# Function theory in sectors and the analytic functional calculus for systems of operators

*Brian Jefferies*

**Abstract**

The connection between holomorphic and monogenic functions in sectors is used to construct an analytic functional calculus for several sectorial operators acting in a Banach space. The results are applied to the $H^\infty$-functional calculus for the differentiation operators on a Lipschitz surface. *MSC (2000): 47A13, 47A60, 30G35, 42B20.*
*Received 28 July 2006 / Accepted 16 November 2006.*

## 1 Introduction

It is well known that there is no 'natural' integral representation formula for holomorphic functions of several complex variables in dimensions greater than one, see [8, p. 25] or [15, p. 144]. However, *Clifford analysis* does possess a natural analogue of the Cauchy integral formula in $\mathbb{C}$; the cost is that in Clifford analysis, regular functions take their values in an anti-commutative algebra. This note is a report on joint work with John Ryan exploring the connection between Clifford analysis and functions of several complex variables by using elementary ideas arising from spectral theory and the functional calculus of systems of operators. The details will appear elsewhere.

The integral representation formula of Clifford analysis has recently been applied to functional calculi for systems of operators by analogy with the Riesz-Dunford functional calculus for a single operator, see [4] for a description and applications to harmonic analysis, PDE and quantum physics. If $\boldsymbol{A} = (A_1, \ldots, A_n)$ is an $n$-tuple of linear operators acting in a Banach space

$X$, we can attempt to form the function $f(\boldsymbol{A})$ of the operators $A_1, \ldots, A_n$ via the higher dimensional analogue

$$f(\boldsymbol{A}) = \int_{\partial\Omega} G_x(\boldsymbol{A})\boldsymbol{n}(x)f(x)\, d\mu(x) \tag{1.1}$$

of the Riesz-Dunford functional calculus, for a suitable open subset $\Omega$ of $\mathbf{R}^{\mathrm{n}+1}$ with smooth oriented boundary $\partial\Omega$, outward unit normal $\boldsymbol{n}(x)$ at $x \in \partial\Omega$ and surface measure $\mu$. The function $f$ is assumed to have suitable decay and be left monogenic in a neighbourhood of $\overline{\Omega}$ in $\mathbf{R}^{\mathrm{n}+1}$, that is, it takes values in a Clifford algebra $C\ell(\mathbb{C}^n)$ generated by the $n$ standard basis vectors in $\mathbb{C}^n$ and satisfies higher dimensional analogues of the Cauchy-Riemann equations. If the operators $A_1, \ldots, A_n$ do not commute with each other, then a *symmetric* functional calculus $f \longmapsto f(\boldsymbol{A})$ is obtained.

The Cauchy kernel $G_x(\boldsymbol{A})$ may be formed in a number of ways. If $\boldsymbol{A}$ satisfies exponential growth estimates, then the Weyl functional calculus $\mathcal{W}_{\boldsymbol{A}}$ is applicable and we can set $G_x(\boldsymbol{A}) := \mathcal{W}_{\boldsymbol{A}}(G_x)$ [5], [4, Section 4.1]. If the spectra of the operators $\langle \boldsymbol{A}, \xi\rangle = \sum_{j=1}^n A_j\xi_j$ lie in a sector in $\mathbb{C}$ and satisfy uniform resolvent bounds for $\xi \in \mathbf{R}^{\mathrm{n}}$ with $|\xi| = 1$, then a plane wave decomposition can be used [6], [4, Chapter 6]. In the commuting case $G_x(\boldsymbol{A})$ may be defined via Taylor's functional calculus [18], [13], [1], [14]. Whichever method is used to obtain the Cauchy kernel, the set $\gamma(\boldsymbol{A}) \subset \mathbf{R}^{\mathrm{n}+1}$ of singularities of the function $x \longmapsto G_x(\boldsymbol{A})$ is called the *monogenic spectrum* of the $n$-tuple $\boldsymbol{A}$, by analogy with the spectrum $\sigma(A)$ of a single operator $A$ interpreted as the set of singularities of its resolvent map $\lambda \longmapsto (\lambda I - A)^{-1}$. If $\boldsymbol{A}$ satisfies exponential growth estimates, then $\gamma(\boldsymbol{A})$ is precisely the support of the Weyl functional calculus [5], [4, Section 4.1].

By viewing $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ as an $n$-tuple of multiplication operators in the Clifford algebra $C\ell(\mathbb{C}^n)$ in equation (1.1), we obtain the formula

$$\tilde{f}(\zeta) = \int_{\partial\Omega_\zeta} G_x(\zeta)\boldsymbol{n}(x)f(x)\, d\mu(x) \tag{1.2}$$

associating a left monogenic function $f$ with its holomorphic counterpart $\tilde{f}$. Cauchy's theorem in Clifford analysis ensures that $\tilde{f}(\xi) = f(\xi)$ for $\xi \in \mathbf{R}^{\mathrm{n}}$ because the Cauchy kernel $G_x(\zeta)$ is the maximal analytic continuation into $\mathbb{C}^n$ of $\xi \longmapsto G_x(\xi)$, $\xi \in \mathbf{R}^{\mathrm{n}}$, $\xi \neq x$. It is the straightforward integral representation theorem of Clifford analysis that facilitates the representation (1.2). The monogenic spectrum $\gamma(\zeta)$ of the complex vector $\zeta \in \mathbb{C}^n$ is an

$(n-1)$ dimensional hypersphere ($n$ odd) or $n$ dimensional ball ($n$ even) in $\mathbf{R}^{n+1}$ centred at $\operatorname{Re}\zeta = (\operatorname{Re}\zeta_1, \ldots, \operatorname{Re}\zeta_n)$ with radius $|\operatorname{Im}\zeta|$ where $\operatorname{Im}\zeta = (\operatorname{Im}\zeta_1, \ldots, \operatorname{Im}\zeta_n)$.

The main result of the paper [3] was that the mapping $f \longmapsto \tilde{f}$ from left monogenic functions $f$ uniformly bounded on subsectors of a fixed sector in $\mathbf{R}^{n+1}$ to holomorphic functions $\tilde{f}$ uniformly bounded on subsectors of a corresponding sector in $\mathbb{C}^n$ is actually a bijection. As a consequence, if $\boldsymbol{D}_\Sigma$ is the $n$-tuple of differentiation operators on a Lipschitz surface $\Sigma$ in $\mathbf{R}^{n+1}$, then the equality $f(\boldsymbol{D}_\Sigma) = \tilde{f}(\boldsymbol{D}_\Sigma)$ extends to all monogenic functions $f$ uniformly bounded on subsectors of a fixed sector in $\mathbf{R}^{n+1}$ determined by the tangent hyperplanes of $\Sigma$. Here $f(\boldsymbol{D}_\Sigma)$ is defined by formula (1.1) in the case that $f$ has decay at zero and infinity and $\tilde{f}(\boldsymbol{D}_\Sigma)$ is defined via the Fourier theory of [9], [12]. It is known that $\boldsymbol{D}_\Sigma$ satisfies "square function estimates", so the mapping $f \longmapsto f(\boldsymbol{D}_\Sigma)$ has a uniformly bounded extension from left monogenic functions with decay at zero and infinity to all left monogenic functions uniformly bounded on a sector containing almost all hyperplanes tangent to $\Sigma$ in its interior.

An essential observation of the paper [3] was that the set of all complex vectors $\zeta \in \mathbb{C}^n$ whose monogenic spectrum $\gamma(\zeta)$ lies in a fixed sector $S_\omega(\mathbf{R}^{n+1})$ of angle $0 < \omega < \pi/2$ in $\mathbf{R}^{n+1}$ coincides with a sector $S_\omega(\mathbb{C}^n)$ in $\mathbb{C}^n$, see Proposition 5.1 below. The sector $S_\omega(\mathbb{C}^n)$ has the property that for each $\zeta \in S_\omega(\mathbb{C}^n)$, the exponential function $e(x,\zeta)$ defined in [9, p. 685] has decay as $|x| \to \infty$ for $x \in S_\nu(\mathbf{R}^{n+1})$, $0 < \nu < \omega$.

The integral representation formula (1.2) is purely local, so the question arises if the mapping $f \longmapsto \tilde{f}$ defined by formula (1.2) is a bijection without any uniform boundedness assumptions and if it is a bijection on domains other than sectors in $\mathbf{R}^{n+1}$ and their counterparts in $\mathbb{C}^n$. In [3], the reconstruction of the monogenic function $f$ from the holomorphic function $\tilde{f}$ is achieved by employing the Fourier theory developed in [9], which is nonlocal in character.

In this note, the representation formula developed in [16], [17] is used to construct the inverse map $\tilde{f} \longmapsto f$ for holomorphic functions $\tilde{f}$ defined in sectors in $\mathbb{C}^n$ onto the vector space of left monogenic functions defined in sectors in $\mathbf{R}^{n+1}$. By avoiding the Fourier theory used in [3], we produce a *local* representation for holomorphic functions defined in open subsets of $\mathbb{C}^n$ onto the space of left monogenic functions defined in the corresponding open subset of $\mathbf{R}^{n+1}$. The correspondence is obtained simply by taking the restriction $f_0$ of a holomorphic function $\tilde{f}$ defined in an open subset of $\mathbb{C}^n$, to its nonempty intersection with $\mathbf{R}^{\mathrm{n}}$ and then, the left monogenic extension $f$ of $f_0$ into $\mathbf{R}^{\mathrm{n+1}}$. In the process, we establish that the Clifford algebra valued function $f$ is actually the restriction to $\mathbf{R}^{\mathrm{n+1}}$ of a complex left regular function defined in an open subset of $\mathbb{C}^{n+1}$.

Besides considering the monogenic spectrum $\gamma(\zeta) \subset \mathbf{R}^{\mathrm{n+1}}$ of the complex vector $\zeta \in \mathbb{C}^n$, the spectrum $\sigma(i\zeta)$ of the element $i\zeta = i(\zeta_1 e_1 + \cdots + \zeta_n e_n)$ of the Clifford algebra $C\ell(\mathbb{C}^n)$ is also relevant to our studies. If $|\zeta|^2_{\mathbb{C}} \neq 0$, then $\sigma(i\zeta) = \{\pm|\zeta|_{\mathbb{C}}\}$ with projections

$$\chi_{\pm}(\zeta) = \frac{1}{2}\left(e_0 \pm i\frac{\zeta}{|\zeta|_{\mathbb{C}}}\right),$$

by which the functional calculus

$$b(i\zeta) = b(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + b(-|\zeta|_{\mathbb{C}})\chi_-(\zeta)$$

is obtained [12, Section 5.2]. The complex sector $S_\omega(\mathbb{C}^n)$ may be viewed in two complementary ways: as the set of all complex vectors $\zeta \in \mathbb{C}^n$ such that the spectrum $\sigma(i\zeta)$ of $i\zeta$ is contained in a double sector $S_\omega(\mathbb{C})$ of angle $\omega$ and following [3], as the set of all complex vectors $\zeta \in \mathbb{C}^n$ such that the monogenic spectrum $\gamma(\zeta)$ of $\zeta$ is contained in a sector $S_\omega(\mathbf{R}^{\mathrm{n+1}})$ of angle $\omega$ in $\mathbf{R}^{\mathrm{n+1}}$. A table of holomorphic functions $b$ uniformly bounded on a double sector in $\mathbb{C}$, their holomorphic several variable counterparts $\zeta \longmapsto b(i\zeta)$ and their Fourier transforms in $\mathbf{R}^{\mathrm{n+1}}$ is given in [9, pp. 701,702]. As a consequence of Theorem 5.1 below, the restriction $\boldsymbol{x} \longmapsto b(i\boldsymbol{x})$, $\boldsymbol{x} \in \mathbf{R}^{\mathrm{n}} \setminus \{0\}$, of any such function to $\mathbf{R}^{\mathrm{n}} \setminus \{0\}$ has a unique left monogenic extension to a corresponding sector in $\mathbf{R}^{\mathrm{n+1}}$.

# 2 Clifford Analysis

The real and imaginary parts of $z \in \mathbb{C}$ are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively and for an element $\zeta = (\zeta_1, \ldots, \zeta_n)$ of $\mathbb{C}^n$, the vector $\mathrm{Re}(\zeta) = (\mathrm{Re}(\zeta_1), \ldots, \mathrm{Re}(\zeta_n)) \in \mathbf{R}^{\mathrm{n}}$ denotes the real part of $\zeta$ and $\mathrm{Im}(\zeta) = (\mathrm{Im}(\zeta_1), \ldots, \mathrm{Im}(\zeta_n)) \in \mathbf{R}^{\mathrm{n}}$ denotes the imaginary part of $\zeta$.

Let $C\ell(\mathbb{C}^n)$ be the *Clifford algebra* generated over the field $\mathbb{C}$ by the standard basis vectors $e_0, e_1, \ldots, e_n$ of $\mathbb{C}^{n+1}$ with conjugation $u \longmapsto \overline{u}$. Then $e_0$ is the unit of $C\ell(\mathbb{C}^n)$, $e_j$ and $e_k$ anticommute for $j, k = 1, \ldots, n$ and $j \neq k$, and $e_j^2 = -1$ for $j = 1, \ldots, n$. The conjugation satisfies $\overline{e_0} = e_0$,

$\overline{e_j} = -e_j$ for $j = 1, \ldots, n$ and $\overline{uv} = \overline{v}\,\overline{u}$ for all $u, v \in C\ell(\mathbb{C}^n)$. Any nonzero element $u = \sum_{j=0}^n u_j e_j$ of $C\ell(\mathbb{C}^n)$ is invertible and $u^{-1} = \overline{u}/(u\overline{u})$ because $u\overline{u} = |u|^2 e_0$. In the case that we don't take the complex conjugate of the complex coefficients of the standard basis, we use the symbol $\overline{u}^{\mathbb{C}} = \sum_S u_S \overline{e_S}$ for $u = \sum_S u_S e_S$ with $u_S \in \mathbb{C}$. Then for the vector $u = \sum_{j=0}^n u_j e_j$ we have

$$u\overline{u}^{\mathbb{C}} = \sum_{j=0}^{n} u_j^2.$$

If $u\overline{u}^{\mathbb{C}}$ is not a negative real number or zero, then the square root of the complex number $u\overline{u}^{\mathbb{C}}$ with positive real part is denoted by $|u|_{\mathbb{C}}$ and $|u|_{\mathbb{C}} = 0$ if $u\overline{u}^{\mathbb{C}} = 0$.

The Clifford algebra $C\ell(\mathbb{C}^n)$ is a complex vector space with a basis $e_S$, $S \subset \{1, \ldots, n\}$ given by $e_S = e_{j_1} \cdots e_{j_k}$ if $S = \{j_1, \ldots, j_k\}$ and $1 \leq j_1 < \cdots < j_k \leq n$ is an ordered enumeration of $S$. If $S = \emptyset$, then $e_\emptyset = e_0$. In particular, $C\ell(\mathbb{C}^n)$ has complex dimension $2^n$. A function $f : U \longrightarrow C\ell(\mathbb{C}^n)$ therefore has a unique representation $f = \sum_S f_S e_S$ in which $f_S : U \longrightarrow \mathbb{C}$ are scalar valued functions for each subset $S$ of $\{1, \ldots, n\}$.

The embedding $z \longmapsto z e_0$, $z \in \mathbb{C}$, identifies $\mathbb{C}$ with a closed commutative subalgebra of $C\ell(\mathbb{C}^n)$ and $\mathbb{C}^{n+1}$ is identified with the closed linear subspace of all elements $z_0 e_0 + z_1 e_1 + \cdots + z_n e_n$ of $C\ell(\mathbb{C}^n)$ with $z_j \in \mathbb{C}$ for $j = 0, 1, \ldots, n$. Then $\mathbb{C}^n$ is always identified with the subspace $\{0\} \times \mathbb{C}^n$ of $\mathbb{C}^{n+1}$ and then with the corresponding subspace of $C\ell(\mathbb{C}^n)$. Similarly, $\mathbf{R}$, $\mathbf{R}^{\mathrm{n}}$ and $\mathbf{R}^{\mathrm{n+1}}$ are identified with the corresponding real linear subspaces of $C\ell(\mathbb{C}^n)$.

The *generalised Cauchy-Riemann operator* is given by $D = \sum_{j=0}^n e_j \frac{\partial}{\partial x_j}$. Let $U \subset \mathbf{R}^{\mathrm{n+1}}$ be an open set. A function $f : U \longrightarrow C\ell(\mathbb{C}^n)$ is called *left monogenic* if $Df = 0$ in $U$ and *right monogenic* if $fD = 0$ in $U$.

The *Cauchy kernel* is given by

$$k(x-y) = \frac{1}{\sigma_n} \frac{\overline{x-y}}{|x-y|^{n+1}}, \quad x, y \in \mathbf{R}^{\mathrm{n+1}}, \mathrm{x} \neq \mathrm{y}, \tag{2.1}$$

with $\sigma_n = 2\pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right)$ the volume of unit $n$-sphere in $\mathbf{R}^{\mathrm{n+1}}$. The function $k$ is both left and right monogenic away from the origin. So, given a left monogenic function $f : U \longrightarrow C\ell(\mathbb{C}^n)$ defined in an open subset $U$ of $\mathbf{R}^{\mathrm{n+1}}$ and an open subset $\Omega$ of $U$ such that the closure $\overline{\Omega}$ of $\Omega$ is contained in $U$, and the boundary $\partial\Omega$ of $\Omega$ is a smooth oriented $n$-manifold, then the Cauchy integral formula [2, Corollary 9.6]

$$f(y) = \int_{\partial\Omega} k(x-y)\boldsymbol{n}(x) f(x)\, d\mu(x), \quad y \in \Omega \tag{2.2}$$

is valid. Here $\boldsymbol{n}(x)$ is the outward unit normal at $x \in \partial\Omega$ and $\mu$ is the volume measure of the oriented manifold $\partial\Omega$. The proof of the Clifford Cauchy integral formula (2.2) is based on Stokes' theorem.

An element $x = (x_0, x_1, \dots, x_n)$ of $\mathbf{R}^{\mathrm{n}+1}$ will often be written as $x = x_0 e_0 + \boldsymbol{x}$ with $\boldsymbol{x} = \sum_{j=1}^n x_j e_j$. The expression $k(x-y)$ will also be written as $G_x(y)$—a more convenient notation when $y$ is replaced by an $n$-tuple of operators.

# 3 The monogenic spectrum of a complex vector

Here we revisit the calculations of [3] in order to set the stage. The Cauchy kernel for $\zeta \in \mathbb{C}^n$ is defined in [3] as the maximal holomorphic extension $\zeta \longmapsto G_x(\zeta)$ of formula (2.1) for $\zeta \in \mathbb{C}^n$:

$$G_x(\zeta) = \frac{1}{\sigma_n} \frac{\overline{x} + \zeta}{|x-\zeta|_{\mathbb{C}}^{n+1}}, \quad x \in \mathbf{R}^{\mathrm{n}+1}, \begin{cases} |x-\zeta|_{\mathbb{C}}^2 \notin (-\infty, 0], & n \text{ even} \\ |x-\zeta|_{\mathbb{C}}^2 \neq 0, & n \text{ odd.} \end{cases} \tag{3.1}$$

Here

$$|x-\zeta|_{\mathbb{C}}^2 = (x-\zeta)(\overline{x-\zeta}^{\mathbb{C}}) = x_0^2 + \sum_{j=1}^n (x_j - \zeta_j)^2$$

for $x = (x_0, x_1, \dots, x_n) \in \mathbf{R}^{\mathrm{n}+1}$ and the complex number $|x-\zeta|_{\mathbb{C}}$ is the square root of $|x-\zeta|_{\mathbb{C}}^2$ with positive real part, coinciding with the holomorphic extension of the modulus function $\xi \longmapsto |x-\xi|$, $\xi \in \mathbf{R}^{\mathrm{n}} \setminus \{\mathrm{x}\}$ in the case $x \in \mathbf{R}^{\mathrm{n}}$. For $\zeta \in \mathbb{C}^n$ fixed, the set $\gamma(\zeta)$ of singularities of the Cauchy kernel $x \longmapsto G_x(\zeta)$, $x \in \mathbf{R}^{\mathrm{n}+1}$, is called the *monogenic spectrum* of the complex vector $\zeta$.

There is a discontinuity in the function $(x, \zeta) \longmapsto |x-\zeta|_{\mathbb{C}}$ on the set

$$\{(x, \zeta) \in \mathbf{R}^{\mathrm{n}+1} \times \mathbb{C}^{\mathrm{n}} : |\mathrm{x}-\zeta|_{\mathbb{C}}^2 \in (-\infty, 0]\}.$$

The analogous reasoning for multiplication by $x \in \mathbf{R}^{\mathrm{n}+1}$ in the algebra $C\ell(\mathbb{C}^n)$ just gives us the Cauchy kernel (2.1), so that $\gamma(x) = \{x\}$, as expected.

Given $\zeta \in \mathbb{C}^n$, if $n$ is even and singularities of (3.1) occur at $x \in \mathbf{R}^{\mathrm{n}+1}$, then $|x-\zeta|_{\mathbb{C}}^2 \in (-\infty, 0]$, otherwise we can simply take the positive square root of $|x-\zeta|_{\mathbb{C}}^2$ in formula (3.1) to obtain a monogenic function of $x$. If $n$ is odd, then the denominator of (3.1) is a power of $|x-\zeta|_{\mathbb{C}}^2$, so $x \longmapsto G_x(\zeta)$ is monogenic provided $|x-\zeta|_{\mathbb{C}}^2$ is nonzero.

To determine the set of $x \in \mathbf{R}^{\mathrm{n}+1}$ where singularities occur, write $\zeta = \xi + i\eta$ for $\xi, \eta \in \mathbf{R}^{\mathrm{n}}$ and $x = x_0 e_0 + \boldsymbol{x}$ for $x_0 \in \mathbf{R}$ and $\boldsymbol{x} \in \mathbf{R}^{\mathrm{n}}$. Then

$$\begin{aligned}
|x-\zeta|_{\mathbb{C}}^2 &= x_0^2 + \sum_{j=1}^n (x_j - \zeta_j)^2 \\
&= x_0^2 + \sum_{j=1}^n (x_j - \xi_j - i\eta_j)^2 \\
&= x_0^2 + |\boldsymbol{x} - \xi|^2 - |\eta|^2 - 2i\langle \boldsymbol{x} - \xi, \eta\rangle. \qquad (3.2)
\end{aligned}$$

Thus, $|x-\zeta|_{\mathbb{C}}^2$ belongs to $(-\infty, 0]$ if and only if $x$ lies in the intersection hyperplane $\langle \boldsymbol{x} - \xi, \eta\rangle = 0$ passing through $\xi$ and with normal $\eta$, and the ball $x_0^2 + |\boldsymbol{x} - \xi|^2 \le |\eta|^2$ centred at $\xi$ with radius $|\eta|$. If $n$ is even, then

$$\gamma(\zeta) = \{x = x_0 e_0 + \boldsymbol{x} \in \mathbf{R}^{\mathrm{n}+1} : \langle \boldsymbol{x} - \xi, \eta\rangle = 0,\ \mathrm{x}_0^2 + |\boldsymbol{x} - \xi|^2 \le |\eta|^2\ \} \quad (3.3)$$

and if $n$ is odd, then

$$\gamma(\zeta) = \{x = x_0 e_0 + \boldsymbol{x} \in \mathbf{R}^{\mathrm{n}+1} : \langle \boldsymbol{x} - \xi, \eta\rangle = 0,\ \mathrm{x}_0^2 + |\boldsymbol{x} - \xi|^2 = |\eta|^2\ \}. \quad (3.4)$$

In particular, if $\mathrm{Im}(\zeta) = 0$, then $\gamma(\zeta) = \{\zeta\} \subset \mathbf{R}^{\mathrm{n}}$.

Off $\gamma(\zeta)$, a calculation shows that the function $x \longmapsto G_x(\zeta)$ is two-sided monogenic, so the Cauchy integral formula gives

$$\tilde{f}(\zeta) = \int_{\partial\Omega} G_x(\zeta)\boldsymbol{n}(x) f(x)\, d\mu(x) \qquad (3.5)$$

for a bounded open neighbourhood $\Omega$ of $\gamma(\zeta)$ with smooth oriented boundary $\partial\Omega$, outward unit normal $\boldsymbol{n}(x)$ at $x \in \partial\Omega$ and surface measure $\mu$. The function $f$ is assumed to be left monogenic in a neighbourhood of $\overline{\Omega}$, but $\zeta \longmapsto \tilde{f}(\zeta)$ is a holomorphic $C\ell(\mathbb{C}^n)$-valued function as the closed set $\gamma(\zeta)$ varies inside $\Omega$. Moreover, $\tilde{f}$ equals $f$ on $\Omega \cap \mathbf{R}^{\mathrm{n}}$ by the usual Cauchy integral formula of Clifford analysis, so if $f$ is, say, the monogenic extension of a polynomial $p : \mathbb{C}^n \longrightarrow \mathbb{C}$ restricted to $\mathbf{R}^{\mathrm{n}}$, then $\tilde{f}(\zeta) = p(\zeta)$, as expected. In this way, for each left monogenic function $f$ defined in a neighbourhood of $\gamma(\zeta)$, in a natural way we associate an holomorphic function $\tilde{f}$ defined in a neighbourhood of $\zeta$.

# 4 Complex Clifford analysis

The *complex generalised Cauchy-Riemann operator* is given by $D_{\mathbb{C}} = \sum_{j=0}^{n} e_j \frac{\partial}{\partial z_j}$. Let $U \subset \mathbb{C}^{n+1}$ be an open set. A function $f : U \longrightarrow C\ell(\mathbb{C}^n)$ is said to be *complex left monogenic* if $D_{\mathbb{C}} f = 0$ in $U$ and *right monogenic* if $f D_{\mathbb{C}} = 0$ in $U$.

The *complex Cauchy kernel* is given by

$$G_z(\zeta) = \frac{1}{\sigma_n} \frac{\overline{z-\zeta}^{\mathbb{C}}}{|z-\zeta|_{\mathbb{C}}^{n+1}}, \tag{4.1}$$

with $z, \zeta \in \mathbb{C}^{n+1}$ and $\sum_{j=0}^{n}(z_j - \zeta_j)^2 \neq 0$ if $n$ is odd and $\sum_{j=0}^{n}(z_j - \zeta_j)^2 \notin (-\infty, 0]$ if $n$ is even. If $n = 2k+1$ is odd, then $|z-\zeta|_{\mathbb{C}}^{n+1} = (\sum_{j=0}^{n}(z_j - \zeta_j)^2)^{k+1}$, while for $n$ even, we take $|z-\zeta|_{\mathbb{C}}$ to be the square root of $\sum_{j=0}^{n}(z_j - \zeta_j)^2$ with positive real part.

For each $\zeta \in \mathbb{C}^{n+1}$, let $N(\zeta)$ denote the set of complex vectors $z \in \mathbb{C}^{n+1}$ at which the complex Cauchy kernel $z \longmapsto G_z(\zeta)$ has a singularity.

Identifying $\mathbb{C}^n$ with $\{0\} \times \mathbb{C}^n \subset \mathbb{C}^{n+1}$, we obtain $\gamma(\zeta) = N(\zeta) \cap \mathbf{R}^{\mathrm{n+1}}$ for each $\zeta \in \mathbb{C}^n$. For each vector $\zeta \in \mathbb{C}^{n+1}$, we set $\gamma_{\mathbb{C}}(\zeta) = N(\zeta) \cap \mathbf{R}^{\mathrm{n+1}}$. The subscript is used to distinguish the set $\gamma_{\mathbb{C}}(\zeta)$ from the monogenic spectrum $\gamma(\zeta')$ of a vector $\zeta' \in \mathbb{C}^n$.

If $\zeta \in \mathbb{C}^{n+1}$ and $\zeta = \xi + i\eta$ for $\xi, \eta \in \mathbf{R}^{\mathrm{n+1}}$ and $n$ is even, then

$$N(\zeta) = \left\{ z = z_0 e_0 + \boldsymbol{z} \in \mathbb{C}^{n+1} : \sum_{j=0}^{n}(z_j - \zeta_j)^2 \in (-\infty, 0] \right\}. \tag{4.2}$$

$$\begin{aligned} \gamma_{\mathbb{C}}(\zeta) &= N(\zeta) \cap \mathbf{R}^{\mathrm{n+1}} \\ &= \{x \in \mathbf{R}^{\mathrm{n+1}} : \langle \mathrm{x} - \xi, \eta \rangle = 0,\ |\mathrm{x} - \xi|^2 \leq |\eta|^2 \}. \end{aligned} \tag{4.3}$$

and if $n$ is odd, then

$$N(\zeta) = \left\{ z = z_0 e_0 + \boldsymbol{z} \in \mathbb{C}^{n+1} : \sum_{j=0}^{n}(z_j - \zeta_j)^2 = 0 \right\}. \tag{4.4}$$

$$\begin{aligned} \gamma_{\mathbb{C}}(\zeta) &= N(\zeta) \cap \mathbf{R}^{\mathrm{n+1}} \\ &= \{x \in \mathbf{R}^{\mathrm{n+1}} : \langle \mathrm{x} - \xi, \eta \rangle = 0,\ |\mathrm{x} - \xi|^2 = |\eta|^2 \}. \end{aligned} \tag{4.5}$$

Let $\Omega$ be a bounded open subset of $\mathbf{R}^{\mathrm{n+1}}$ with smooth oriented boundary $\partial\Omega$ and suppose that $\Omega$ intersects $\mathbf{R}^{\mathrm{n}}$ and $f$ is left monogenic in a neighbourhood of $\overline{\Omega}$. Then the function $\tilde{f}$ defined by the Cauchy integral formula (3.5) is holomorphic in the set $\{\zeta \in \mathbb{C}^n : \gamma(\zeta) \subset \Omega\}$ and equals $f$ on $\Omega \cap \mathbf{R}^{\mathrm{n}}$.

In order to determine the range of the mapping $f \longmapsto \tilde{f}$, as in [16, 17], we note that $\tilde{f}$ has a complex left monogenic extension $\tilde{f}^{\mathbb{C}}$ in $\mathbb{C}^{n+1}$ defined in the component $\kappa(\Omega) \subset \mathbb{C}^{n+1}$ of $\mathbb{C}^{n+1} \setminus N(\partial\Omega)$ containing $\Omega$ by virtue of the formula

$$\tilde{f}^{\mathbb{C}}(\zeta) = \int_{\partial\Omega} G_x(\zeta)\boldsymbol{n}(x)f(x)\,d\mu(x), \qquad \zeta \in \kappa(\Omega). \tag{4.6}$$

The domain $\kappa(\Omega)$ is an example of a *cell of harmonicity* discussed in this context in [16, 17].

Let $U$ be a nonempty open subset of $\mathbf{R}^{\mathrm{n}+1}$. Because a vector $\zeta \in \mathbb{C}^{n+1}$ belongs to $N(\partial U)$ if and only if $\gamma_{\mathbb{C}}(\zeta) = N(\zeta) \cap \mathbf{R}^{\mathrm{n}+1}$ intersects $\partial U$, the equality

$$\mathbb{C}^{n+1} \setminus N(\partial U) = \{\zeta \in \mathbb{C}^{n+1} : \gamma_{\mathbb{C}}(\zeta) \subset \mathbf{R}^{\mathrm{n}+1} \setminus \partial \mathrm{U}\}$$

holds. The disjoint open sets $U$ and $\mathbf{R}^{\mathrm{n}+1}\setminus\overline{\mathrm{U}}$ cannot disconnect the set $\gamma_{\mathbb{C}}(\zeta)$, so either $\gamma_{\mathbb{C}}(\zeta) \subset U$ or $\gamma_{\mathbb{C}}(\zeta) \subset \mathbf{R}^{\mathrm{n}+1} \setminus \overline{\mathrm{U}}$. For any nonempty connected open subset $U$ of $\mathbf{R}^{\mathrm{n}+1}$, the component of the open set $\{\zeta \in \mathbb{C}^{n+1} : \gamma_{\mathbb{C}}(\zeta) \subset U\}$ containing $U$ is denoted by $\kappa(U)$. The following observation is an immediate consequence of the Cauchy integral formula (4.6).

**Proposition 4.1.** *Let $U$ be a nonempty connected open subset of $\mathbf{R}^{\mathrm{n}+1}$. Then $\kappa(U) \cap \mathbf{R}^{\mathrm{n}+1} = \mathrm{U}$ and every left monogenic function $f$ defined in $U$ has a unique complex left monogenic extension $\tilde{f}^{\mathbb{C}}$ to $\kappa(U)$. The mapping $f \longmapsto \tilde{f}^{\mathbb{C}}$ is a bijection. The inverse map is the restriction map of complex left monogenic functions defined in $\kappa(U)$ to $\kappa(U) \cap \mathbf{R}^{\mathrm{n}+1} = \mathrm{U}$.*

The question remains as to when a holomorphic function defined on $\kappa(U) \cap (\{0\} \times \mathbb{C}^n)$ has a complex left monogenic extension to $\kappa(U)$. In the course of the proof of Theorem 5.1 below, we shall show that a holomorphic function defined on

$$\kappa(S^\circ_\omega(\mathbf{R}^{\mathrm{n}+1})) \cap (\{0\} \times \mathbb{C}^{\mathrm{n}}) = \{0\} \times \mathrm{S}^\circ_\omega(\mathbb{C}^{\mathrm{n}})$$

has a complex left monogenic extension to $\kappa(S^\circ_\omega(\mathbf{R}^{\mathrm{n}+1}))$. However, the simplest example is when $U$ is the open unit ball centred at zero in $\mathbf{R}^{\mathrm{n}+1}$, where the set $\kappa(U)$ is computed below.

**Example 4.1.** Let $n = 1, 2, \ldots$ and $B_1(0) = \{x \in \mathbf{R}^{\mathrm{n}+1} : |\mathrm{x}| < 1\}$. Then $\kappa(B_1(0)) = L_{n+1}$ where

$$L_{n+1} = \{\zeta \in \mathbb{C}^{n+1} : |\zeta|^2 + \sqrt{|\zeta|^4 - ||\zeta|^2_{\mathbb{C}}|^2} < 1 \} \tag{4.7}$$

is the *Lie ball* in $\mathbb{C}^{n+1}$. Let $\zeta \in \mathbb{C}^{n+1}$ and suppose that $\zeta = \xi + i\eta$ with $\xi, \eta \in \mathbf{R}^{\mathrm{n+1}}$. If $\eta = 0$, then $\gamma_{\mathbb{C}}(\zeta) = \{\xi\}$ so that $B_1(0) \subset \kappa(B_1(0))$. Moreover,

$$L_{n+1} \cap (\{0\} \times \mathbb{C}^n) = \{0\} \times L_n, \quad n = 1, 2, \dots .$$

To establish the identity $\kappa(B_1(0)) = L_{n+1}$, suppose that $\eta \neq 0$. According to equations (4.3) and (4.5), the set $\gamma_{\mathbb{C}}(\zeta)$ is an $(n-1)$-dimensional ball or sphere with radius $|\eta|$ in $\mathbf{R}^{\mathrm{n+1}}$, lying in the hyperplane with normal $\eta$ and passing through $\xi \in \mathbf{R}^{\mathrm{n+1}}$. Let $0 \leq \angle(\xi, \eta) \leq \pi$ be the angle between $\xi$ and $\eta$ in $\mathbf{R}^{\mathrm{n+1}}$, that is $\langle \xi, \eta \rangle = |\xi|.|\eta| \cos(\angle(\xi, \eta))$. The projection of $\xi$ onto $\{\eta\}^{\perp}$ has length $|\xi| \sin(\angle(\xi, \eta))$ and the projection of $\xi$ onto $\eta$ has length $|\xi| \cos(\angle(\xi, \eta))$. The projection of $\gamma_{\mathbb{C}}(\zeta)$ onto $\{\eta\}^{\perp}$ is a ball or sphere whose maximum distance from the origin is $|\xi| \sin(\angle(\xi, \eta)) + |\eta|$ in the direction of the projection of $\xi$ onto $\{\eta\}^{\perp}$. Because $\{\eta\}^{\perp}$ is distant $|\xi| |\cos(\angle(\xi, \eta))|$ from the hyperplane in $\mathbf{R}^{\mathrm{n+1}}$ in which $\gamma_{\mathbb{C}}(\zeta)$ lies, the maximum distance from the origin of points belonging to $\gamma_{\mathbb{C}}(\zeta)$ is $\sqrt{|\xi|^2 \cos^2(\angle(\xi, \eta)) + (|\xi| \sin(\angle(\xi, \eta)) + |\eta|)^2}$, so

$$\begin{aligned}
&\{\zeta \in \mathbb{C}^{n+1} : \gamma_{\mathbb{C}}(\zeta) \subset B_1(0)\} \\
&\quad = \{\zeta \in \mathbb{C}^{n+1} : \zeta = \xi + i\eta,\ \eta \neq 0,\ |\xi|^2 + |\eta|^2 + 2|\xi||\eta| \sin(\angle(\xi, \eta)) < 1\} \cup B_1(0) \\
&\quad = \{\zeta \in \mathbb{C}^{n+1} : \zeta = \xi + i\eta,\ |\xi|^2 + |\eta|^2 + 2(|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2)^{\frac{1}{2}} < 1\} \\
&\quad = \{\zeta \in \mathbb{C}^{n+1} : |\zeta|^2 + \sqrt{|\zeta|^4 - ||\zeta|_{\mathbb{C}}^2|^2} < 1\ \}
\end{aligned}$$

is a connected open set and the equality (4.7) follows. Consequently, any left monogenic function $f : B_1(0) \longrightarrow C\ell(\mathbb{C}^n)$ has a unique complex left monogenic extension $\tilde{f}^{\mathbb{C}} : L_{n+1} \longrightarrow C\ell(\mathbb{C}^n)$ to the Lie ball $L_{n+1}$ in $\mathbb{C}^{n+1}$ [16, Proposition 7].

The complex left monogenic function $\tilde{f}^{\mathbb{C}} : \kappa(\Omega) \to C\ell(\mathbb{C}^n)$ defined by formula (4.6) has another representation which is best described by first interpreting formula (4.6) in terms of differential forms.

The boundary $\partial\Omega$ of $\Omega$ is assumed to be an orientable smooth $n$-manifold in $\mathbf{R}^{\mathrm{n+1}}$. The $\mathbf{R}^{\mathrm{n+1}}$-valued $n$-form $\boldsymbol{\omega}(dx)$ is defined by

$$\boldsymbol{\omega}(dx) = \sum_{j=0}^{n} (-1)^j e_j dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n. \tag{4.8}$$

Here the symbol $\widehat{dx_j}$ means that the factor $dx_j$ is simply omitted from the wedge product. Given an $n$-dimensional orientable submanifold $M$ of $\mathbf{R}^{\mathrm{n+1}}$, the pullback of $\boldsymbol{\omega}(dx)$ onto $M$ by the embedding of $M$ into $\mathbf{R}^{\mathrm{n+1}}$ is denoted by the same symbol. In terms of this differential form, equation (4.6) may be rewritten as

$$\tilde{f}^{\mathbb{C}}(\zeta) = \int_{\partial\Omega} G_x(\zeta)\boldsymbol{\omega}(dx)f(x), \qquad \zeta \in \kappa(\Omega). \tag{4.9}$$

Now suppose that $M$ is a real $n$-dimensional orientable submanifold of $\mathbb{C}^{n+1}$. The $\mathbb{C}^{n+1}$-valued $n$-form $\boldsymbol{\omega}(d\zeta)$ is defined by

$$\boldsymbol{\omega}(d\zeta) = \sum_{j=0}^{n}(-1)^j e_j d\zeta_0 \wedge \cdots \wedge \widehat{d\zeta_j} \wedge \cdots \wedge d\zeta_n. \tag{4.10}$$

The pullback of $\boldsymbol{\omega}(d\zeta)$ onto $M$ by the embedding of $M$ into $\mathbb{C}^{n+1}$ is denoted by the same symbol. Let $\zeta \in \kappa(\Omega)$. If $M \subset \kappa(\Omega)$ and $\partial\Omega$ are homologous in $\kappa(\Omega) \setminus N(\zeta)$, then by Stokes' Theorem we obtain

$$\tilde{f}^{\mathbb{C}}(\zeta) = \int_{M} G_z(\zeta)\boldsymbol{\omega}(dz)\tilde{f}^{\mathbb{C}}(z) \tag{4.11}$$

from equation (4.9), because $D_{\mathbb{C}}\tilde{f}^{\mathbb{C}} = 0$, so that $G_z(\zeta)\boldsymbol{\omega}(dz)\tilde{f}^{\mathbb{C}}(z)$ is a closed $C\ell(\mathbb{C}^n)$-valued differential form in $\kappa(\Omega) \setminus N(\zeta)$.

The sum $A + B$ of two subsets $A, B$ of a vector space is the set

$$A + B = \{a + b : a \in A,\ b \in B\ \}.$$

For each $r > 0$, let

$$\mathbb{S}^n(r) = \{x \in \mathbf{R}^{\mathrm{n+1}} : |\mathrm{x}| = \mathrm{r}\ \}$$

be the $n$-dimensional hypersphere of radius $r$ in $\mathbf{R}^{\mathrm{n+1}}$. The hypersphere $\mathbb{S}^n(r)$ is identified with a subset of $\mathbb{C}^{n+1}$ via the embedding of $\mathbf{R}^{\mathrm{n+1}}$ in $\mathbb{C}^{n+1}$. It has the orientation induced from the standard orientation of $\mathbf{R}^{\mathrm{n+1}}$ and the outward unit normal. An application of Stokes' theorem as $\varepsilon \to 0^+$ gives the following representation.

**Proposition 4.2.** *Let $\zeta \in \mathbb{C}^{n+1}$. If $f : U \to C\ell(\mathbb{C}^n)$ is a complex left monogenic function defined in a neighbourhood $U$ of $\zeta$ in $\mathbb{C}^{n+1}$, then there exists $\varepsilon > 0$ such that $\zeta + \mathbb{S}^n(\varepsilon) \subset U \setminus N(\zeta)$ and*

$$f(\zeta) = \int_{\zeta+\mathbb{S}^n(\varepsilon)} G_z(\zeta)\boldsymbol{\omega}(dz)f(z). \tag{4.12}$$

If $\Omega$ is any nonempty open connected subset of $\mathbf{R}^{n+1}$ and $f:\kappa(\Omega)\longrightarrow C\ell(\mathbb{C}^n)$ is a complex left monogenic function, then for each $\zeta\in\kappa(\Omega)\setminus\Omega$, following [16] we determine a simple real $n$-cycle $M(\zeta)\subset\kappa(\Omega)\setminus N(\zeta)$ close to $\gamma_{\mathbb{C}}(\zeta)\subset\mathbf{R}^{n+1}$ in an $n$-dimensional complex affine subspace of $\mathbb{C}^{n+1}$ and homologous to the $n$-sphere $\zeta+\mathbb{S}^n(\varepsilon)$ in the image of $\kappa(\Omega)\setminus N(\zeta)$ in complex projective space $\mathbb{CP}^n$. An application of Stokes' theorem ensures that the representation

$$f(\zeta)=\int_{M(\zeta)}G_z(\zeta)\boldsymbol{\omega}(dz)f(z) \tag{4.13}$$

is valid. The point of difference with the representation (4.12) is that $\zeta+\mathbb{S}^n(\varepsilon)$ lies in the $(n+1)$-dimensional affine subspace $\zeta+\mathbf{R}^{n+1}$ of $\mathbb{C}^{n+1}$. We look at the cases of $n$ odd and even separately.

## 4.1 $n$ odd

Let $\zeta\in\mathbb{C}^{n+1}\setminus\mathbf{R}^{n+1}$. By equation (4.5), the monogenic spectrum $\gamma_{\mathbb{C}}(\zeta)=N(\zeta)\cap\mathbf{R}^{n+1}$ of $\zeta$ is an $(n-1)$-dimensional sphere in $\mathbf{R}^{n+1}$ in the hyperplane orthogonal to $\eta=\operatorname{Im}\zeta\neq 0$ and passing through $\xi=\operatorname{Re}\zeta$. For each $x\in\gamma_{\mathbb{C}}(\zeta)$, let $\mathbb{C}_x(\zeta)$ denote the complex line

$$\{z\in\mathbb{C}^{n+1}:z=\xi+\lambda(x-\xi),\ \lambda\in\mathbb{C}\ \} \tag{4.14}$$

passing through $x$ and its antipodal point in $\gamma_{\mathbb{C}}(\zeta)$. For each $0<\varepsilon<|\eta|$, let

$$\Sigma_\varepsilon^{(n+1)}(\zeta) = \bigcup_{x\in\gamma_{\mathbb{C}}(\zeta)}\{z\in\mathbb{C}_x(\zeta):|z-x|<\varepsilon\ \} \tag{4.15}$$

$$\partial\Sigma_\varepsilon^{(n+1)}(\zeta) = \bigcup_{x\in\gamma_{\mathbb{C}}(\zeta)}\{z\in\mathbb{C}_x(\zeta):|z-x|=\varepsilon\ \}. \tag{4.16}$$

Then $\partial\Sigma_\varepsilon^{(n+1)}(\zeta)$ is a real $n$-dimensional submanifold of $\mathbb{C}^{n+1}$—an $S^1$-*fibration* of $\gamma_{\mathbb{C}}(\zeta)\subset\mathbf{R}^{n+1}$ mentioned in [16, p. 416]. The set $\partial\Sigma_\varepsilon^{(n+1)}(\zeta)$ is the boundary of the open set $\Sigma_\varepsilon^{(n+1)}(\zeta)$ containing $\gamma_{\mathbb{C}}(\zeta)$ and contained in an $\varepsilon'$-neighbourhood of $\gamma_{\mathbb{C}}(\zeta)$ in $\mathbb{C}^{n+1}$ for all $\varepsilon'>\varepsilon$. It is oriented from the standard orientations of $\mathbf{R}^{n+1}$ and the unit circles in $\mathbb{C}_x(\zeta)$ for each $x\in\gamma_{\mathbb{C}}(\zeta)$, so that $\partial\Sigma_\varepsilon^{(n+1)}(\zeta)$ has the orientation induced by the $(n+1)$-form $d\zeta_0\wedge\cdots\wedge d\zeta_n$.

**Proposition 4.3.** *Suppose that $n\in\mathbf{N}$ is odd and $\Omega$ is an open subset of $\mathbf{R}^{n+1}$. Let $\zeta\in\kappa(\Omega)\setminus\Omega$ and suppose that the convex hull of $\gamma_{\mathbb{C}}(\zeta)$ in $\mathbf{R}^{n+1}$ is* *contained in $\Omega$. If $f : \kappa(\Omega) \to C\ell(\mathbb{C}^n)$ is a complex left monogenic function, then there exists $\varepsilon > 0$ such that $\partial\Sigma_\varepsilon^{(n+1)}(\zeta) \subset \kappa(\Omega) \setminus N(\zeta)$ and*

$$f(\zeta) = \int_{\partial\Sigma_\varepsilon^{(n+1)}(\zeta)} G_z(\zeta)\boldsymbol{\omega}(dz)f(z). \tag{4.17}$$

*Proof.* First, we show that the half-open line segment $[w, \zeta) = \{\lambda w+(1-\lambda)\zeta : 0 < \lambda \leq 1\ \}$ joining $w \in \partial\Sigma_\varepsilon^{(n+1)}(\zeta)$ and $\zeta$ lies in the complement of the null cone $N(\zeta)$ of $\zeta$. Let $\zeta = \xi + i\eta$ with $\xi, \eta \in \mathbf{R}^{n+1}$. Representing $w$ as $w = \xi + u(1 + z/|u|)$ for $u \in \{\eta\}^\perp$ in $\mathbf{R}^{n+1}$ with $|u| = |\eta|$ and $z \in \mathbb{C}$ with $|z| = \varepsilon$, we have

$$\begin{aligned}
|\lambda w + (1-\lambda)\zeta - \zeta|^2_{\mathbb{C}} &= \lambda^2|w - \zeta|^2_{\mathbb{C}} \\
&= \lambda^2| - i\eta + u(1 + z/|u|)|^2_{\mathbb{C}} \\
&= \lambda^2(-|\eta|^2 + |u|^2(1 + z/|u|)^2) \\
&= \lambda^2(2|\eta| + z)z, \quad 0 < \lambda \leq 1.
\end{aligned}$$

Because $|2|\eta| + z| \geq 2|\eta| - \varepsilon > |\eta| > 0$, it follows that $[w, \zeta) \subset \mathbb{C}^{n+1} \setminus N(\zeta)$.

Next we see that $[w, \zeta) \subset \kappa(\Omega)$. Let $\hat{u} = u/|u|$. For each $0 < \lambda \leq 1$, the set

$$\gamma_{\mathbb{C}}(\lambda w + (1-\lambda)\zeta) = \gamma_{\mathbb{C}}(\zeta + \lambda(-i\eta + u(1 + z/|u|)))$$

is the $(n-1)$-dimensional sphere centred at $\xi + \lambda(u + \mathrm{Re}(z)\hat{u})$, with radius $|(1-\lambda)\eta + \lambda\,\mathrm{Im}(z)\hat{u}|$ and contained in the hyperplane

$$\{x \in \mathbf{R}^{n+1} : \langle \mathrm{x} - (\xi + \lambda(\mathrm{u} + \mathrm{Re(z)\hat{u}})), (1-\lambda)\eta + \lambda\,\mathrm{Im(z)\hat{u}}\rangle = 0\ \}.$$

Some open neighbourhood of the convex hull of $\gamma_{\mathbb{C}}(\zeta)$ is contained in $\Omega$, so for $\varepsilon$ sufficiently small, $\gamma_{\mathbb{C}}(\lambda w+(1-\lambda)\zeta)$ is contained in $\Omega$ for each $0 \leq \lambda \leq 1$, proving that that $[w, \zeta] \subset \kappa(\Omega)$ for every $w \in \partial\Sigma_\varepsilon^{(n+1)}(\zeta)$. Consequently, we can translate $\partial\Sigma_\varepsilon^{(n+1)}(\zeta)$ along the line segments $[w, \zeta) \subset \kappa(\Omega) \setminus N(\zeta)$ for $w \in \partial\Sigma_\varepsilon^{(n+1)}(\zeta)$.

Now suppose that $\varepsilon' > 0$ is so small that $\zeta+B_{\varepsilon'}(0) \subset \kappa(\Omega)$, where $B_{\varepsilon'}(0)$ is the closed ball of radius $\varepsilon' > 0$ centred at zero in $\mathbb{C}^{n+1}$. Choose $\delta > 0$ so that $\delta(2|\eta|+\varepsilon) < \varepsilon'$. For each $w \in \partial\Sigma_\varepsilon^{(n+1)}(\zeta)$ there exists $u \in \mathbf{R}^{n+1}$ as above such that $w - \zeta = -i\eta + u(1 + z/|u|)$. Then $\delta|w - \zeta| \leq \delta(|\eta| + |u(1 + z/|u|)|) < \varepsilon'$ and $\partial\Sigma_\varepsilon^{(n+1)}(\zeta)$ is homologous to the cycle

$$\{\zeta + \delta(w - \zeta) : w \in \partial\Sigma_\varepsilon^{(n+1)}(\zeta)\ \} \subset B_{\varepsilon'}(0)$$

in $\kappa(\Omega) \setminus N(\zeta)$.

For each $u \in \{\eta\}^\perp$ with $|u| = |\eta|$, let $\hat{u} = u/|u|$ and $\hat{\eta} = \eta/|\eta|$ be unit vectors in the $u$ and $\eta$ directions and let $V$ denote the real three dimensional subspace spanned by the vectors $\hat{u}, i\hat{u}$ and $i\hat{\eta}$. According to formula (4.4), we have

$$\begin{aligned} V \cap N(0) &= \{\zeta \in V : |\zeta|^2_{\mathbb{C}} = 0\} \\ &= \{z\hat{u} + ai\hat{\eta} : z \in \mathbb{C},\ a \in \mathbf{R},\ \mathrm{z}^2 - \mathrm{a}^2 = 0\} \\ &= \{a(\hat{u} \pm i\hat{\eta}) :\ a \in \mathbf{R}\} \end{aligned}$$

The cycle defined by $-i\eta \pm u + z\hat{u}$, $z \in \mathbb{C}$ with $|z| = \varepsilon$ and the positive orientation can be deformed in $V \setminus N(0)$ into the circle of radius $|\eta| + \varepsilon$ centred at $-i\eta$ and then into $z\hat{u}$, $z \in \mathbb{C}$ with $|z| = \varepsilon'$, so that $\partial\Sigma^{(n+1)}_\varepsilon(\zeta)$ is homologous to the cycle

$$\zeta + \{z\hat{u} : u \in \mathbf{R}^{\mathrm{n}+1},\ \mathrm{u} \in \{\eta\}^\perp \setminus \{0\},\ \mathrm{z} \in \mathbb{C},\ |\mathrm{z}| = \varepsilon'\} \tag{4.18}$$

in $\kappa(\Omega) \setminus N(\zeta)$. Moreover, the $n$-form $\dfrac{\overline{z}^{\mathbb{C}}}{|z|^{n+1}_{\mathbb{C}}}\boldsymbol{\omega}(dz)$ is homogeneous on $\mathbb{C}^{n+1}$ and so defines a closed $n$-form on complex projective space $\mathbb{CP}^n$. Because the images of the cycle (4.18) and $\zeta + \mathbb{S}^n(\varepsilon)$ in $\mathbb{CP}^n$ are homologous and $\partial\Sigma^{(n+1)}_\varepsilon(\zeta)$ and (4.18) are homologous, the equality (4.17) is now a consequence of Stokes' theorem and Proposition 4.2. $\square$

## 4.2 $n$ even

Let $\zeta \in \mathbb{C}^{n+1} \setminus \mathbf{R}^{\mathrm{n}+1}$. By equation (4.3), the monogenic spectrum $\gamma_{\mathbb{C}}(\zeta) = N(\zeta) \cap \mathbf{R}^{\mathrm{n}+1}$ of $\zeta$ is an $n$-dimensional ball in $\mathbf{R}^{\mathrm{n}+1}$ in the hyperplane orthogonal to $\eta = \operatorname{Im}\zeta \neq 0$ and passing through $\xi = \operatorname{Re}\zeta$. In particular, $\gamma_{\mathbb{C}}(\zeta)$ is a convex subset of $\mathbf{R}^{\mathrm{n}+1}$.

For each $x \in \mathbf{R}^{\mathrm{n}+1} \setminus \{\xi\}$, let $\mathbb{C}_x(\zeta)$ denote the complex line (4.14). In polar coordinates, we have

$$\mathbb{C}_x(\zeta) = \left\{ z \in \mathbb{C}^{n+1} : z = \xi + re^{i\theta}\frac{x-\xi}{|x-\xi|},\ r \geq 0,\ 0 \leq \theta < 2\pi \right\}.$$

For each $\varepsilon > 0$, let

$$\Sigma_\varepsilon^{(n+1)}(\zeta) = \bigcup_{x\in\partial\gamma_{\mathbb{C}}(\zeta)} \left\{ z \in \mathbb{C}_x(\zeta) : \frac{1}{r^2} > \frac{\sin^2\theta}{\varepsilon^2} + \frac{\cos^2\theta}{(1+\varepsilon^2)|\eta|^2},\ 0 \le \theta < 2\pi \right\} \tag{4.19}$$

$$\partial\Sigma_\varepsilon^{(n+1)}(\zeta) = \bigcup_{x\in\partial\gamma_{\mathbb{C}}(\zeta)} \left\{ z \in \mathbb{C}_x(\zeta) : \frac{1}{r^2} = \frac{\sin^2\theta}{\varepsilon^2} + \frac{\cos^2\theta}{(1+\varepsilon^2)|\eta|^2},\ 0 \le \theta < 2\pi \right\} \tag{4.20}$$

The ellipsoid $\partial\Sigma_\varepsilon^{(n+1)}(\zeta)$ is an $n$-cycle in $\mathbb{C}^{n+1}$. The set $\partial\Sigma_\varepsilon^{(n+1)}(\zeta)$ is the boundary of the open set $\Sigma_\varepsilon^{(n+1)}(\zeta)$ containing $\gamma_{\mathbb{C}}(\zeta)$ and contained in an $\varepsilon'$-neighbourhood of $\gamma_{\mathbb{C}}(\zeta)$ in $\mathbb{C}^{n+1}$ for $\varepsilon' > \varepsilon$; $\partial\Sigma_\varepsilon^{(n+1)}(\zeta)$ has the orientation induced by the $(n+1)$-form $d\zeta_0 \wedge \cdots \wedge d\zeta_n$. If $n$ is odd and the convex hull of $\gamma_{\mathbb{C}}(\zeta)$ in $\mathbf{R}^{n+1}$ is contained in $\Omega$, then the cycles $\partial\Sigma_\varepsilon^{(n+1)}(\zeta)$ defined by (4.16) and (4.20) are homologous in $\kappa_{\mathbb{C}}(\Omega) \setminus N(\zeta)$. A proof similar to the case for $n$ odd gives the following result.

**Proposition 4.4.** *Suppose that $n \in \mathbf{N}$ is even and $\Omega$ is a nonempty connected open subset of $\mathbf{R}^{n+1}$. Let $\zeta \in \kappa(\Omega) \setminus \Omega$. If $f : \kappa(\Omega) \to C\ell(\mathbb{C}^n)$ is a complex left monogenic function, then there exists $\varepsilon > 0$ such that $\partial\Sigma_\varepsilon^{(n+1)}(\zeta) \subset \kappa(\Omega) \setminus N(\zeta)$ and*

$$f(\zeta) = \int_{\partial\Sigma_\varepsilon^{(n+1)}(\zeta)} G_z(\zeta)\boldsymbol{\omega}(dz)f(z). \tag{4.21}$$

In the case of $n$ odd or even, for $\zeta = \xi + i\eta \in \kappa(\Omega)\setminus\Omega$, the set $\Sigma_\varepsilon^{(n+1)}(\zeta)$ lies in a small neighbourhood of $\gamma_{\mathbb{C}}(\zeta)$ contained in the intersection of $\kappa(\Omega)\setminus N(\zeta)$ with the $n$-dimensional complex hyperplane $\xi + \mathbb{C}\{\eta\}^\perp$ in $\mathbb{C}^{n+1}$. Because the representations (4.17) and (4.21) depend only on the values of $f$ in an $n$-dimensional complex hyperplane in $\mathbb{C}^{n+1}$, they may be modified to obtain the complex monogenic extension to $\mathbb{C}^{n+1}$ of a holomorphic function defined on an open subset of $\mathbb{C}^n$. We show how this is done for the case of sectors in the next section.

# 5 Holomorphic and Monogenic Functions on Sectors

Let

$$\begin{aligned} S_\nu(\mathbf{R}^{\mathrm{n+1}}) &= \{x \in \mathbf{R}^{\mathrm{n+1}} : \mathrm{x} = \mathrm{x}_0\mathrm{e}_0 + \boldsymbol{x},\ |\mathrm{x}_0| \le \tan\nu|\boldsymbol{x}|\ \}, \\ S_\nu^\circ(\mathbf{R}^{\mathrm{n+1}}) &= \{x \in \mathbf{R}^{\mathrm{n+1}} : \mathrm{x} = \mathrm{x}_0\mathrm{e}_0 + \boldsymbol{x},\ |\mathrm{x}_0| < \tan\nu|\boldsymbol{x}|\ \}. \end{aligned}$$

It is clear that if $\zeta = \xi + i\eta$ lies in a sector in $\mathbb{C}^n$, say, $|\eta| \le |\xi|\tan\nu$, then the monogenic spectrum $\gamma(\zeta)$ lies in a corresponding sector in $\mathbf{R}^{\mathrm{n+1}}$. More precisely, we have

**Proposition 5.1** ( [4, Proposition 6.10])**.** *Let $\zeta \in \mathbb{C}^n \setminus \{0\}$ and $0 < \omega < \pi/2$. Then $\gamma(\zeta) \subset S_\omega(\mathbf{R}^{\mathrm{n+1}})$ if and only if*

$$|\zeta|_{\mathbb{C}}^2 \ne (-\infty, 0] \quad \text{and} \quad |\operatorname{Im}(\zeta)| \le \operatorname{Re}(|\zeta|_{\mathbb{C}})\tan\omega. \tag{5.1}$$

Let $S_\omega(\mathbb{C}^n)$ denote the set of complex vectors $\zeta \in \mathbb{C}^n$ satisfying the inequality (5.1), that is, the complex sector

$$S_\omega(\mathbb{C}^n) = \{\zeta \in \mathbb{C}^n : \gamma(\zeta) \subset S_\omega(\mathbf{R}^{\mathrm{n+1}})\} \tag{5.2}$$

in $\mathbb{C}^n$. The interior of $S_\omega(\mathbb{C}^n)$ is written as $S_\omega^\circ(\mathbb{C}^n)$. For $n = 1$, we have

$$\begin{aligned} S_\omega(\mathbb{C}) &= \{z \in \mathbb{C} : |\operatorname{Im} z| \le \tan\omega|\operatorname{Re} z|\ \}, \\ S_\omega^\circ(\mathbb{C}) &= \{z \in \mathbb{C} : |\operatorname{Im} z| < \tan\omega|\operatorname{Re} z|\ \}. \end{aligned}$$

**Theorem 5.1.** *Let $n$ be a nonnegative integer and $0 < \omega < \pi/2$. If $\tilde{f}$ is a $C\ell(\mathbb{C}^n)$-valued holomorphic function defined on $S_\omega^\circ(\mathbb{C}^n)$, then there exists a unique left monogenic function $f$ defined on $S_\omega^\circ(\mathbf{R}^{\mathrm{n+1}})$ such that $\tilde{f}(x) = f(x_1e_1 + \cdots + x_ne_n)$ for all $x \in \mathbf{R}^{\mathrm{n}}$, $x \ne 0$. The linear map $\tilde{f} \longmapsto f$ is continuous for the compact-open topology.*

*Sketch of the Proof.* We describe the case that $n$ is an odd integer. The case of $n$ even is similar but a little more complicated. A calculation like that in (4.3) shows that for each $\zeta \in \mathbb{C}^{n+1}$, $z \in \mathbb{C}^n$ and $\theta \in \mathbf{R}$, the intersection $N(\zeta) \cap (z + e^{i\theta}\mathbf{R}^{\mathrm{n}})$ is either empty, a point or an $(n-2)$ sphere contained in a real $(n-1)$ dimensional hyperplane $H_{z,\zeta,\theta}$ in $\mathbb{C}^n$. Suppose that

$$\inf_\theta \operatorname{diam}(N(\zeta) \cap (z + e^{i\theta}\mathbf{R}^{\mathrm{n}})) > 2\varepsilon > 0,$$

so that the radius of the $(n-2)$ sphere $N(\zeta)\cap(z+e^{i\theta}\mathbf{R}^{\mathrm{n}})$ is bounded below by $\varepsilon$ as $\theta\in\mathbf{R}$ varies. For each $u\in N(\zeta)\cap(z+e^{i\theta}\mathbf{R}^{\mathrm{n}})$ with $u=z+e^{i\theta}x$, let $C_u(z,\zeta,\theta)$ denote the 2 dimensional plane in $\mathbb{C}^n$ passing through $u$ and the centre of $N(\zeta)\cap(z+e^{i\theta}\mathbf{R}^{\mathrm{n}})$ and parallel to $H^{\perp}_{z,\zeta,\theta}$ in $z+e^{i\theta}\mathbf{R}^{\mathrm{n}}$. Then

$$\Gamma^{(n+1)}_{\varepsilon}(\zeta,z) = \bigcup_{\theta\in(-\pi,\pi]}\ \bigcup_{u\in N(\zeta)\cap(z+e^{i\theta}.\mathbf{R}^{\mathrm{n}})} \{w\in C_u(z,\zeta,\theta):|w-u|<\varepsilon\}, \tag{5.3}$$

$$\partial\Gamma^{(n+1)}_{\varepsilon}(\zeta,z) = \bigcup_{\theta\in(-\pi,\pi]}\ \bigcup_{u\in N(\zeta)\cap(z+e^{i\theta}.\mathbf{R}^{\mathrm{n}})} \{w\in C_u(z,\zeta,\theta):|w-u|=\varepsilon\}. \tag{5.4}$$

The subset $\Gamma^{(n+1)}_{\varepsilon}(\zeta,z)$ of $\mathbb{C}^n$ is a real $(n+1)$-dimensional manifold embedded in $\mathbb{C}^n$ with an $n$-dimensional boundary $\partial\Gamma^{(n+1)}_{\varepsilon}(\zeta,z)$. Furthermore, the formula

$$f(\zeta')=\int_{\partial\Gamma^{(n+1)}_{\varepsilon}(\zeta,z)} G_w(\zeta')\boldsymbol{\omega}(dw)\tilde{f}(w) \tag{5.5}$$

defines a complex left monogenic function for every $\zeta'\in\mathbb{C}^{n+1}$ such that $N(\zeta')$ intersects $\Gamma^{(n+1)}_{\varepsilon}(\zeta,z)$ but is disjoint from $\partial\Gamma^{(n+1)}_{\varepsilon}(\zeta,z)$ [16]. Because $\tilde{f}$ has a unique complex left monogenic extension from an open subset $U$ of its domain in $\mathbb{C}^n$ to some neighbourhood of $U$ in $\mathbb{C}^{n+1}$ [16, p. 422], by Stokes' theorem the same representation holds for all orientable $n$-dimensional manifolds homologous in $\mathbb{C}^n\setminus N(\zeta')$ to $\partial\Gamma^{(n+1)}_{\varepsilon}(\zeta,z)$. As $\zeta\in\mathbb{C}^{n+1}$ and $z\in S^{\circ}_{\omega}(\mathbb{C}^n)$ vary, we obtain a well-defined complex left monogenic function $f$ [16, p. 418].

Let $U_\omega(\mathbb{C}^{n+1})$ denote the set of all $\zeta\in\mathbb{C}^{n+1}$ for which there exists $z\in S^{\circ}_{\omega}(\mathbb{C}^n)$ such that

$$\inf_{\theta}\operatorname{diam}(N(\zeta)\cap(z+e^{i\theta}\mathbf{R}^{\mathrm{n}}))>0 \text{ and } \mathrm{N}(\zeta)\cap(\mathrm{z}+\mathrm{e}^{i\theta}\mathbf{R}^{\mathrm{n}})\subset \mathrm{S}^{\circ}_{\omega}(\mathbb{C}^{\mathrm{n}}),\ \forall\theta\in\mathbf{R}. \tag{5.6}$$

By virtue of the formula (5.5), we obtain a unique complex left monogenic extension of $\tilde{f}$ to $U_\omega(\mathbb{C}^{n+1})$ for which the linear map $\tilde{f}\longmapsto f$ is continuous for the compact-open topology for holomorphic functions defined on $S^{\circ}_{\omega}(\mathbb{C}^n)$, to the space of complex left monogenic functions defined on $U_\omega(\mathbb{C}^{n+1})$. To complete the proof, it suffices to show that $S^{\circ}_{\omega}(\mathbf{R}^{\mathrm{n+1}})\setminus\mathbf{R}^{\mathrm{n}}\subset \mathrm{U}_\omega(\mathbb{C}^{\mathrm{n+1}})\cap(\mathbf{R}^{\mathrm{n+1}}\setminus\mathbf{R}^{\mathrm{n}})$.

Let $x=x_0e_0+\boldsymbol{x}\in S^{\circ}_{\omega}(\mathbf{R}^{\mathrm{n+1}})$ with $x_0\neq 0$. Then $\boldsymbol{x}\neq 0$. Let $z=\boldsymbol{x}(1+i\tan\beta)$ for $\tan^{-1}(|x_0|/|\boldsymbol{x}|)<\beta<\omega$. The centre of the $(n-2)$-sphere $N(x)\cap(z+e^{i\theta}\mathbf{R}^{\mathrm{n}})$ is

$$z-e^{i\theta}\sin\theta\operatorname{Im}z\left(1-\frac{x_0^2}{|\operatorname{Im}z|^2}\right)$$

and it has radius $\sqrt{(\sin^2\theta x_0^2 + \cos^2\theta |\operatorname{Im} z|^2)(1 - x_0^2/|\operatorname{Im} z|^2)} > 0$, so that for any $\varepsilon > 0$, there exists $\tan^{-1}(|x_0|/|\boldsymbol{x}|) < \beta < \omega$ such that $N(x) \cap (z + e^{i\theta}\mathbf{R}^{\mathrm{n}})$ is contained in a ball of radius $\varepsilon$ about $z$ in $\mathbb{C}^n$, for every $\theta \in \mathbf{R}$. Because $|z|_{\mathbb{C}} = |\boldsymbol{x}|(1 + i\tan\beta)$, an application of Proposition 5.1 guarantees that $z \in S^\circ_\omega(\mathbb{C}^n)$.

With a sufficiently small choice of $\beta > \tan^{-1}(|x_0|/|\boldsymbol{x}|)$, we can ensure that

$$\bigcup_{\theta\in\mathbf{R}} \gamma(N(x) \cap (z + e^{i\theta}\mathbf{R}^{\mathrm{n}})) \subset \mathrm{S}^\circ_\omega(\mathbf{R}^{\mathrm{n}+1}).$$

This shows that $S^\circ_\omega(\mathbf{R}^{\mathrm{n}+1}) \setminus \mathbf{R}^{\mathrm{n}} \subset \mathrm{U}_\omega(\mathbb{C}^{\mathrm{n}+1}) \cap (\mathbf{R}^{\mathrm{n}+1} \setminus \mathbf{R}^{\mathrm{n}})$. □

**Corollary 5.1.** *The linear map $f \longmapsto \tilde{f}$ defined by formula (1.2) maps the space of all left monogenic functions defined on $S^\circ_\omega(\mathbf{R}^{\mathrm{n}+1})$ bijectively onto the space of all $C\ell(\mathbb{C}^n)$-valued holomorphic functions defined on $S^\circ_\omega(\mathbb{C}^n)$. It is continuous between the compact-open topologies on each space.*

A $C\ell(\mathbb{C}^n)$-valued real analytic function defined on an open subset $U$ of $\mathbf{R}^{\mathrm{n}}$ necessarily has a left monogenic extension to an open subset of $\mathbf{R}^{\mathrm{n}+1}$ containing $U$ by taking an expansion about each point of $U$ in monogenic polynomials [2, Theorem 14.8]. In particular, the product $\phi.\psi : V \cap \mathbf{R}^{\mathrm{n}} \to \mathrm{C}\ell(\mathbb{C}^{\mathrm{n}})$ of two left monogenic functions defined in an open subset $V$ of $\mathbf{R}^{\mathrm{n}+1}$ intersecting $\mathbf{R}^{\mathrm{n}} \equiv \{0\} \times \mathbf{R}^{\mathrm{n}}$ has a unique left monogenic extension $\phi \cdot_\ell \psi$ to a neighbourhood of $V \cap \mathbf{R}^{\mathrm{n}}$ called the (left) *Cauchy-Kowalewski product* of $\phi$ and $\psi$. The following corollary shows that in the case $V = S^\circ_\omega(\mathbf{R}^{\mathrm{n}+1})$, the product is actually defined on all of $V$.

**Corollary 5.2.** *Let $\phi, \psi$ be left monogenic functions defined on $S^\circ_\omega(\mathbf{R}^{\mathrm{n}+1})$. Then there exists a unique left monogenic function $\phi \cdot_\ell \psi$ defined on $S^\circ_\omega(\mathbf{R}^{\mathrm{n}+1})$ such that $\phi \cdot_\ell \psi(x) = \phi(x).\psi(x)$ for every $x \in \mathbf{R}^{\mathrm{n}} \setminus \{0\}$.*

*Proof.* Let $\tilde{\phi}$ and $\tilde{\psi}$ be the holomorphic counterparts of the left monogenic functions $\phi$ and $\psi$ defined by formula (1.2). Then the product function $\tilde{\phi}.\tilde{\psi}$ is certainly a $C\ell(\mathbb{C}^n)$-valued holomorphic function defined on the open sector $S^\circ_\omega(\mathbb{C}^n)$. According to Corollary 5.1, there exists a unique left monogenic function defined on $S^\circ_\omega(\mathbf{R}^{\mathrm{n}+1})$, which we denote by $\phi \cdot_\ell \psi$, such that $(\phi \cdot_\ell \psi)^\sim = \tilde{\phi}.\tilde{\psi}$ as holomorphic functions defined on $S^\circ_\omega(\mathbb{C}^n)$. An appeal to the Cauchy integral formula (2.2) of Clifford analysis ensures that

$$\phi \cdot_\ell \psi(x) = (\phi \cdot_\ell \psi)^\sim(x) = \phi(x).\psi(x)$$

for every $x \in \mathbf{R}^{\mathrm{n}} \setminus \{0\}$. □

# 6 The analytic functional calculus for systems of operators of type $\omega$

As mentioned in the Introduction, one the difficulties in forming a function $f(\boldsymbol{A})$ of the $n$-tuple $\boldsymbol{A} = (A_1, \ldots, A_n)$ of commuting linear operators acting in a Banach space $X$ for a holomorphic function $f$ of $n$ complex variables, is the absence of a suitably general integral representation formula for functions of several complex variables. Now we formulate an analytic functional calculus $\tilde{f} \longmapsto \tilde{f}(\boldsymbol{A})$ for holomorphic functions $\tilde{f}$ defined on a sector $S^\circ_\omega(\mathbb{C}^n)$ by using the representation (1.1) with the left monogenic counterpart $f : S^\circ_\omega(\mathbf{R}^{n+1}) \to \mathbb{C}\ell(\mathbb{C}^n))$ of $\tilde{f}$ obtained from Theorem 5.1 above.

Suppose that $T : \mathcal{D}(T) \longrightarrow \mathcal{H}$ is a single closed densely defined linear operator acting in the Hilbert space $\mathcal{H}$. The spectrum of $T$ is denoted by $\sigma(T)$. If $0 \le \omega < \pi/2$, then $T$ is said to be of *type* $\omega$, if $\sigma(T) \subset S_\omega(\mathbb{C})$ and for each $\nu > \omega$, there exists $C_\nu > 0$ such that

$$\|(zI - T)^{-1}\| \le C_\nu |z|^{-1}, \quad z \notin S_\nu(\mathbb{C}). \tag{6.1}$$

Then the bounded linear operator $f(T)$ is defined by the Riesz-Dunford formula

$$f(T) = \frac{1}{2\pi i} \int_C (\lambda I - T)^{-1} f(\lambda)\, d\lambda. \tag{6.2}$$

for any function $f$ satisfying the bounds

$$|f(z)| \le K_\nu \frac{|z|^s}{1 + |z|^{2s}}, \quad z \in S^\circ_\nu(\mathbb{C}).$$

The contour $C$ can be taken to be $\{z \in \mathbb{C} : |\operatorname{Im}(z)| = \tan\theta |\operatorname{Re}(z)|\ \}$, with $\omega < \theta < \nu$.

The operator $T$ of type $\omega$ is said to have a *bounded $H^\infty$-functional calculus* if for each $\omega < \nu < \pi/2$, there exists an algebra homomorphism $f \longmapsto f(T)$ from $H^\infty(S^\circ_\nu(\mathbb{C}))$ to $\mathcal{L}(\mathcal{H})$ agreeing with (6.2) and a positive number $C_\nu$ such that $\|f(T)\| \le C_\nu \|f\|_\infty$ for all $f \in H^\infty(S^\circ_\nu(\mathbb{C}))$.

The following result is from [10, Theorem 6.2.2]

**Theorem 6.1.** *Suppose that $T$ is a one-to-one operator of type $\omega$ in $\mathcal{H}$. Then $T$ has a bounded $H^\infty$-functional calculus if and only if for every $\omega < \nu < \pi/2$, there exists $c_\nu > 0$ such that $T$ and its adjoint $T^*$ satisfy the square function* *estimates*

$$\int_0^\infty \|\psi_t(T)u\|^2 \frac{dt}{t} \leq c_\nu \|u\|^2, \quad u \in \mathcal{H}, \tag{6.3}$$

$$\int_0^\infty \|\psi_t(T^*)u\|^2 \frac{dt}{t} \leq c_\nu \|u\|^2, \quad u \in \mathcal{H}, \tag{6.4}$$

*for some function (every function) $\psi \in H^\infty(S^\circ_\nu(\mathbb{C}))$, which satisfies*

$$\int_0^\infty \psi^3(t)\frac{dt}{t} = \int_0^\infty \psi^3(-t)\frac{dt}{t} = 1, \text{ and} \tag{6.5}$$

$$|\psi(z)| \leq C_\nu \frac{|z|^s}{1+|z|^{2s}}, \quad z \in S^\circ_\nu(\mathbb{C}), \tag{6.6}$$

*for some $s > 0$. Here $\psi_t(z) = \psi(tz)$ for $z \in S^\circ_\nu(\mathbb{C})$.*

We now use formula (1.1) to generalise this result to $n$-tuples of commuting operators acting in a Hilbert space $\mathcal{H}$.

The $(n-1)$-sphere in $\mathbf{R}^n$ is denoted by $\mathbb{S}^{n-1}$. The set of $s \in \mathbb{S}^{n-1}$ with nonzero coordinates $s_j$ for every $j = 1, \ldots, n$ is denoted by $\mathbb{S}^{n-1}_0$. Then $\mathbb{S}^{n-1}_0$ is a dense open subset of $\mathbb{S}^{n-1}$ with full surface measure.

**Definition 6.1.** Let $X$ be a Banach space and let $\boldsymbol{A} = (A_1, \ldots, A_n)$ be an $n$-tuple of densely defined linear operators $A_j : \mathcal{D}(A_j) \longrightarrow X$ acting in $X$ such that $\cap_{j=1}^n \mathcal{D}(A_j)$ is dense in $X$ and let $0 \leq \omega < \frac{\pi}{2}$. Then $\boldsymbol{A}$ is said to be *uniformly of type* $\omega$ if for all $s \in \mathbb{S}^{n-1}_0$, the densely defined operator $\langle A, s\rangle := \sum_{j=1}^n A_j s_j$ is closed, $\sigma(\langle A, s\rangle) \subset S_\omega(\mathbb{C})$ and for each $\nu > \omega$, there exists $C_\nu > 0$ such that

$$\|(zI - \langle A, s\rangle)^{-1}\| \leq C_\nu |z|^{-1}, \quad z \notin S_\nu(\mathbb{C}),\ s \in \mathbb{S}^{n-1}_0. \tag{6.7}$$

It follows that $s \longmapsto \langle A, s\rangle$ is continuous on $\mathbb{S}^{n-1}_0$ in the sense of strong resolvent convergence [7, Theorem VIII.1.5].

If $\boldsymbol{A}$ is uniformly of type $\omega$, it turns out that we can define the Cauchy kernel $G_x(\boldsymbol{A})$ for the $n$-tuple $\boldsymbol{A}$ by the plane wave formula

$$G_x(\boldsymbol{A}) = \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n \operatorname{sgn}(x_0)^{n-1} \int_{\mathbb{S}^{n-1}} (e_0+is)\left(\langle \boldsymbol{x}, s\rangle I - \langle A, s\rangle - x_0 sI\right)^{-n} ds \tag{6.8}$$

for all $x = x_0e_0 + \boldsymbol{x}$ with $x_0$ a nonzero real number, $\boldsymbol{x} \in \mathbf{R}^n$ and $x \notin S_\omega(\mathbf{R}^{n+1})$. Here $\mathbb{S}^{n-1}$ is the unit $(n-1)$-sphere in $\mathbf{R}^n$, $ds$ is surface measure and the inverse power $(\langle \boldsymbol{x}I - A, s\rangle - x_0 sI)^{-n}$ is taken in the Clifford module $\mathcal{L}(X)\otimes C\ell(\mathbb{C}^n)$, which is identified with the set $\mathcal{L}_{(n)}(X_{(n)})$ of all left module homomorphisms of $X_{(n)} = X\otimes C\ell(\mathbb{C}^n)$, see [4, Equation (6.14)].

Suppose that $0<\omega<\nu<\pi/2$ and $f$ is a left monogenic function defined on $S^\circ_\nu(\mathbf{R}^{\mathrm{n}+1})$ such that for every $\omega<\theta<\nu$ there exists $C_\theta>0$ and $s>0$ such that

$$|f(x)|\le C_\theta \frac{|x|^s}{1+|x|^{2s}},\quad x\in S^\circ_\theta(\mathbf{R}^{\mathrm{n}+1})\setminus \mathrm{S}_\omega(\mathbf{R}^{\mathrm{n}+1}). \tag{6.9}$$

Now if $\omega<\theta<\nu$ and

$$H_\theta = \{x\in\mathbf{R}^{\mathrm{n}+1} : \mathrm{x} = \mathrm{x}_0\mathrm{e}_0 + \boldsymbol{x},\ |\mathrm{x}_0|/|\mathrm{x}| = \tan\theta\}\subset \mathrm{S}^\circ_\nu(\mathbf{R}^{\mathrm{n}+1}), \tag{6.10}$$

then it is easy to check that $x\longmapsto G_x(A)\boldsymbol{n}(x)f(x)$ is integrable with respect to $n$-dimensional surface measure $\mu$ on $H_\theta$. Therefore, we define

$$f(\boldsymbol{A}) = \int_{H_\theta} G_x(\boldsymbol{A})\boldsymbol{n}(x)f(x)\,d\mu(x). \tag{6.11}$$

The hypersurface $H_\theta$ can be varied in the region where $x\longmapsto G_x(\boldsymbol{A})$ is two-sided monogenic in the Clifford module $\mathcal{L}(X)\otimes C\ell(\mathbb{C}^n)$ and $f$ is left monogenic in $S^\circ_\nu(\mathbf{R}^{\mathrm{n}+1})$.

**Theorem 6.2.** *Let $\boldsymbol{A}=(A_1,\dots,A_n)$ be an $n$-tuple of densely defined commuting linear operators $A_j:\mathcal{D}(A_j)\longrightarrow\mathcal{H}$ acting in a Hilbert space $\mathcal{H}$ such that $\cap_{j=1}^n\mathcal{D}(A_j)$ is dense in $\mathcal{H}$. Suppose that $0\le\omega<\frac{\pi}{2}$ and $\boldsymbol{A}$ is uniformly of type $\omega$.*

*Suppose in addition, that $T=i(A_1e_1+\cdots+A_ne_n)$ is a one-to-one operator of type $\omega$ acting in $\mathcal{H}_{(n)}=C\ell(\mathbb{C}^n)\otimes\mathcal{H}$ and $T$ has an $H^\infty$-functional calculus. Then the $n$-tuple $\boldsymbol{A}$ has a bounded $H^\infty$-functional calculus on $S^\circ_\nu(\mathbb{C}^n)$ for any $\omega<\nu<\pi/2$, that is, there exists a homomorphism $b\longmapsto b(\boldsymbol{A})$, $b\in H^\infty(S^\circ_\nu(\mathbb{C}^n))$, from $H^\infty(S^\circ_\nu(\mathbb{C}^n))$ into $\mathcal{L}_{(n)}(\mathcal{H}_{(n)})$ and there exists $C_\nu>0$ such that $\|b(\boldsymbol{A})\|\le C_\nu\|b\|_\infty$ for all $b\in H^\infty(S^\circ_\nu(\mathbb{C}^n))$.*

*Moreover, if $f$ is the unique two-sided monogenic function defined on $S^\circ_\nu(\mathbf{R}^{\mathrm{n}+1})$ such that $b=\tilde f$, as in Theorem 5.1, and there exists $C_\nu>0$, $s>0$ such that*

$$|b(\zeta)|\le C_\nu\frac{|\zeta|^s}{1+|\zeta|^{2s}},\quad \zeta\in S^\circ_\nu(\mathbb{C}^n),$$

*then $f$ satisfies the bound (6.9) and $b(\boldsymbol{A})=f(\boldsymbol{A})$ is given by formula (6.11).*

*Proof.* By assumption, the operator $T$ has an $H^\infty$-functional calculus, so there exists a function $\psi \in H^\infty(S^\circ_\nu(\mathbb{C}))$ satisfying conditions (6.3) – (6.6). In [3] a special choice of $\psi$ was made, but we now see that this is not necessary.

Following [10, Theorem 6.4.3], our aim is to define $b(\boldsymbol{A})$ for $b \in H^\infty(S^\circ_\nu(\mathbb{C}^n))$, by the formula

$$(b(\boldsymbol{A})u, v) = \int_0^\infty \Big((b\phi_t)(\boldsymbol{A})\psi_t(T)u, \psi_t(T)^* v\Big)\frac{dt}{t} \tag{6.12}$$

for all $u, v \in \mathcal{H}_{(n)}$. The function $\phi : S^\circ_\nu(\mathbb{C}^n) \longrightarrow \mathbb{C}$ is constructed from $\psi$ by setting

$$\phi(\zeta) = \psi\{i\zeta\} = \psi(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + \psi(-|\zeta|_{\mathbb{C}})\chi_-(\zeta),$$

for all $\zeta \in S^\circ_\nu(\mathbb{C}^n)$, by the functional calculus for multiplication by $i\zeta$. Let $\phi_t(\zeta) = \phi(t\zeta)$ for all $t > 0$ and $\zeta \in S^\circ_\nu(\mathbb{C}^n)$.

Let $b\cdot_\ell \phi_t$ be the left monogenic function defined on $S^\circ_\nu(\mathbf{R}^{n+1})$ by Corollary 5.2, that is, the Cauchy-Kowalewski product of the left monogenic counterparts of $b$ and $\phi_t$. Moreover, the proof of Theorem 5.1 shows that $b\cdot_\ell \phi_t$ satisfies the bound (6.9) with $C_\theta$ proportional to $\|b\|_\infty$, so $(b\phi_t)(\boldsymbol{A}) = b\cdot_\ell\phi_t(\boldsymbol{A})$ is defined by formula (6.11) and we have

$$\begin{aligned}
&\int_0^\infty \left|\Big((b\phi_t)(\boldsymbol{A})\psi_t(T)u, \psi_t(T)^* v\Big)\right|\frac{dt}{t} \\
&\le \sup_{t>0}\|(b\phi_t)(\boldsymbol{A})\|\left\{\int_0^\infty \|\psi_t(T)u\|^2\frac{dt}{t}\right\}^{1/2}\left\{\int_0^\infty \|\psi_t(T)^* v\|^2\frac{dt}{t}\right\}^{1/2} \\
&\le C'\|b\|_\infty\|u\|\,\|v\|.
\end{aligned}$$

Because $\int_0^\infty \psi^3(t)\frac{dt}{t} = 1$, we obtain the desired functional calculus by analogy with [10, Theorem 6.4.3]. □

The assumptions of Theorem 6.2 are satisfied if the $n$-tuple $\boldsymbol{A}$ consists of differentiation operators on a Lipschitz surface [9]. The commutativity of the operators is most easily seen from the representation in [9, p. 708].

# References


[1] D. Albrecht, Integral formulae for special cases of Taylor's functional calculus, *Studia Math.* **105** (1993), 51–68.

[2] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*, Research Notes in Mathematics 76, Pitman, Boston/London/Melbourne, 1982.

[3] B. Jefferies, Function Theory in Sectors, *Studia Math.*, **163** (2004), 257–287.

[4] B. Jefferies, *Spectral Properties of Noncommuting Operators*, Lecture Notes in Mathematics **1843**, Springer-Verlag, Berlin/Heidelberg/New York, 2004.

[5] B. Jefferies and A. McIntosh, The Weyl calculus and Clifford analysis, *Bull. Austral. Math. Soc.* **57** (1998), 329–341.

[6] B. Jefferies, A. McIntosh and J. Picton-Warlow, The monogenic functional calculus, *Studia Math.* **136** (1999), 99–119.

[7] T. Kato, *Perturbation Theory for Linear Operators*, 2nd Ed., Springer-Verlag, Berlin/Heidelberg/New York, 1980.

[8] S.G. Krantz, *Function Theory of Several Complex Variables*, 2nd Ed., Amer. Math. Soc., Providence, 1992.

[9] C. Li, A. McIntosh, T. Qian, Clifford algebras, Fourier transforms and singular convolution operators on Lipschitz surfaces, *Rev. Mat. Iberoamericana* **10** (1994), 665–721.

[10] C. Li, A. McIntosh, Clifford algebras and $H_\infty$ functional calculi of commuting operators, *Clifford algebras in analysis and related topics* (Fayetteville, AR, 1993), 89–101, Stud. Adv. Math., CRC, Boca Raton, FL, 1996.

[11] A. McIntosh, Operators which have an $H_\infty$-functional calculus, in: Miniconference on Operator Theory and Partial Differential Equations 1986, 212–222 *Proc. Centre for Mathematical Analysis* **14**, ANU, Canberra, 1986.

[12] A. McIntosh, Clifford algebras, Fourier theory, singular integrals, and harmonic functions on Lipschitz domains, *Clifford algebras in analysis and related topics* (Fayetteville, AR, 1993), 33–87, Stud. Adv. Math., CRC, Boca Raton, FL, 1996.

[13] A. McIntosh and A. Pryde, A functional calculus for several commuting operators, *Indiana U. Math. J.* **36** (1987), 421–439.

[14] V. Müller, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras*, Operator Theory Adv. Appl. **139**, Birkhaüser, Basel-Boston-Berlin, 2003.

[15] R.M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*, Graduate Texts in Mathematics **108**, Springer-Verlag, New York, 1986.

[16] J. Ryan, Complex Clifford analysis and domains of holomorphy, *J. Austral. Math. Soc. Ser. A* **48** (1990), 413–433.

[17] J. Ryan, Cells of harmonicity and generalized Cauchy integral formulae, *Proc. London Math. Soc.* **60** (1990), 295–318.

[18] J.L. Taylor, The analytic-functional calculus for several commuting operators, *Acta Math.* **125** (1970), 1–38.



**Brian Jefferies**, School of Mathematics, UNSW, Kensington 2052 NSW, Australia. b.jefferies@unsw.edu.au