounded operators, introduced by D.R. Smart ([18]), are defined as having a functional calculus for the absolutely continuous functions on some compact interval. They coincide with the operators having, in some weaker sense, a spectral decomposition that converges only conditionally. It is shown in [3] that on spaces with property (Δ), the sum of two commuting scalar-type spectral operators is always well-bounded. (The class of spaces with property (Δ), which was introduced by N. Kalton and L. Weis in [10], includes all UMD spaces.) On the other hand, T.A Gillespie proved in [7] that the sum of two commuting well-bounded operators is not always well-bounded, even on a Hilbert space.

We address now the question of the well-boundedness of the sum of a real scalar-type spectral operator and a commuting well-bounded operator. It follows from the same work of T.A. Gillespie [7] that the answer is positive for Hilbert spaces. However, it is shown in [4], how one may construct a counterexample in any reflexive non-Hilbertian Banach lattice. The aim of this note is to prove that the answer is positive however on an abstract class of Banach spaces which includes L^1 and $C(K)$ spaces.

Before proceeding, we would like to point out that much of this theory bears a close resemblance with that which arises from questions concerning whether the sum of two commuting unbounded operators either has an H^∞ functional calculus or is sectorial. We refer the reader to [1], [9], [10], [11] and [16] for the relevant definitions and, among other things, theorems analogous to the above mentioned results. One common ground for these two theories is certainly the classical work on unconditional bases as is expounded in [13].

2 Notation

Throughout this paper, X will denote a complex Banach space, B_X its closed unit ball and $B(X)$ the algebra of all bounded linear operators on X. Let Σ be the family of all Borel subsets of $\mathbb{C}$.

An operator $T \in B(X)$ is said to be *scalar-type spectral* if there exists a spectral measure $\mathcal{F}$ defined on Σ, whose values are projections in $B(X)$ and satisfying the following properties:
(i) $\|\mathcal{F}\| = \sup\{\|\mathcal{F}(A)\|,\ A \in \Sigma\} < +\infty$.
(ii) $T\mathcal{F}(A) = \mathcal{F}(A)T$, for all $A \in \Sigma$.
(iii) $\sigma(T|\mathcal{F}(A)X) \subset \overline{A}$, for all $A \in \Sigma$.
(iv) $\mathcal{F}$ is countably additive in the strong operator topology.
(v) $T = \int \lambda\, \mathcal{F}(d\lambda)$.
If in addition $\sigma(T) \subset \mathbb{R}$, then T is said to be *real scalar-type spectral.*
Every scalar-type spectral operator T admits a functional calculus defined on the space $B_\infty(\sigma(T))$ of all bounded Borel measurable functions on $\sigma(T)$ by the formula

$$f(T) = \int_{\sigma(T)} f(\lambda)\, \mathcal{F}(d\lambda)$$

and satisfying the standard estimate

$$\|f(T)\| \leq 4\|\mathcal{F}\| \sup_{\lambda \in \sigma(T)} |f(\lambda)|.$$

Details can be found in [5] (the constant 4 can be replaced by 2 if f is real valued).

An operator $S \in B(X)$ is said to be *well-bounded* if there exist a constant K and a compact interval $J = [a, b]$ such that for all complex polynomials p,

$$\|p(S)\| \leq K\|p\|_{AC(J)}, \quad \text{where} \quad \|p\|_{AC(J)} = \sup_{t \in J} |p(t)| + \int_a^b |p'(t)|\, dt.$$

Equivalently, there is a Banach algebra homomorphism $f \mapsto f(T)$, from the algebra $AC(J)$ of all absolutely continuous functions on J into $B(X)$, extending the natural definition for polynomials and satisfying:

$$\forall f \in AC(J), \quad \|f(T)\| \leq K\|f\|_{AC(J)}.$$

On a general Banach space X, an operator S is well-bounded if and only if it admits a so-called *decomposition of the identity.* This a family $(H(t))_{t\in\mathbb{R}} \subset$

$B(X^*)$ of projections enjoying a few more properties and providing the following integral representation for the functional calculus: for any f in $AC(J)$, $x \in X$ and $x^* \in X^*$,

$$\langle f(T)x, x^* \rangle = f(b)\langle x, x^* \rangle - \int_a^b \langle x, H(t)x^* \rangle f'(t)\, dt.$$

Since we shall not need to use this decomposition of the identity, we refer the reader to [5] for the complete definition.

We now recall that a Banach space X is a *Grothendieck space* (in short GT-space) if there is a constant C such that for every bounded linear operator T from X to ℓ_2 and all $x_1, .., x_n \in X$:

$$\sum_{k=1}^{n} \|Tx_k\|_{\ell_2} \leq C\, \|T\| \sup_{x^* \in B_{X^*}} \sum_{k=1}^{n} |x^*(x_k)|.$$

Such an operator is called absolutely summing (see [17] for a complete study of this notion).

3 A stronger functional calculus for scalar-type spectral operators on certain Banach spaces

The key step is to show that the unconditionality of the spectral decomposition of a scalar-type spectral operator is automatically strengthened in a GT-space. This is clearly inspired by the last section of [10] and by the fundamental work of J. Lindenstrauss and A. Pełczyński [12] on the uniqueness of unconditional bases in ℓ_1. In fact, our next proposition is just a variation of Corollary 8 of Theorem 6.1 in [12].

Proposition 3.1. *Suppose that X is a GT-space and $\mathcal{F}$ is a bounded finitely additive spectral measure defined on some algebra of subsets of $\mathbb{C}$. Then there is a constant C such that for any $x \in X$ and any $A_1, .., A_n \in \Sigma$ which are pairwise disjoint:*

$$\sum_{j=1}^{n} \|\mathcal{F}(A_j)x\| \leq C\|x\|.$$

In particular, if $\mathcal{F}$ is the spectral measure of a scalar-type spectral operator T on X, then for any $x \in X$, the X-valued vector measure μ_x, defined by

$\mu_x(A) = \mathcal{F}(A)x$, is a measure of bounded variation, whose total variation is dominated by $C\|x\|$.

Proof. Let $A_1, .., A_n \in \Sigma$ be pairwise disjoint. For every $u \in X$ and $1 \le k \le n$, we pick $u_k^* = u_k^*(u) \in B_{X^*}$ such that $\langle \mathcal{F}(A_k)u, u_k^* \rangle = \|\mathcal{F}(A_k)u\|$. Then, for $a = (a_k)_{k=1}^n \in \mathbb{C}^n$, we define $T_{u,a} : X \to \ell_2^n$ by

$$\forall y \in X, \qquad T_{u,a}y = \big(a_k \langle \mathcal{F}(A_k)y, u_k^* \rangle\big)_{k=1}^n.$$

We clearly have

$$\forall y \in X, \qquad \|T_{u,a}y\|_{\ell_2^n} \le \|a\|_{\ell_2^n} \|\mathcal{F}\| \, \|y\|.$$

Since X is a GT-space, there exists $C_1 > 0$ such that for all $a \in \ell_2^n$, $u \in X$ and $u_1, .., u_n \in X$,

$$\sum_{j=1}^n \|T_{u,a}u_j\|_{\ell_2^n} \le C_1 \, \|\mathcal{F}\| \, \|a\|_{\ell_2^n} \sup_{x^* \in B_{X^*}} \sum_{j=1}^n |x^*(u_j)|. \tag{3.1}$$

We now apply the above inequality for $u_j = \mathcal{F}(A_j)u$ and $a_j = \|\mathcal{F}(A_j)u\|$. We have that

$$T_{u,a}u_j = \big(a_k \langle \mathcal{F}(A_k)\mathcal{F}(A_j)u, u_k^* \rangle\big)_{k=1}^n = \big(\delta_{k,j} \|\mathcal{F}(A_j)u\|^2\big)_{k=1}^n.$$

Cancelling $\|a\|_{\ell_2^n}$ from both sides of (3.1) gives

$$\forall u \in X, \qquad (\sum_{j=1}^n \|\mathcal{F}(A_j)u\|^2)^{1/2} \le C_1 \, \|\mathcal{F}\| \sup_{x^* \in B_{X^*}} \sum_{j=1}^n |x^*(u_j)|.$$

Note now that for any $x^* \in B_{X^*}$ there exist complex numbers of modulus one, $\alpha_1, .., \alpha_n$, so that $\sum_{j=1}^n |x^*(u_j)| = |\sum_{j=1}^n \langle \alpha_j \mathcal{F}(A_j)u, x^* \rangle|$. Then, it follows from the disjointness of the A_j's and the functional calculus bounds for T that

$$\forall u \in X, \qquad \sup_{x^* \in B_{X^*}} \sum_{j=1}^n |x^*(u_j)| \le 4\|\mathcal{F}\|\|u\|$$

and so

$$\forall u \in X, \qquad (\sum_{j=1}^n \|\mathcal{F}(A_j)u\|^2)^{1/2} \le 4\, C_1 \, \|\mathcal{F}\|^2 \, \|u\|. \tag{3.2}$$

Suppose now that $x \in X$. Denote by T_x the operator $T_{x,a}$ where $a = (1, \dots, 1)$. For $y \in X$, inequality (3.2) implies

$$\|T_x y\| \leq (\sum_{j=1}^{n} \|\mathcal{F}(A_j)y\|^2)^{1/2} \leq 4\, C_1 \, \|\mathcal{F}\|^2 \, \|y\|.$$

Note that if $x_j = \mathcal{F}(A_j)x$ then $\|T_x x_j\|_{\ell_2^n} = \|\mathcal{F}(A_j)x\|$. Using again the fact that X is a GT-space we get that

$$\sum_{j=1}^{n} \|\mathcal{F}(A_j)x\| = \sum_{j=1}^{n} \|T_x x_j\|_{\ell_2^n} \leq C_1 \|T_x\| \sup_{x^* \in B_{X^*}} \sum_{j=1}^{n} |x^*(x_j)| \leq 16\, C_1^2 \, \|\mathcal{F}\|^3 \|x\|.$$

$\square$

We will also need the following dual statement.

Proposition 3.2. *Suppose X^* is a GT-space. Let T be a scalar-type spectral operator on X and let $\mathcal{F}$ be its spectral measure. Then there is a constant C such that for any $x^* \in X^*$ and any $A_1, .., A_n \in \Sigma$ which are pairwise disjoint:*

$$\sum_{j=1}^{n} \|\mathcal{F}(A_j)^* x^*\| \leq C\|x^*\|.$$

In other words, for any $x^* \in X^*$, the total variation of the X^*-valued finitely additive vector measure ν_{x^*}, defined by $\nu_{x^*}(A) = \mathcal{F}(A)^* x^*$, is dominated by $C\|x^*\|$.

Proof. We just apply Proposition 3.1 to $\mathcal{F}^*$. $\square$

We now need to introduce more notation. For $T \in B(X)$, we denote by $\{T\}'$ the commutant of T (namely, the closed subalgebra of $B(X)$ consisting of all operators commuting with T). Let $\mathcal{S}_T$ denote the algebra of all $\{T\}'$-valued Borel simple functions defined on $\sigma(T)$, and let $B_\infty(\sigma(T), \{T\}')$ be the uniform closure of $\mathcal{S}_T$. We can now state the main result of this section.

Theorem 3.1. *Suppose that X or X^* is a GT-space (for instance X is an L^1-space, or X is a $C(K)$-space). Let T be a scalar-type spectral operator on X and let $\mathcal{F}$ be its spectral measure. For any finite families $(A_i)_{i=1}^n$ of pairwise disjoint Borel subsets of $\sigma(T)$ and $(S_i)_{i=1}^n$ in $\{T\}'$, we define $\Phi(\sum_{i=1}^n S_i \mathbb{1}_{A_i}) = \sum_{i=1}^n S_i \mathcal{F}(A_i)$. Then Φ can be extended into a bounded algebra homomorphism from $B_\infty(\sigma(T), \{T\}')$ into $B(X)$.*

Proof. Since T is a spectral operator, each operator that commutes with T also commutes with its spectral measure (see [6] or Theorem 6.6 in [5]). It is therefore simple to check that Φ is an algebra homomorphism on $\mathcal{S}_T$. The conclusion will therefore follow immediately once we can show that Φ is bounded on $\mathcal{S}_T$. Suppose then that $f = \sum_{i=1}^n S_i \mathbb{1}_{A_i} \in \mathcal{S}$.
If X is a GT-space then it follows from Proposition 3.1 that for all $x \in X$,

$$\|\Phi(f)x\| \leq \sum_{i=1}^n \|S_i \mathcal{F}(A_i)x\| \leq C \sup_{1\leq i\leq n} \|S_i\| \, \|x\| = C\|f\|_\infty \|x\|.$$

If X^* is a GT-space, we apply Proposition 3.2 and the fact that S_i and $\mathcal{F}(A_i)$ commute to obtain that

$$\forall x \in X \quad \forall x^* \in X^* \quad |\langle \Phi(f)x, x^*\rangle| \leq C\|f\|_\infty \|x\| \, \|x^*\|.$$

In each case, our estimate clearly yields the conclusion. □

4 Application to the well-boundedness of sums of operators

Our result is the following

Theorem 4.1. *Suppose that X or X^* is a GT-space. Let T be a real scalar-type operator on X and let S be a well-bounded operator on X which commutes with T. Then $S + T$ is a well-bounded operator on X.*

Proof. Let Φ be the functional calculus map from $B_\infty(\sigma(T), \{T\}')$ into $B(X)$, associated with T in Theorem 3.1. Note first that if $f(\lambda) = g(\lambda)Id_X$, where Id_X is the identity operator on X and g is bounded, Borel measurable and scalar valued, then $\Phi(f) = g(T)$. On the other hand, if $U \in \{T\}'$ and $f(\lambda) = U$ for all $\lambda \in \sigma(T)$, then $\Phi(f) = U$.
For p a complex polynomial, define the map $f_p : \sigma(T) \to \{T\}'$ by $f_p(\lambda) = p(S + \lambda)$. Combining the above remarks, with the fact that Φ is an algebra homomorphism, we get that $\Phi(f_p) = p(S + T)$. It follows that there is a constant C_1 such that for every polynomial p:

$$\|p(S + T)\| \leq C_1 \sup_{\lambda\in\sigma(T)} \|f_p(\lambda)\|_{B(X)}. \tag{4.1}$$

Since S is well-bounded, there exist a compact interval J and a constant C_2 such that for any complex polynomial p, $\|p(S)\| \leq C_2\|p\|_{AC(J)}$. Let K be a compact interval containing $\sigma(T)+J$. It is a standard fact [5, Lemma 18.7] that for all $\lambda \in \sigma(T)$ and all complex polynomials p,

$$\|f_p(\lambda)\|_{B(X)} = \|p(S+\lambda)\| \leq C_2\|p\|_{AC(K)}.$$

Combining this with (4.1) shows that there is a constant C such that for every polynomial p, $\|p(S+T)\| \leq C\|p\|_{AC(K)}$, and therefore that $S+T$ is a well-bounded operator. □

Remark 4.1. Under the same assumptions, one can show with a similar proof that $q(S,T)$ is well-bounded for every real polynomial q.

Remark 4.2. We conclude this note by showing that if T is a scalar-type spectral operator on a Hilbert space H, then it also admits a functional calculus defined on $B_\infty(\sigma(T), \{T\}')$. This gives an alternative quick proof of Gillespie's result ([7]).

So, let $(A_i)_{i=1}^n$ be pairwise disjoint Borel subsets of $\sigma(T)$, $(S_i)_{i=1}^n$ in $\{T\}'$. For $f = \sum_{i=1}^n S_i \mathbb{1}_{A_i}$, set $\Phi(f) = \sum_{i=1}^n S_i \mathcal{F}(A_i)$. Writing

$$\forall t \in [0,1] \quad \forall x \in H \quad \Phi(f)x = \big(\sum_{i=1}^n r_i(t)\mathcal{F}(A_i)\big)\big(\sum_{i=1}^n r_i(t)\mathcal{F}(A_i)S_i x\big),$$

noting that $\|\sum_{i=1}^n r_i(t)\mathcal{F}(A_i)\| \leq 2\|\mathcal{F}\|$ and using the parallelogram law, we obtain that for any $x \in H$:

$$\|\Phi(f)x\| \leq 2\|\mathcal{F}\|(\int_0^1 \|\sum_{i=1}^n r_i(t)\mathcal{F}(A_i)S_i x\|^2\,dt)^{1/2} = 2\|\mathcal{F}\|(\sum_{i=1}^n \|\mathcal{F}(A_i)S_i x\|^2)^{1/2}.$$

But

$$\sum_{i=1}^n \|\mathcal{F}(A_i)S_i x\|^2 \leq \|f\|_\infty^2(\sum_{i=1}^n \|\mathcal{F}(A_i)x\|^2) = \|f\|_\infty^2(\int_0^1 \|\sum_{i=1}^n r_i(t)\mathcal{F}(A_i)x\|^2\,dt).$$

Thus

$$\|\Phi(f)x\| \leq 4\|\mathcal{F}\|^2\|f\|_\infty\|x\|.$$

This finishes our proof.

Acknowledgements. We wish to thank the referee for his helpful comments.

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