Vector-valued singular integrals, and the border between the one-parameter and the multi-parameter theories

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Abstract

We survey the vector-valued theory of Fourier multipliers and singular integrals, especially concentrating on identifying the border between the one-parameter theory valid in UMD spaces and the multi-parameter theory valid in UMD spaces with property (α). Some new results are also proved which clarify this question.
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1 Introduction

After its introduction in the early 1980's by D. L. Burkholder [10], the probabilistic *UMD property* (unconditionality of martingale differences; see Def. 2.1) has become the central notion in Harmonic Analysis of functions with values in infinite-dimensional spaces. Indeed, several results from the classical Littlewood–Paley and Calderón–Zygmund theories, including their more recent extensions, remain valid in the context of X-valued functions if and only if the Banach space X has UMD. Out of the vast amount of examples, we record here the continuity in $L^p(\mathbf{R}, X)$ of (the tensor extension of) the Hilbert transform [6, 11], as well as the extensions to $L^p(\mathbf{R}^n, X)$ of the Marcinkiewicz–Mihlin multiplier theorem [8, 45] and the David–Journé $T(1)$ theorem [24]. While the mentioned results were all obtained by the end

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of the 1980's, there has been a revival of interest in the related questions since the turn of the millennium. This has been boosted by the realization of the connection of another probabilistic notion, the *R-boundedness* (Def. 2.3), to the continuity of more general vector-valued Calderón–Zygmund transformations with an operator-valued kernel [14, 57], as well as the successful applications of these ideas to Partial Differential Equations. The UMD spaces have retained their central position also for these recent developments.

However, another probabilistic Banach space condition, *property* (α) (see Def. 2.2), has also frequently appeared in the assumptions of various results on vector-valued Harmonic Analysis, and it is now known that in many cases it cannot be avoided. Already in the late 1980's it was shown that the more general (compared to Mihlin's) Marcinkiewicz–Lizorkin multiplier theorem is not valid in certain UMD spaces [61], and more recently it was realized [43] that a characterization of the spaces with this multiplier theorem is UMD combined with (α). Since this observation, several further results have considerably clarified the interplay of the UMD and (α) properties, and the need for the latter has been clearly related to the "multi-parameter" nature of certain results [34, 36, 37]. The present article has a two-fold purpose:

- to survey the state-of-art of the theory of vector-valued Fourier multipliers and singular integrals, with emphasis on the rôle of property (α) (and heavily biased towards the author's own interests), and
- to supplement some new results, which establish a fairly sharp border between the parts of the theory requiring or not requiring (α).

The paper will concentrate on the vector-valued Calderón–Zygmund theory *per se*. For the applications to Partial Differential Equations we refer to the recent monograph [19] and the lecture notes [42]. The latter also contains a detailed presentation of the H^∞-calculus of sectorial operators in UMD spaces (not dealt with here), for which there is also the recent survey [58].

2 Probabilistic preliminaries

We first recall the two fundamental Banach space properties which determine the behaviour of singular integrals of vector-valued functions:

Definition 2.1 ([10]). A Banach space X is UMD if for some (and then all) $p \in]1,\infty[$ there is a $C < \infty$ such that

$$\Big(\mathsf{E}\Big|\sum_{k=1}^{N}\epsilon_k d_k\Big|_X^p\Big)^{1/p} \leq C\Big(\mathsf{E}\Big|\sum_{k=1}^{N} d_k\Big|_X^p\Big)^{1/p}$$

whenever $N \in \mathbf{Z}_+$, $(\epsilon_k)_{k=1}^N \in \{-1,+1\}^N$, and $(d_k)_{k=1}^N \in L^p(\Omega,\mathscr{F},\mathsf{P};X)^N$ is a martingale difference sequence on some probability space $(\Omega,\mathscr{F},\mathsf{P})$ with expectation $\mathsf{E} := \int_\Omega \mathsf{dP}$; i.e., there are sub-$\sigma$-algebras $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \cdots \subseteq \mathscr{F}_N \subseteq \mathscr{F}$ so that $d_k \in L^p(\Omega,\mathscr{F}_k,\mathsf{P};X)$ and $\mathsf{E}(1_A d_k) = 0$ for all $A \in \mathscr{F}_{k-1}$, for all $k = 1,\cdots,N$.

Examples of UMD spaces include the reflexive Lebesgue L^p, Lorentz $L^{p,q}$ and Schatten–von Neumann $\mathscr{C}^q$ spaces, $1 < p,q < \infty$. If X is any UMD space, so are its dual X', the Bôchner spaces $L^p(\mu,X)$ for $1 < p < \infty$, and the closed subspaces and quotients of X. UMD spaces are (super)reflexive, and they have non-trivial *type* (and then also *Fourier-type*) as well as *cotype*. There are a number of useful surveys of UMD spaces [12, 13, 51].

In the following definition, and always thereafter, the $\varepsilon_i, \tilde{\varepsilon}_j, \varepsilon_k^{(1)}, \cdots$ are i.i.d. (independent identically distributed) random signs with distribution $\mathsf{P}(\varepsilon_i = +1) = \mathsf{P}(\varepsilon_i = -1) = 1/2$. They are called the Rademacher variables.

Definition 2.2 ([47]). A Banach space X has property (α) if there is a $C < \infty$ such that

$$\mathsf{E}\Big|\sum_{i,j=1}^{N}\varepsilon_i\tilde{\varepsilon}_j\alpha_{ij}x_{ij}\Big|_X \leq C\mathsf{E}\Big|\sum_{i,j=1}^{N}\varepsilon_i\tilde{\varepsilon}_j x_{ij}\Big|_X$$

whenever $N \in \mathbf{Z}_+$, $(x_{ij})_{i,j=1}^N \in X^{N\times N}$ and $(\alpha_{ij})_{i,j=1}^N \in \{-1,+1\}^{N\times N}$.

This property holds for the commutative L^p spaces for all $1 \leq p < \infty$, and is also inherited from X by $L^p(\mu,X)$ for p in the same range; on the other hand, the (infinite-dimensional) non-commutative $\mathscr{C}^q$ spaces have (α) only when $q = 2$. Unlike UMD, property (α) is not self-dual (e.g., ℓ^1 has (α) while ℓ^∞ does not); however, it is important that the joint property "UMD and (α)" is inherited by the dual space. Every Banach space with a *local unconditional structure* (l.u.st.), in particular every Banach lattice, has property (α) if and only if it has finite cotype. A good reference is Pisier's original paper [47], where property (α) was introduced.

We also recall the main property that one typically needs to impose on the range of operator-valued singular integral kernels:

Definition 2.3 ([1]). An operator collection $\mathscr{T} \subset \mathscr{L}(X)$ is R-bounded if there is a $C < \infty$ such that

$$\mathsf{E}\Big|\sum_{k=1}^{N} \varepsilon_k T_k x_k\Big|_X \leq C\mathsf{E}\Big|\sum_{k=1}^{N} \varepsilon_k x_k\Big|_X$$

whenever $N \in \mathbf{Z}_+$, $(x_k)_{k=1}^N \in X^N$ and $(T_k)_{k=1}^N \in \mathscr{T}^N$. The least number C is called the R-bound and denoted by $\mathscr{R}(\mathscr{T})$.

Even earlier, this notion made anonymous appearances in [8, 61], and Bourgain [8] proved the useful fact that $\mathscr{R}(\overline{\mathrm{abco}\mathscr{T}}) \leq 2\mathscr{R}(\mathscr{T})$, where the bar designates the strong operator closure, and abco stands for the absolute (or balanced) convex hull. A scalar-valued version of this inequality was implicitly used already in Marcinkiewicz' original proof of his multiplier theorem [44], which may explain why R-boundedness has become such a central concept in the operator-valued extensions of this classical result. This notion is studied in detail in [14, 57].

Another frequently used result in connection with the randomized norms is the inequality of Kahane which provides the second, non-trivial comparison in the following chain, where $K_p < \infty$ is constant only depending on p:

$$\mathsf{E}\Big|\sum_{k=1}^{N} \varepsilon_k x_k\Big|_X \leq \Big(\mathsf{E}\Big|\sum_{k=1}^{N} \varepsilon_k x_k\Big|_X^p\Big)^{1/p} \leq K_p\mathsf{E}\Big|\sum_{k=1}^{N} \varepsilon_k x_k\Big|_X, \qquad 1 < p < \infty.$$

This implies that one can replace the L^1 norms in Definitions 2.2 and 2.3 by other L^p norms for $1 < p < \infty$. (The p invariance of Def. 2.1 is due to a different reason.) Another useful inequality of Kahane is the contraction principle, for which the notion of R-boundedness gives the compact formulation $\mathscr{R}(\Lambda \cdot \mathrm{id}_X) \leq 2\sup\{|\lambda| : \lambda \in \Lambda\}$, whenever $\Lambda \subset \mathbf{C}$.

It is sometimes handy to transform R-boundedness on X into usual boundedness on a certain larger space, $\mathrm{Rad}(X)$. The use of this space (well-known in the Banach space theory) in the context of R-boundedness originates from Girardi and Weis [27].

Definition 2.4. The Rademacher space $\mathrm{Rad}(X)$ is the completion of all finitely non-zero sequences $(x_j)_{j=-\infty}^{\infty} \in X^{\mathbf{Z}}$ in any of the following equivalent norms, where $p \in [1, \infty[$:

$$\big\|(x_j)_{-\infty}^{\infty}\big\|_{\mathrm{Rad}^p(X)} := \Big(\mathsf{E}\Big|\sum_{j=-\infty}^{\infty} \varepsilon_j x_j\Big|_X^p\Big)^{1/p}.$$

Unless otherwise said, we use the L^1 norm on $\mathrm{Rad}(X)$.

Let us identify a finitely non-zero sequence $(T_j)_{-\infty}^{\infty} \in \mathscr{L}(X)^{\mathbf{Z}}$ with the operator $\tilde{T}(x_j)_{-\infty}^{\infty} := (T_j x_j)_{-\infty}^{\infty}$ on $\mathrm{Rad}(X)$. For $\mathscr{T} \subset \mathscr{L}(X)$, let us denote

$$\tilde{\mathscr{T}} := \{(T_j)_{-\infty}^{\infty} \in \mathscr{T}^{\mathbf{Z}} \text{ finitely non-zero}\}.$$

Then clearly $\mathscr{R}(\mathscr{T}) = \sup\{\|\tilde{T}\|_{\mathscr{L}(\mathrm{Rad}(X))} : \tilde{T} \in \tilde{\mathscr{T}}\} \leq \mathscr{R}(\tilde{\mathscr{T}})$. One of the main implications, and in fact a characterization, of property (α) of X is the converse estimate $\mathscr{R}(\tilde{\mathscr{T}}) \leq C\mathscr{R}(\mathscr{T})$ for $\mathscr{T} \subset \mathscr{L}(X)$, where C depends on the (α) property constant of X only [14].

3 Littlewood–Paley decompositions

The first step in Bourgain's [8] approach to the estimation of singular integrals in UMD spaces is transforming the defining unconditionality property of martingale differences into another unconditionality estimate of more analytic flavour. This is an analogue of the classical inequality of Littlewood and Paley concerning the dyadic spectral decomposition of a function. For an interval $I \subset \mathbf{R}$, we denote $\Delta[I] := \mathscr{F}^{-1} 1_I \mathscr{F}$, where $\mathscr{F}$ is the Fourier transform. Let $\mathscr{I} := \{\eta\left[2^k, 2^{k+1}\right[: \eta \in \{-1, +1\}, k \in \mathbf{Z}\}$ be the collection of dyadic intervals on $\mathbf{R}$. The vector-valued Littlewood–Paley inequality is the following:

Theorem 3.1 ([8, 45]). *Let X be a UMD space and $1 < p < \infty$. Then there are constant $0 < c \leq C < \infty$ such that*

$$c\,\|f\|_p \leq \mathsf{E}\Big\|\sum_{I \in \mathscr{I}} \varepsilon_I \Delta[I] f\Big\|_p \leq C\,\|f\|_p\,, \qquad f \in L^p(\mathbf{R}, X).$$

Conversely, this estimate implies that X is UMD and $1 < p < \infty$.

The proof of Bourgain [8] consists of writing the UMD inequality for the translated dyadic filtrations $(\mathscr{D}_k - u)_{k=0}^{\infty}$ of $\mathbf{R}$, where $\mathscr{D}_k := \{2^{-k}\left[j, j+1\right[: j \in \mathbf{Z}\}$ and $0 \leq u < 1$, and averaging over the values of the translation parameter u. This yields an inequality similar to that of Theorem 3.1 but with smooth and overlapping cut-offs (decaying like $(2^k\xi)^2$ resp. $(2^k\xi)^{-2}$ as $\xi \to 0$ resp. $\xi \to \infty$) in place of the indicators of the intervals $\left[2^k, 2^{k+1}\right[$. The desired sharp cut-offs are then reached by a perturbation argument, which uses the R-boundedness of the family of spectral projections $\Delta[J]$, where J ranges over all intervals of $\mathbf{R}$. This R-boundedness is a consequence of the boundedness of the single projection $\Delta[\mathbf{R}_+] = (\mathrm{id} + \mathbf{i}H)/2$, where H is the Hilbert transform, the identity $\Delta[a + J] = e^{\mathbf{i}2\pi ax}\Delta[J]e^{-\mathbf{i}2\pi ax}$, and the basic properties of R-bounded sets.

A different approach to the vector-valued Littlewood–Paley decomposition is due to McConnell [45], who proved a Mihlin-type multiplier theorem directly from the UMD inequality by means of heavy stochastic machinery, and derived Theorem 3.1 as a corollary of his multiplier estimate. In Bourgain's approach, on the other hand, Theorem 3.1 is used to obtain the multiplier theorem (see next section), which turns out to be sharper than that proved by McConnell.

Theorem 3.1 shows that on the one-dimensional Euclidean domain $\mathbf{R}$, the classical spectral decomposition extends to the UMD-valued situation, and in fact reflects the one-parameter decomposition postulated in the definition of UMD. When we want to move to $\mathbf{R}^n$ with $n > 1$, however, a one-parameter decomposition cannot adequately capture the full n-dimensional structure of the product domain $\mathbf{R}^n = \mathbf{R} \times \cdots \times \mathbf{R}$, and this is where property (α) enters the scene. Of course, by simply iterating Theorem 3.1, we obtain

$$c \left\| f \right\|_p \leq \mathsf{E} \Big\| \sum_{I \in \mathscr{I}^n} \varepsilon_{I_1}^{(1)} \cdots \varepsilon_{I_n}^{(n)} \Delta[I] f \Big\|_p \leq C \left\| f \right\|_p, \quad f \in L^p(\mathbf{R}^n, X), \tag{3.1}$$

where $\mathscr{I}^n := \{I = I_1 \times \cdots \times I_n : I_1, \cdots, I_n \in \mathscr{I}\}$, and $\varepsilon_{I_1}^{(1)}, \cdots, \varepsilon_{I_n}^{(n)}$ are i.i.d. sequences of Rademacher variables. The products $\varepsilon_{I_1}^{(1)} \cdots \varepsilon_{I_n}^{(n)}$, however, are not quite the same as one independent sequence $\varepsilon_I = \varepsilon_{I_1 \times \cdots \times I_n}$ indexed by the product intervals. Property (α), on the other hand, is precisely the condition under which the two random sums are equivalent, and Zimmermann proved the following:

Theorem 3.2 ([61]). *Let $n > 1$. There are constant $0 < c \leq C < \infty$ such that the following estimates hold, if and only if X is a UMD space with property (α) and $1 < p < \infty$:*

$$c \left\| f \right\|_p \leq \mathsf{E} \Big\| \sum_{I \in \mathscr{I}^n} \varepsilon_I \Delta[I] f \Big\|_p \leq C \left\| f \right\|_p, \quad f \in L^p(\mathbf{R}^n, X).$$

All hope is not lost in general UMD spaces, either, but we have to content ourselves with a coarser decomposition, the so-called *blocking by squares* of

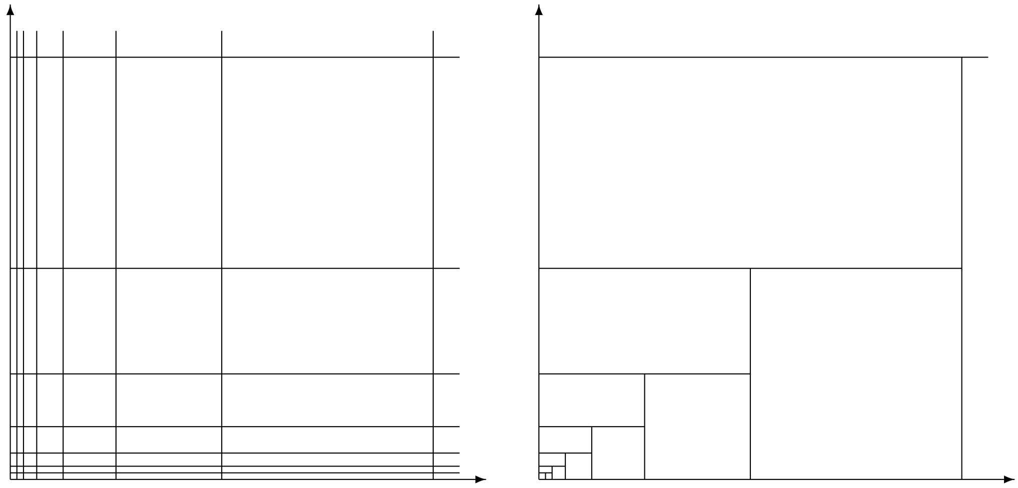

Figure 1: The dyadic product decomposition (left) and its blocking by squares (right).

the product decomposition $\mathscr{I}^n$. This consists of the intervals

$$\prod_{i=1}^{r-1} \eta_i \,]0, 2^{k_i+1}[\,\times \eta_r \,[2^{k_r}, 2^{k_r+1}[\,\times \prod_{i=r+1}^{n} \eta_i \,]0, 2^{k_i}[\tag{3.2}$$

with $\eta \in \{-1, +1\}^n$, $k \in \mathbf{Z}^n$. Denoting the set of these intervals by $\mathscr{I}_n$, the result reads:

Theorem 3.3 ([61]). *Let X be a UMD space, $1 < p < \infty$ and $n \geq 1$. Then there are constants $0 < c \leq C < \infty$ such that*

$$c \left\| f \right\|_p \leq \mathsf{E} \Big\| \sum_{I \in \mathscr{I}_n} \varepsilon_I \Delta[I] f \Big\|_p \leq C \left\| f \right\|_p, \quad f \in L^p(\mathbf{R}^n, X).$$

The two decompositions for $n = 2$ are illustrated in Fig. 1. Unconditional decompositions and their products and blockings are studied in detail in Witvliet's thesis [59].

4 Vector-valued Fourier multipliers

While various classes of Calderón–Zygmund operators and their generalizations have been treated in UMD spaces by now, many of the typical vector-valued phenomena are most easily illustrated in the context of Fourier multipliers, whose boundedness properties are very closely related to the unconditionality of Littlewood–Paley decompositions. In fact, the two ingredients used in Bourgain's approach to the vector-valued Mihlin multiplier theorem are Theorem 3.1 and an R-boundedness estimate for the class of multipliers of bounded variation. The point is then to apply this R-boundedness result to the "dyadic pieces" $m1_I$ of a Mihlin multiplier to obtain the boundedness of the full multiplier m. Essentially the same idea goes through for operator-valued multipliers; we only need to strengthen bounded variation to R-bounded variation:

Definition 4.1. We say that a set $\mathscr{M}$ of functions $m : \mathbf{R}^n \to \mathscr{L}(X)$ has *uniformly R-bounded variation* if there exists a fixed R-bounded set $\mathscr{T} \subset \mathscr{L}(X)$, and for each $m \in \mathscr{M}$ a probability measure μ on $[-\bar{\infty}, \bar{\infty}[$ and a strongly measurable $\tau : [-\bar{\infty}, \bar{\infty}[\to \mathscr{T}$, such that for all $x \in X$

$$m(\xi)x = \int_{[-\bar{\infty},\xi]} \tau(y)x\, \mu(\mathrm{d}y).$$

We denoted $\bar{\infty} := (\infty, \cdots, \infty)$ and $[-\bar{\infty}, \xi] := [-\infty, \xi_1] \times \cdots \times [-\infty, \xi_n]$. Typical examples of R-bounded variation arise from multipliers m supported on intervals $J = J_1 \times \cdots \times J_n$ and having derivatives $D^\alpha m(\xi)$, $\alpha \in \{0,1\}^n$, $\xi \in J$, such that $\| \|D^\alpha m(\xi)\|_{\mathscr{T}} \|_{L^1(J^\alpha)} \leq C < \infty$, where $\|\cdot\|_{\mathscr{T}}$ is the Minkowski functional of an R-bounded set $\mathscr{T}$, and $J^\alpha := \prod_{i:\alpha_i=1} J_i \times \prod_{i:\alpha_i=0} \{\inf J_i\}$.

Theorem 4.1 ([8, 42, 61]). *Let X be a UMD space and $1 < p < \infty$. If $\mathscr{M}$ is a uniformly R-bounded collection of functions on $\mathbf{R}^n$, then the set of Fourier multiplier operators $T_m := \mathscr{F}^{-1} m \mathscr{F}$, $m \in \mathscr{M}$, is R-bounded on $\mathscr{L}(L^p(\mathbf{R}^n, X))$.*

This result is actually equivalent to the $L^p(\mathbf{R}, X)$ boundedness of the Hilbert transform, since H itself has a scalar multiplier of bounded variation, whereas every multiplier T_m with $m \in \mathscr{M}$ belongs to $\overline{\mathrm{abco}}(\mathscr{T} \cdot \mathscr{S})$, where $\mathscr{T}$ is the R-bounded set appearing in Def. 4.1, and $\mathscr{S} := \{\Delta[\xi + \mathbf{R}^n_+] : \xi \in \mathbf{R}^n\}$ whose R-boundedness follows from the boundedness of H.

We can now formulate a generic multiplier theorem:

Theorem 4.2. *Let X be a UMD space and $1 < p < \infty$. Let $\mathscr{J}$ be a collection of intervals $\subset \mathbf{R}^n$ such that $\{\Delta[J] : J \in \mathscr{J}\}$ satisfies, for some*

$0 < c \leq C < \infty$, *the unconditionality estimate*

$$c \|f\|_p \leq \mathsf{E}\Big\| \sum_{J \in \mathscr{J}} \varepsilon_J \Delta[J] f \Big\|_p \leq C \|f\|_p, \qquad f \in L^p(\mathbf{R}^n, X).$$

If a multiplier $m : \mathbf{R}^n \to \mathscr{L}(X)$ *has the property that the* $m 1_J$, $J \in \mathscr{J}$, *are of uniformly R-bounded variation, then* $T_m \in \mathscr{L}(L^p(\mathbf{R}^n, X))$.

In fact, $K := \mathscr{R}(T_{m1_J} : J \in \mathscr{J}) < \infty$ by Theorem 4.1, and then

$$\begin{aligned} \|T_m f\|_p &\leq \frac{1}{c}\mathsf{E}\Big\| \sum_{J \in \mathscr{J}} \varepsilon_J \Delta[J] T_m f \Big\|_p = \frac{1}{c}\mathsf{E}\Big\| \sum_{J \in \mathscr{J}} \varepsilon_J T_{m1_J} \Delta[J] f \Big\|_p \\ &\leq \frac{1}{c} K \mathsf{E}\Big\| \sum_{J \in \mathscr{J}} \varepsilon_J \Delta[J] f \Big\|_p \leq \frac{C}{c} K \|f\|_p. \end{aligned}$$

The following two results, which provide vector-valued extensions of the classical Mihlin and Marcinkiewicz–Lizorkin multiplier theorems, are now consequences of the generic Theorem 4.2 and the Littlewood–Paley decompositions of the previous section:

Theorem 4.3 ([8, 54, 57, 61])**.** *Let* $n \geq 1$. *If (and only if)* X *is a UMD space and* $1 < p < \infty$, *then every multiplier* $m : \mathbf{R}^n \setminus \{0\} \to \mathscr{L}(X)$ *such that*

$$\mathscr{R}(|\xi|^{|\alpha|} D^\alpha m(\xi) : \alpha \in \{0,1\}^n, \xi \in \mathbf{R}^n \setminus \{0\}) < \infty$$

satisfies $T_m \in \mathscr{L}(L^p(\mathbf{R}^n, X))$.

Theorem 4.4 ([54, 61])**.** *Let* $n > 1$. *If (and only if)* X *is a UMD space with property* (α) *and* $1 < p < \infty$, *then every multiplier* $m : \mathbf{R}^n \setminus \{0\} \to \mathscr{L}(X)$ *such that*

$$\mathscr{R}(|\xi^\alpha| D^\alpha m(\xi) : \alpha \in \{0,1\}^n, \xi \in (\mathbf{R} \setminus \{0\})^n) < \infty$$

satisfies $T_m \in \mathscr{L}(L^p(\mathbf{R}^n, X))$.

The case of Theorem 4.3 when $n = 1$ and m is scalar-valued was proved by Bourgain [8] and extended to $n > 1$ by Zimmermann [61]; a slightly weaker statement had already been obtained by McConnell [45] using different methods. The operator-valued statement was first achieved by Weis [57] for $n = 1$ and extended to $n > 1$ by Štrkalj and Weis [54]; other proofs of this result are given in [19, 29, 42]. Zimmermann's original formulation of the scalar-multiplier case of Theorem 4.4 was in the slightly smaller class of UMD spaces with l.u.st.; a similar but yet more restricted statement in Banach lattices was proved in [21], based on [45]. The class of UMD spaces with (α) used by Štrkalj and Weis is the largest possible, as observed by Lancien [43].

Finally, it should be pointed out that the R-boundedness assumptions are not only a matter of technical convenience, but a necessity to obtain multiplier theorems, as shown by Clément and Prüss [15]:

Theorem 4.5 ([15]). *Let $n \geq 1$ and $1 < p < \infty$. There is a constant $C < \infty$ such that if X is an arbitrary Banach space and $m \in L^1_{\text{loc}}(\mathbf{R}^n, \mathscr{L}(X))$, then*

$$\mathscr{R}(m(\xi) : \xi \in \mathbf{R}^n \text{ a Lebesgue point of } m) \leq C \, \|T_m\|_{\mathscr{L}(L^p(\mathbf{R}^n,X))} .$$

5 Bootstrapping and induction

In the context of vector-valued estimates, various possibilities of self-improvement are looming around. A basic observation is the isomorphic (thanks to Kahane's inequality) identification of spaces

$$\text{Rad}(L^p(\mathbf{R}^n, X)) \eqsim L^p(\mathbf{R}^n, \text{Rad}(X)),$$

which gives rise to an identification of operators $(T_{m_j})_{-\infty}^{\infty} \eqsim T_{(m_j)_{-\infty}^{\infty}}$, where on the right we have the Fourier multiplier with the sequence-valued kernel $\xi \mapsto M(\xi) := (m_j(\xi))_{-\infty}^{\infty} \in \mathscr{L}(\text{Rad}(X))$. Thus, proving the R-boundedness of a family of Fourier multiplier operators amounts to checking the boundedness of the single $\mathscr{L}(\text{Rad}(X))$-valued multiplier $M(\xi)$. In the presence of property (α), it is possible to conclude that M inherits the required R-boundedness estimates from the original multipliers m_j. These observations lead to the following:

Theorem 5.1 ([9, 27, 41, 56]). *Let $n \geq 1$. If (and only if) $1 < p < \infty$ and X is a UMD-space with property (α), then every family $\mathscr{M}$ of $\mathscr{L}(X)$-valued multipliers on $\mathbf{R}^n$ such that*

$$\mathscr{R}(|\xi^\alpha| \, D^\alpha m(\xi) : \alpha \in \{0,1\}^n, \xi \in (\mathbf{R} \setminus \{0\})^n, m \in \mathscr{M}) < \infty$$

induces an R-bounded family of operators $\{T_m : m \in \mathscr{M}\} \subset \mathscr{L}(L^p(\mathbf{R}^n, X))$.

This was first proved for scalar-valued multipliers by Venni [56]; the operator-valued extension was found by Bu [9] and by Girardi and Weis [27] (apparently independently). The necessity of (α) is shown in [41].

Another basic identification is the Fubini isometry

$$L^p(\mathbf{R}^{n+1}, X) \eqsim L^p(\mathbf{R}, L^p(\mathbf{R}^n, X)),$$

where $L^p(\mathbf{R}^n, X)$ has UMD and/or (α) if X has, and $1 < p < \infty$. This again gives an identification of Fourier multipliers,

$$T_{\xi\in\mathbf{R}^{n+1}\mapsto m(\xi)\in\mathscr{L}(X)} \eqsim T_{\xi_1\in\mathbf{R}\mapsto T_{m(\xi_1,\cdot)}\in\mathscr{L}(L^p(\mathbf{R}^n,X))}.$$

As it turns out, this kind of identification can be used, as done in [31], to reprove Theorems 4.4 and 5.1 for general $n \geq 1$ using only Theorem 5.1 for $n = 1$ and a simple induction on the dimension. While the inductive method is not necessary for getting these results, as we saw, it may offer at least some conceptual simplification, which has been exploited in proving new estimates for singular integrals by Portal and the author [38]. It seems that the inductive method is most useful in the presence of (α), for otherwise it is difficult to ensure the required R-bounds of the sequence-valued multipliers, but it may also be useful in some special situations without (α), as we see in the proof of Proposition 6.1 below.

6 Scope of the two multiplier theorems

The Mihlin Multiplier Theorem 4.3 is a result typical of the classical Calderón–Zygmund theory, which deals with classes of operators invariant under the natural one-parameter family of dilations $\xi \mapsto \delta\xi$, $\delta > 0$, of $\mathbf{R}^n$. The Marcinkiewicz–Lizorkin Multiplier Theorem 4.4, on the other hand, is typical of the multi-parameter or product theory, allowing independent dilations $\xi \mapsto (\delta_1\xi_1, \cdots, \delta_n\xi_n)$, $\bar{\delta} = (\delta_1, \cdots, \delta_n) > \bar{0}$, in the different coordinate directions. Since $|\xi^\alpha| = |\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}| \leq |\xi|^{|\alpha|}$ (typically with strict inequality), Theorem 4.4 imposes less stringent conditions on the multiplier m, at the cost of restricting the admissible Banach spaces.

Since the main interest in results like Theorems 4.3 and 4.4 comes from their applications, it is natural to enquire about the difference of the two conditions in practice: What are the typical multipliers for which one needs to use the more general Marcinkiewicz–Lizorkin theorem? In his classic book

Singular integrals, Stein [52] gives two examples of multipliers failing the Mihlin condition but falling under the scope of the Marcinkiewicz–Lizorkin theorem. The first one is the multiplier

$$m(\xi) = \frac{\xi_1}{\xi_1 + i(\xi_2^2 + \xi_3^2 + \cdots + \xi_n^2)} \tag{6.1}$$

related to parabolic equations, whereas the other example

$$m(\xi) = \frac{|\xi_1|^{\alpha_1}|\xi_2|^{\alpha_2}\cdots|\xi_n|^{\alpha_n}}{(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^{|\alpha|/2}} = \frac{|\xi^\alpha|}{|\xi|^{|\alpha|}} = \prod_{i=1}^{n}\Big(\frac{|\xi_i|}{|\xi|}\Big)^{\alpha_i}, \tag{6.2}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \geq \bar{0}$, "is not untypical of a class arising in connection with the study of spaces of fractional potentials".

We will study the multipliers of the parabolic type more systematically in the next section and now take a closer look at the second example (6.2). This multiplier has the curious feature of formally belonging to the one-parameter class, having invariance under the standard dilations $\xi \mapsto \delta\xi$, but failing the Mihlin conditions for the derivatives. Nevertheless, there holds:

Proposition 6.1. *For the multiplier m in* (6.2)*, we have $T_m \in \mathscr{L}(L^p(\mathbf{R}^n, X))$ for all $1 < p < \infty$ and all UMD spaces X.*

Proof. Let us first observe that it suffices to treat $m(\xi) = m_\epsilon(\xi) := (|\xi_1| / |\xi|)^\epsilon$, $\epsilon > 0$: this is the special case with $\alpha = (\epsilon, 0, \cdots, 0)$, but if we know the boundedness of these multipliers, the general case of (6.2) also follows, since it is just the product of our special multiplier and its rotated versions. We make use of the identification $T_m \eqsim T_{\xi_1 \mapsto T_{m(\xi_1,\cdot)}}$, and verify the operator-valued Mihlin conditions for the one-dimensional multiplier $M(\xi_1) := T_{m(\xi_1,\cdot)}$.

Let us show that $\{M(\xi_1) : \xi_1 \in \mathbf{R}\backslash\{0\}\}$ is R-bounded in $\mathscr{L}(L^p(\mathbf{R}^{n-1}, X))$, for which it suffices by Theorem 4.1 that the scalar multipliers $m(\xi_1, \cdot)$ have uniformly bounded variation. For $\alpha \in \{0\} \times \{0,1\}^{n-1}$, it is easy to compute

$$D^\alpha m_\epsilon(\xi) = C_{\epsilon,\alpha} m_\epsilon(\xi)\frac{\xi^\alpha}{|\xi|^{|\alpha|}} = C_{\epsilon,\alpha}\prod_{i:\alpha_i=1}\frac{|\xi_1|^{\epsilon/|\alpha|}\xi_i}{(\xi_1^2 + |\xi'|^2)^{\epsilon/2|\alpha|+1}},$$

where $\xi' := (\xi_2, \cdots, \xi_n)$, and then

$$\begin{aligned}
\int_{\mathbf{R}^\alpha} |D^\alpha m(\xi)| \,\mathrm{d}\xi_\alpha &\leq C_{\epsilon,\alpha}\prod_{i:\alpha_i=1}\int_{-\infty}^{\infty}\frac{|\xi_1|^{\epsilon/|\alpha|}|\xi_i|}{(\xi_1^2 + \xi_i^2)^{\epsilon/2|\alpha|+1}}\,\mathrm{d}\xi_i \\
&= C_{\epsilon,\alpha}\Big(\int_{-\infty}^{\infty}\frac{|x| \,\mathrm{d}x}{(1 + x^2)^{1+\epsilon/2|\alpha|}}\Big)^{|\alpha|},
\end{aligned}$$

which is a finite constant. We still need the R-boundedness of $\xi_1 M'(\xi_1)$, but this is immediate from the work already done, once we observe that $\xi_1 D_1 m_\epsilon(\xi) = \epsilon\big(m_\epsilon(\xi) - m_{\epsilon+2}(\xi)\big)$. □

Thus we observe that even if the full Marcinkiewicz–Lizorkin Multiplier Theorem 4.4 does not apply in a general UMD space, we can still go somewhat beyond the Mihlin Multiplier Theorem 4.3. The next section elaborates on this theme even more.

7 Parabolic theory of multipliers

The multiplier (6.1) is not well-behaved under the radial dilations, but it has the property of being invariant under the *anisotropic* one-parameter dilations $\xi \mapsto (\delta^2\xi_1, \delta\xi_2, \cdots, \delta\xi_n)$, $\delta > 0$. It turns out that this is essentially as good as the radial dilations, since there is also an anisotropic version of the Littlewood–Paley inequality, which can be proved almost by a repetition of the construction of the blocking-by-squares decomposition of Theorem 3.3. The key observations to this end are the facts that

- the one-dimensional Littlewood–Paley Theorem 3.1 remains valid if we replace the dyadic intervals $\pm\left[2^k, 2^{k+1}\right[$ by any lacunary intervals $\pm\left[\lambda^k, \lambda^{k+1}\right[$, $\lambda > 1$, (this follows easily from the case $\lambda = 2$) and
- in constructing the blocking by squares of the product decomposition, we may take different one-dimensional decompositions in the different coordinate directions.

Using the decomposition with $\lambda = 2^{\theta_i}$ in the ith coordinate, this construction results in the decomposition $\mathscr{I}_n(\theta)$, whose intervals are

$$\prod_{i=1}^{r-1} \eta_i \left]0, 2^{\theta_i(k_i+1)}\right[\times \eta_r \left[2^{\theta_r k_r}, 2^{\theta_r(k_r+1)}\right[\times \prod_{i=r+1}^{n} \eta_i \left]0, 2^{\theta_i k_i}\right[,$$

where $\eta \in \{-1,+1\}^n$ and $k \in \mathbf{Z}^n$. This decomposition for $n = 2$ is illustrated in Fig. 2. We obtain the following theorems:

Theorem 7.1 ([34]). *Let $n \geq 1$, $\theta = (\theta_1, \cdots, \theta_n) > \bar{0}$, X be a UMD space and $1 < p < \infty$. Then there are $0 < c \leq C < \infty$ such that*

$$c\,\|f\|_p \leq \mathsf{E}\Big\|\sum_{I\in\mathscr{I}_n(\theta)} \varepsilon_I \Delta[I] f\Big\|_p \leq C\,\|f\|_p, \qquad f \in L^p(\mathbf{R}^n, X).$$

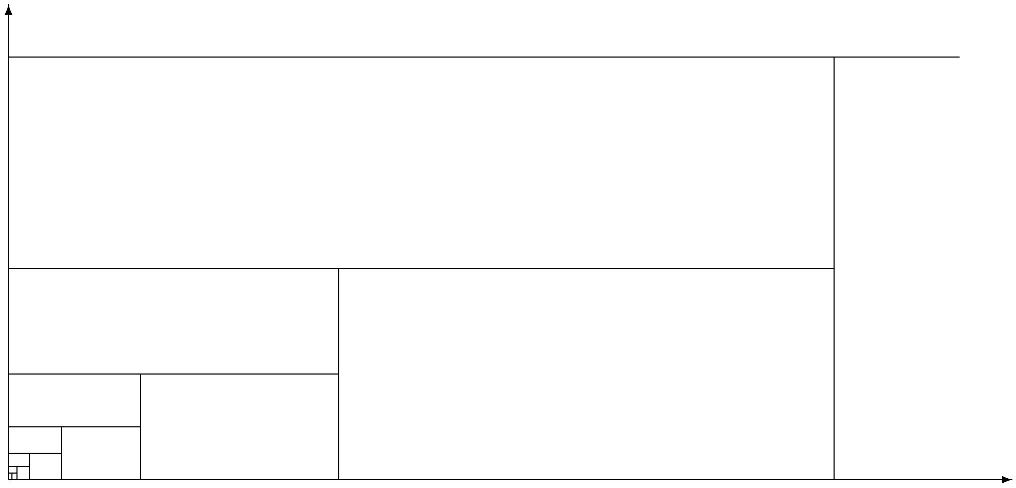

Figure 2: The anisotropic blocking decomposition with $(2^{\theta_1}, 2^{\theta_2}) = (\frac{5}{2}, 2)$.

Theorem 7.2 ([34]). *Let $n \geq 1$, $\theta = (\theta_1, \cdots, \theta_n) > \bar{0}$, X be a UMD space and $1 < p < \infty$. Then every multiplier $m : \mathbf{R}^n \setminus \{0\} \to \mathscr{L}(X)$ such that*

$$\mathscr{R}(\varrho_\theta(\xi)^{\theta \cdot \alpha} D^\alpha m(\xi) : \alpha \in \{0,1\}^n, \xi \in \mathbf{R}^n \setminus \{0\}) < \infty,$$

where $\varrho_\theta(\xi)$ is the unique positive solution of $\sum_{i=1}^n \xi_i^2 \varrho_\theta(\xi)^{-2\theta_i} = 1$, satisfies $T_m \in \mathscr{L}(L^p(\mathbf{R}^n, X))$.

Theorem 7.2 follows from 7.1 in the same way as Theorem 4.3 follows from 3.3. It contains Theorem 4.3 as the special case $\theta = (1, \cdots, 1)$, but also applies to more general multipliers of the parabolic type, like (6.1).

8 Multipliers for Sobolev-type inequalities

The following question of generalized Sobolev-type inequalities leads to an interesting class of Fourier multipliers: Find the conditions under which the following estimate for partial derivatives holds for all test functions, say $u \in \mathscr{D}(\mathbf{R}^n, X)$:

$$\left\| D^\beta u \right\|_p \leq C \sum_{\alpha \in \mathscr{A}} \left\| D^\alpha u \right\|_p . \tag{8.1}$$

The classical problem for $X = \mathbf{C}$ has the simple answer, for $p \in]1, \infty[$, that we must have $\beta \in \operatorname{conv} \mathscr{A}$, the convex hull of $\mathscr{A}$ (see e.g. [5]). The necessity follows by writing (8.1) for appropriate sums of translates and dilates of u. A sketch for the sufficiency is as follows: By the boundedness of $\Delta[\mathbf{R}_{\eta_1} \times \cdots \times \mathbf{R}_{\eta_n}]$, $\eta \in \{-,+\}^n$, which commute with D^α, and by symmetry we may assume that $\mathscr{F}u$ is supported in $\mathbf{R}^n_+$. By induction on the size of $\mathscr{A}$, it suffices to treat $\#\mathscr{A} = 2$. The task is reduced to proving that

$$m(\xi) = \frac{\xi^{t\alpha_0+(1-t)\alpha_1}}{\xi^{\alpha_0}+\xi^{\alpha_1}} 1_{\mathbf{R}^n_+}(\xi) = \frac{1_{\mathbf{R}^n_+}(\xi)}{\xi^{(1-t)(\alpha_0-\alpha_1)}+\xi^{t(\alpha_1-\alpha_0)}} \tag{8.2}$$

is a Fourier multiplier of $L^p(\mathbf{R}^n, X)$. It is readily checked to be a Marcinkiewicz–Lizorkin multiplier, so we have (8.1) in UMD spaces with (α). But does it hold in general UMD spaces?

For $n = 1$, clearly yes. For $n > 1$, we may observe that $m(\lambda^\theta \xi) = m(\xi)$ for all $\lambda > 0$, provided that we choose θ so that $\theta \cdot (\alpha_0 - \alpha_1) = 0$. This is always possible with some $\theta \neq \bar{0}$, but may be not with $\theta > \bar{0}$. The homogeneity of m is not in general of the parabolic type covered by Theorem 7.2, which leads to the question: Does there exist a “hyperbolic” version of Theorem 7.2, allowing $\theta \in (\mathbf{R} \setminus \{0\})^n$, and still valid in all UMD spaces?

The answer to this general question turns out to be negative, as demonstrated in the following section, but for more specific reasons there nevertheless holds:

Theorem 8.1 ([36]). *Let X be a UMD space, $1 < p < \infty$, and $\beta \in \operatorname{conv} \mathscr{A}$. Then there is $C < \infty$ such that (8.1) holds for all $u \in \mathscr{D}(\mathbf{R}^n, X)$.*

The proof exploits the Littlewood–Paley type estimate (3.1), the fact that the multipliers ξ^α are products $\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ of one-dimensional multipliers, and finally a complex interpolation argument.

9 No hyperbolic theory of multipliers

We now prove the impossibility of the hyperbolic theory of multipliers that one could have hoped for by the considerations in the previous section. Let us first look at some consequences that such a theory would have. If $m \in C^2(\mathbf{R} \setminus \{0\})$ satisfies the second-order Mihlin-type condition $\left|\xi^k D^k m(\xi)\right| \leq C$ for $k = 0, 1, 2$, then $(\xi_1, \xi_2) \in \mathbf{R}^2 \mapsto m(\xi_1 \xi_2)$ is a hyperbolic multiplier with dilation invariance under $\xi \mapsto (\lambda \xi_1, \lambda^{-1} \xi_2)$. If it was to be bounded on $L^p(\mathbf{R}^2, X) \eqsim L^p(\mathbf{R}, L^p(\mathbf{R}, X))$, then by Theorem 4.5, the set of operators

$T_{m(\xi_1\cdot)}$ induced by the dilations of m would be R-bounded in $\mathscr{L}(L^p(\mathbf{R},X))$. That such a result can only hold under the property (α) is the main content of the following:

Proposition 9.1. *There exists a Mihlin multiplier $m \in C^\infty(\mathbf{R}\setminus\{0\})$ satisfying $\left|\xi^k D^k m(\xi)\right| \leq C_k$ for all $k \in \mathbf{N}$, such that the following hold for every $1<p<\infty$ and every Banach space X:*

- *$T_m \in \mathscr{L}(L^p(\mathbf{R},X))$ if and only if X is UMD, and*
- *$\{T_{m(2^j\cdot)} : j \in \mathbf{Z}\}$ is R-bounded on $L^p(\mathbf{R},X)$ if and only if X is UMD with (α).*

Proof. The building blocks of our construction are a function $\hat{\psi} \in \mathscr{D}(\mathbf{R})$ supported in $[1/2,2]$ and satisfying $\sum_{j=-\infty}^{\infty}\hat{\psi}(2^j\xi) = 1_{]0,\infty[}(\xi)$, and a sequence $(\alpha_j)_{j=-\infty}^{\infty} \in \{0,1\}^{\mathbf{Z}}$ chosen so as to contain as a subsequence every one of the (countably many) finite bit sequences. Our multiplier is then defined as

$$m(\xi) := \sum_{j\in\mathbf{Z}} \alpha_j \hat{\psi}(2^{-j}\xi).$$

This clearly satisfies the Mihlin conditions of any order, so the asserted boundedness follows Theorem 4.3 and the R-boundedness from Theorem 5.1.

Concerning the necessity of UMD, we observe that it suffices to prove $\|\mathscr{F}^{-1}(1_{]0,\infty[}\hat{f})\|_p \leq C\,\|f\|_p$ for all $f \in \mathscr{S}(\mathbf{R},X)$ with $\operatorname{supp}\hat{f}$ a compact subset of $\mathbf{R}\setminus\{0\}$. Given such an f, let us denote

$$\rho := \sup\{|\xi|\,/\,|\eta| : \xi,\eta \in \operatorname{supp}\hat{f}\} < \infty.$$

By definition, $(\alpha_j)_{j=-\infty}^{\infty}$ contains a subsequence of 1's of length $N+1$ where $N > \log_2\rho$, say $\alpha_j \equiv 1$ for $j \in [j_0, j_0+N]\cap\mathbf{Z}$. An appropriate dilation $\hat{g} := \hat{f}(\lambda\cdot)$, $\lambda>0$, has the positive half of its support on $[2^{j_0}, \rho 2^{j_0}] \subset [2^{j_0}, 2^{j_0+N}]$, where $m(\xi)\equiv 1$. Thus $\|\mathscr{F}^{-1}(1_{]0,\infty[}\hat{g})\|_p = \|T_m g\|_p \leq C\,\|g\|_p$, and by dilation invariance we have the same with f in place of g.

We then come to the necessity of (α) for the R-boundedness of the dilated multipliers. Recall that this is equivalent to the boundedness of the operator-valued Fourier multiplier T_M on $L^p(\mathbf{R},\operatorname{Rad}(X))$, where

$$\begin{aligned} M(\xi)(x_j)_{-\infty}^{\infty} &:= (m(2^j\xi)x_j)_{-\infty}^{\infty} \\ &= (\alpha_{i+j}x_j)_{j=-\infty}^{\infty}, \qquad \text{if} \quad \xi = 2^i. \end{aligned}$$

For this, it is necessary by Theorem 4.5 that the essential range of M is R-bounded on $\mathrm{Rad}(X)$. This requires in particular that

$$\mathsf{E}\tilde{\mathsf{E}}\Big|\sum_{i\in F}\sum_{j\in G}\varepsilon_i\tilde{\varepsilon}_j\alpha_{i+j}x_{i,j}\Big|_X \leq C\mathsf{E}\tilde{\mathsf{E}}\Big|\sum_{i\in F}\sum_{j\in G}\varepsilon_i\tilde{\varepsilon}_j x_{i,j}\Big|_X \tag{9.1}$$

for all finite $F, G \subset \mathbf{Z}$ and $x_{i,j} \in X$. But we may choose $F = \{n_0 + Nk : k = 0, \cdots, N-1\}$, $G = \{\ell : \ell = 0, \cdots, N-1\}$, so that

$$\{i + j : i \in F, j \in G\} = [n_0, n_0 + N^2 - 1] \cap \mathbf{Z}$$

will be any desired sequence of N^2 consecutive integers, and the corresponding $\alpha_{i+j} = \alpha_{n_0+Nk+\ell}$ may be chosen as an arbitrary $N \times N$ array of bits. Thus (9.1) becomes precisely the defining condition of property (α). $\square$

We saw earlier that the UMD condition alone suffices for much of the one-parameter theory of multipliers, which goes somewhat beyond the standard Calderón–Zygmund theory related to the radial dilations. In contrast to this, the previous Proposition shows that we cannot go much further: there is no "hyperbolic one-parameter theory" of multipliers on its own right; it only exists as a special case of the multi-parameter theory, which requires the additional property (α).

Incidentally, the fact that hyperbolic dilations, although having only one independent variable, should already be regarded as part of the multi-parameter theory, has been observed in other contexts, too. As pointed out in [22] in connection to near-L^1 estimates for maximal functions

$$M_{\mathscr{R}}f(x) := \sup_{x\in R\in\mathscr{R}} \frac{1}{|R|}\int_R |f(y)|\,\mathrm{d}y$$

related to different families of rectangles $\mathscr{R}$, "the two-dimensional collection of rectangles of the form $s \times 1/s$ is already a two-parameter family."

We have now found a fairly sharp border between the multiplier theory valid in all UMD spaces and only in UMD spaces with (α), but some questions still remain. E.g., it would be interesting to know if the multiplier in Prop. 9.1 could be replaced by some more naturally occurring one. Observe that the m constructed in the proof in some sense contains all information in the universe, and one may wonder if a little less would be sufficient.

Furthermore, we have mainly excluded the possibility of certain general statements in all UMD spaces, but the continuity or its failure of a particular operator arising from a specific application is not implied by these results, leaving the possibility for various interesting special theorems, like the inequality (8.1) discussed in the previous section.

10 Other Banach space properties

In the remaining three sections, we briefly survey other developments in vector-valued Harmonic Analysis, which are independent of the above discussion of the border between one-parameter and multi-parameter theories. We first consider the effect on the multiplier theory of some further Banach space properties (besides UMD and (α)), which have been studied in a number of papers.

Fourier-type

In the scalar-valued context, it is a classical result of Hörmander that Mihlin's multiplier theorem remains valid when changing the set of derivatives for which the estimate $|\xi|^{|\alpha|}|D^\alpha m(\xi)| \leq C$ is required from $\alpha \in \{0,1\}^n$ to $|\alpha| \leq \lfloor n/2 \rfloor + 1$. In the vector-valued case, a similar reduction of the needed total order of differentiation is caused by the Fourier-type of the Banach space X. Recall that X has Fourier-type $t \in [1,2]$ if the Hausdorff–Young inequality $\|\mathscr{F} f\|_{t'} \leq C\|f\|_t$ holds for $f \in L^t(\mathbf{R}^n, X)$ for one (and then all) $n \in \mathbf{Z}_+$.

Theorem 10.1 ([26, 30, 33]). *Let $n \geq 1$, $1 < p < \infty$, and let X be a UMD space (resp. UMD space with (α)) with Fourier-type $t \in]1,2]$. If*

$$\mathscr{R}(|\xi|^{|\alpha|} D^\alpha m(\xi) : \xi \in \mathbf{R}^n \setminus \{0\}) < \infty,$$
$$\big(\textit{resp. } \mathscr{R}(|\xi^\alpha| D^\alpha m(\xi) : \xi \in (\mathbf{R} \setminus \{0\})^n) < \infty\big),$$

for all $\alpha \in \{0,1\}^n$ with $|\alpha| \leq \lfloor n/t \rfloor + 1$, then $T_m \in \mathscr{L}(L^p(\mathbf{R}^n, X))$.

The original realization of the condition $|\alpha| \leq \lfloor n/t \rfloor + 1$ under Fourier-type t is due to Girardi and Weis [26]. The possibility of intersecting this with Mihlin's resp. Marcinkiewicz–Lizorkin's condition was observed in [30] resp. [33]. The assumptions may be further weakened somewhat by considering appropriate fractional order smoothness, which allows to approach the critical index n/t.

Lattices

Rubio de Francia [51] wrote about twenty years ago: "Our present knowledge of the properties and structure of UMD lattices is deeper than for general UMD spaces." The state of affairs is still very much the same today. The additional tools available in UMD lattices, especially maximal functions and techniques using Muckenhoupt's A_p weights, have made it possible to prove results like the following, which remain interesting open problems for general UMD spaces:

Theorem 10.2 ([51]). *Every UMD lattice X is a complex interpolation space $[H, Y]_\theta$, $0 < \theta < 1$, between a Hilbert space H and another UMD lattice Y.*

Theorem 10.3 ([50, 51]). *Let X be a UMD lattice and $1 < p < \infty$. If $f \in L^p(\mathbf{T}, X)$, then the Fourier series of f converges to f almost everywhere.*

Theorem 10.3 was first proved by Rubio de Francia [50] for UMD spaces with an *unconditional basis* and extended by the same author [51] to the generality stated above. Other results in UMD lattices or UMD spaces with an unconditional basis have been proved in [7, 28, 55].

Interpolation spaces

Theorem 10.2 motivated the following definition by Berkson and Gillespie:

Definition 10.1 ([2]). The class $\mathcal{I}$ consists of those UMD spaces X which are isomorphic to a closed subspace of a complex interpolation space $[H, Y]_\theta$, $0 < \theta < 1$, between a Hilbert space H and another UMD space Y.

By Theorem 10.2, every UMD lattice belongs to $\mathcal{I}$, but $\mathcal{I}$ also contains other UMD spaces like the Schatten–von Neumann ideals $\mathscr{C}^p = [\mathscr{C}^2, \mathscr{C}^q]_\theta$, $1/p = (1-\theta)/2 + \theta/q$. More generally, the interpolation properties of non-commutative spaces coincide with those of their commutative analogues under fairly broad conditions [20], which implies the membership in $\mathcal{I}$ for many further operator spaces.

In some cases [2, 35, 38], improved results (compared to general UMD spaces) have been proved in the spaces of class $\mathcal{I}$ by interpolating with the estimates available in arbitrary UMD spaces and the stronger ones that one can get in a Hilbert space. Thus it would be interesting to know if $\mathcal{I}$ actually contains all UMD spaces.

Littlewood–Paley–Rubio property

Berkson, Gillespie and Torrea have recently introduced the following notion:

Definition 10.2 ([3, 25]). Let $2 \leq p < \infty$. A Banach space X has the Littlewood–Paley–Rubio property LPR_p if there is $C < \infty$ so that for every collection $\mathscr{J}$ of disjoint intervals $J \subset \mathbf{R}$ there holds:

$$\mathsf{E}\Big\| \sum_{J \in \mathscr{J}} \varepsilon_J \Delta[J] f \Big\|_p \leq C \, \|f\|_p \,, \qquad f \in L^p(\mathbf{R}, X).$$

The scalar field $X = \mathbf{C}$, and then by Fubini also $L^p(\mu)$, satisfies LPR_p by an inequality of Rubio de Francia [49]. This inequality was used by Coifman, Rubio de Francia and Semmes [16] to improve the Marcinkiewicz multiplier theorem. Similarly, the LPR_p property of a Banach space X implies an improvement of the vector-valued multiplier theorem, which was obtained by Potapov and the present author:

Theorem 10.4 ([38]). *Let $1 \leq s < 2 \leq p < \infty$ and X be a Banach space with LPR_p. Let $m : \mathbf{R} \to \mathscr{L}(X)$ be a function such that for all dyadic intervals $I = \pm \left[2^k, 2^{k+1}\right[$ we have*

$$\sup \Big(\|f(\xi_0)\|_{\mathscr{T}}^s + \sum_{j=1}^{N} \|f(\xi_{j-1}) - f(\xi_j)\|_{\mathscr{T}}^s \Big)^{1/s} \leq c < \infty,$$

where $\mathscr{T}$ is an R-bounded set, and the supremum is over all partitions $\inf I = \xi_0 < \xi_1 < \cdots < \xi_N = \sup I$. *Then $T_m \in \mathscr{L}(L^p(\mathbf{R}, X))$.*

The case $s = 1$ above is the (vector-valued) Marcinkiewicz multiplier theorem valid in every UMD space and $1 < p < \infty$ (although we have only given the somewhat weaker formulation in Theorem 4.3 above). The assumption becomes weaker with increasing s. If we also assume that $X \in \mathfrak{I}$, then we may take $s = 2$ in Theorem 10.4.

11 Singular convolution operators

In the vector-valued treatment of Calderón–Zygmund operators beyond those represented by Mihlin or Marcinkiewicz–Lizorkin type multipliers, the following "restricted R-boundedness" estimate for the translations $\tau_h f := f(\cdot - h)$ has become an indispensable companion to the Littlewood–Paley inequalities discussed earlier:

Theorem 11.1 ([8]). *Let X be a UMD space and $1 < p < \infty$. Then there is a constant $C < \infty$ such that, whenever the functions $f_j \in L^p(\mathbf{R}^n, X)$ and the colinear points $h_j \in \mathbf{R}^n$, $j \in \mathbf{Z}$, satisfy*

$$\operatorname{supp} \mathscr{F} f_j \subseteq \{\xi : |\xi| \leq 2^j\}, \qquad |h_j| \leq K 2^{-j}$$

for some $K \geq 2$, there holds

$$\mathsf{E}\Big\| \sum_j \varepsilon_j \tau_{h_j} f_j \Big\|_p \leq C \log K \cdot \mathsf{E}\Big\| \sum_j \varepsilon_j f_j \Big\|_p.$$

This result of Bourgain's [8] had apparently no scalar-valued predecessor, but the scalar version of the theorem was independently discovered around the same time by Yamazaki [60]. The original statement in [8] concerns $n = 1$, but the transference to $n > 1$ is standard and may be found from [26].

Like the proof of Theorem 3.1, also that of Theorem 11.1 starts from the defining inequality of UMD spaces, but now applied to a less obvious choice of the filtration. Every second σ-algebra will be generated by simple intervals $N^{-1}\,[k, k+1[$, $k \in \mathbf{Z}$, but the atoms of the intermediate σ-algebras will be unions $N^{-1}\big([k, k+1[\cup [n+k, n+k+1[\big)$, $k \equiv 0, 1, \cdots, n-1 \pmod{2n}$, of two separated intervals, where the separation reflects the translations that we are aiming at. Like in the case of Theorem 3.1, the estimate obtained from the martingale inequality introduces extra smoothing in addition to the desired translations. Moreover, we can only reach translations which satisfy certain algebraic restrictions. In order to remove these deficiencies, a perturbation argument based on Theorem 4.1 is required. To make this argument, we have to split the functions f_j into approximately $\log K$ subsets for separate treatment, which gives rise to the logarithmic factor in the final estimate. The scalar-valued version in [60] is somewhat easier, since the trivial case $p = 2$ can be extrapolated to the whole range of $1 < p < \infty$ by standard Calderón–Zygmund methods.

The first application of Theorem 11.1 is to the boundedness of singular integrals of convolution type:

Theorem 11.2 ([8, 39]). *Let X be a UMD space and $1 < p < \infty$. Let $K \in C(\mathbf{R}^n \setminus \{0\}, \mathscr{L}(X))$ satisfy the size and cancellation conditions*

$$\mathscr{R}(|x|^n K(x), |x|^{n+\delta} |y|^{-\delta} [K(x+y) - K(x)] : |x| > 2|y| > 0) < \infty$$

for some $\delta > 0$, and $\mathscr{R}(\int_{r<|x|<R} K(x)\,\mathrm{d}x : R > r > 0) < \infty$, as well as the existence of the limit $\lim_{\epsilon\downarrow 0} \int_{\epsilon<|x|<1} K(x)\,\mathrm{d}x$ in the strong operator topology.

Then the convolution $f \mapsto K * f$ (initially defined on a test-function space) extends to a bounded linear operator on $L^p(\mathbf{R}^n, X)$.

Bourgain [8] first demonstrated the use of his Theorem 11.1 to obtain the scalar-kernel version of Theorem 11.2 for $n = 1$, and this was extended to the n-dimensional operator-kernel setting by Weis and the present author in [39], where also a more general but rather technical sufficient condition of Hörmander-type is given for the kernel K. As with multipliers, one gets the R-boundedness of families of convolution operators under the additional assumption of property (α). One should also note that the improved estimates for multipliers involving the Fourier-type (discussed in the previous section) are actually based on convolution-kernel estimates and, at the bottom, on Theorem 11.1.

In the proof of results like Theorem 11.2, the translation inequality of Theorem 11.1 assumes, to some extent, the rôle played by maximal function estimates in the scalar-valued theory. In fact, the proof of the scalar-valued case of Theorem 11.2 by Littlewood–Paley theory is something like the following: First, by the Littlewood–Paley inequality, we have

$$\begin{aligned}\|K * f\|_p &\lesssim \Big\| \Big(\sum_{I \in \mathscr{I}_n} |\Delta[I](K * f)|^2 \Big)^{1/2} \Big\|_p \\ &= \Big\| \Big(\sum_{I \in \mathscr{I}_n} |(\varphi_I * K) * (\Delta[I] f)|^2 \Big)^{1/2} \Big\|_p,\end{aligned}$$

where the φ_I have Fourier transforms $\hat{\varphi}_I$, which are smoothed versions of 1_I, say $1_I \leq \hat{\varphi}_I \leq 1_{I^*}$, where $I^* \supset I$ is a slightly larger rectangle. The assumptions on the kernel K imply that the convolution operators $(\varphi_I * K)*$ are point-wise dominated by the Hardy–Littlewood maximal function, so that the Fefferman–Stein maximal inequality and the reverse Littlewood–Paley estimate give

$$\lesssim \Big\| \Big(\sum_{I \in \mathscr{I}_n} [M(\Delta[I] f)]^2 \Big)^{1/2} \Big\|_p \lesssim \Big\| \Big(\sum_{I \in \mathscr{I}_n} |\Delta[I] f|^2 \Big)^{1/2} \Big\|_p \lesssim \|f\|_p .$$

The first and last estimates and the equality in the above chain remain valid in the UMD-valued situation, thanks to Theorem 3.3, just by replacing the square sums $\big(\sum |\Delta[I] g|^2 \big)^{1/2}$ by the randomized sums $\mathsf{E} \left| \sum \varepsilon_I \Delta[I] g \right|_X$, where $g \in \{f, K * f\}$; however, a suitable analogue of the maximal inequality is not known, except in the lattice setting. Thus the two estimates in the middle are replaced by the following computation:

$$\begin{aligned}
&\mathsf{E}\Big\| \sum_{I\in\mathscr{I}_n} \varepsilon_I(\varphi_I * K) * (\Delta[I]f)\Big\|_p = \mathsf{E}\Big\| \sum_{I\in\mathscr{I}_n} \varepsilon_I \int (\varphi_I * K)(y)\tau_y(\Delta[I]f)\,\mathrm{d}y\Big\|_p \\
&= \mathsf{E}\Big\| \int \sum_{I\in\mathscr{I}_n} \varepsilon_I 2^{-j_I n}(\varphi_I * K)(2^{-j_I}y)\tau_{2^{-j_I}y}(\Delta[I]f)\,\mathrm{d}y\Big\|_p,
\end{aligned}$$

where $2^{j_I} \eqsim$ the side-length of I and the integrals are over $\mathbf{R}^n$. The estimate now continues with

$$\begin{aligned}
&\lesssim \int \mathscr{R}(2^{-j_I n}(\varphi_I * K)(2^{-j_I}y) : I \in \mathscr{I}_n)\mathsf{E}\Big\| \sum_{I\in\mathscr{I}_n} \varepsilon_I \tau_{2^{-j_I}y}(\Delta[I]f)\Big\|_p \,\mathrm{d}y \\
&\lesssim \int \mathscr{R}(2^{-j_I n}(\varphi_I * K)(2^{-j_I}y) : I \in \mathscr{I}_n)\log(2+|y|)\,\mathrm{d}y \cdot \mathsf{E}\Big\| \sum_{I\in\mathscr{I}_n} \varepsilon_I \Delta[I]f\Big\|_p,
\end{aligned}$$

where the final estimate utilized Theorem 11.1. The above integral will be finite under the assumptions made on K, whereas the last randomized norm is $\lesssim \|f\|_p$ by Theorem 3.3.

12 General Calderón–Zygmund operators

The fundamental tools of vector-valued Harmonic Analysis introduced in the earlier section have been successfully exploited to treat also the generalized, non-translation-invariant, Calderón–Zygmund operators. There is the following version of the David–Journé $T(1)$ theorem [17]:

Theorem 12.1 ([24, 40]). *Let X be a UMD space. Let $K \in C(\mathbf{R}^n \times \mathbf{R}^n \setminus \{x = y\}, \mathscr{L}(X))$ satisfy the estimate*

$$\begin{aligned}
&\mathscr{R}(\,|x-y|^n K(x,y), |x-y|^{n+\delta}|x-x'|^{-\delta}[K(x,y)-K(x',y)], \\
&\qquad |x-y|^{n+\delta}|x-x'|^{-\delta}[K(y,x)-K(y,x')] : |x-y| > 2|x-x'| > 0) < \infty.
\end{aligned}$$

Let $T : \mathscr{S}(\mathbf{R}^n) \to \mathscr{L}(\mathscr{S}(\mathbf{R}^n), \mathscr{L}(X))$ be a linear operator such that

$$\langle \phi_1, T\phi_0\rangle = \iint \phi_1(x)K(x,y)\phi_0(y)\,\mathrm{d}x\,\mathrm{d}y \tag{12.1}$$

for all disjointly supported $\phi_0, \phi_1 \in \mathscr{D}(\mathbf{R}^n)$, and such that

$$\mathscr{R}(r^n \langle \phi_1(r \cdot +h), T(\phi_0(r \cdot +h)) \rangle : r > 0, h \in \mathbf{R}^n) \leq C < \infty \tag{12.2}$$

for all "bump functions" $\phi_i \in \mathscr{D}(B(0,1))$ with $\|D^\alpha \phi_i\|_\infty \leq 1$ for all $|\alpha| \leq N$ (some large number). If $T(1) = 0$, $T'(1) = 0$ (in the sense of distributions modulo constants), then T extends to an operator in $\mathscr{L}(L^p(\mathbf{R}^n, X))$ for $1 < p < \infty$.

This result was first obtained by Figiel [24] for scalar-valued kernels using a clever decomposition of the operator and martingale estimates. For scalar kernels, one can even replace the conditions $T(1) = 0$, $T'(1) = 0$ by $T(1), T'(1) \in \mathrm{BMO}(\mathbf{R}^n)$, and this condition is both necessary and sufficient for the conclusion $T \in \mathscr{L}(L^p(\mathbf{R}^n, X))$ under the other stated assumptions, so that a full analogue of the David–Journé theorem is valid. The operator-valued extension, and a new proof building on the Fourier-analytic techniques discussed in the previous sections, was recently found by Weis and the present author [40]. It is possible to state *sufficient* BMO-type conditions for $T(1)$ and $T'(1)$ even in the operator setting, but they are probably *far* stronger than necessary. The problem may be reduced to the question of boundedness of operator-valued *paraproducts*, but the precise condition for this is unknown already in infinite-dimensional Hilbert spaces [4, 46].

While Figiel's martingale approach to the $T(1)$ theorem differs quite a lot from the techniques discussed in this paper, there still exist parallel ingredients. In particular, a key rôle is played by estimates for translations of the Haar functions from [23], which are analogous to Bourgain's Translation Theorem 11.1. With appropriate modifications of Figiel's ideas, it is also possible to get an operator-valued extension of the $T(b)$ theorem of David, Journé and Semmes [18]:

Theorem 12.2 ([32]). *Let X be a UMD space and let K and T be as in Theorem 12.1, except that in (12.1) we have $b_i\phi_i$ in place of ϕ_i, and in (12.2) $b_i(\cdot)\phi_i(r \cdot +h)$ in place of $\phi_i(r \cdot +h)$,*[1] *where $b_0, b_1 \in L^\infty(\mathbf{R}^n)$ satisfy the accretivity condition $\operatorname{Re} b_i \geq c > 0$. If $T(b_0) = 0$ and $T'(b_1) = 0$, then $T \in \mathscr{L}(L^p(\mathbf{R}^n, X))$ for $1 < p < \infty$.*

[1] The analogue of (12.2) is assumed in a stronger form in [32], where also the ϕ_i are required to vary *inside* the R-bound; however, it is easy to see that only the uniform boundedness over the ϕ_i is actually required in the proof.

Again, for a scalar-valued kernel it is sufficient (and necessary) for the conclusion that $T(b_0), T'(b_1) \in \mathrm{BMO}(\mathbf{R}^n)$, and there is an analogous sufficient condition in the operator-valued case. The accretivity assumption may be replaced by more general *para-accretivity*; see [32]. The scalar-kernel case of Theorem 12.2 can be easily deduced from Theorem 12.1 and the original $T(b)$ theorem from [18].

Various results have also been proved for vector-valued pseudo-differential operators, of which the following is representative:

Theorem 12.3 ([37, 48, 53])**.** *Let X be a UMD space and $1 < p < \infty$. Let $a \in L^\infty(\mathbf{R}^n \times \mathbf{R}^n, \mathscr{L}(X))$ satisfy*

$$\mathscr{R}\Big((1+|\xi|)^k D^k_{\xi_i} a(x,\xi), (1+|\xi|)^k \frac{D^k_{\xi_i} a(x,\xi) - D^k_{\xi_i} a(y,\xi)}{|x-y|^\delta} : \xi \in \mathbf{R}^n\Big) \leq C < \infty$$

for all $x, y \in \mathbf{R}^n$, $i = 1, \cdots, n$ and $k = 0, 1, \cdots, n+1$. Then the pseudo-differential operator

$$Tf(x) := \int_{\mathbf{R}^n} a(x,\xi) \hat{f}(\xi) e^{\mathbf{i}x\cdot\xi} \,\mathrm{d}\xi$$

extends to $T \in \mathscr{L}(L^p(\mathbf{R}^n, X))$.

First results on operator-symbol pseudo-differential operators were obtained in Štrkalj's thesis [53] and worked out in a slightly different form by Portal and Štrkalj [48]. The above statement is a special case of the recent results of Portal and the author [37]; in comparison to [48], fewer derivatives are required, but somewhat more R-bounds (instead of just uniform bounds) are imposed.

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