# On an operator-valued $T(1)$ theorem by Hytönen and Weis

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**Abstract**

We consider generalized Calderón-Zygmund operators whose kernel takes values in the space of all continuous linear operators between two Banach spaces. In the spirit of the $T(1)$ theorem of David and Journé we prove boundedness results for such operators on vector-valued Riesz potential spaces. This improves and generalizes a result by Hytönen and Weis.
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## 1 Introduction

In this paper we want to study non-convolution type singular integrals of the form

$$(Tf)(u) = \int_{\mathbb{R}^N} K(u,v)f(v)dv \tag{1.1}$$

(see e.g. [19, 20]). Inspired by the famous $T(1)$ theorem of G. David and J.-L. Journé [4, 5] on the $L_2$ boundedness of operators (1.1) T. Figiel [8] has proved a $T(1)$ theorem for $X$-valued $L_p$ functions, where $X$ has the UMD-property. The kernels $K$ are still scalar valued, however. Recently T. Hytönen and L. Weis [13] gave a new proof of Figiel's $T(1)$ theorem and extended it to operator-valued kernels $K$.

On the other hand, various authors (e.g. [9, 11, 18, 21, 23]) obtained results of the same spirit as the $T(1)$ theorem of David and Journé for other scalar-valued function spaces, including homogeneous Besov, Triebel-Lizorkin and Riesz potential spaces. In [12] T. Hytönen and the author give a new prove of the $T(1)$ theorem for Triebel-Lizorkin spaces, using methods from [13]. And in [15] we show an operator-valued version of the $T(1)$ theorem for vector-valued Besov spaces.

In this paper, we prove the following operator-valued $T(1)$ theorem for vector-valued Riesz potential spaces. The definition of these spaces, as well as of the various conditions appearing in the theorem, are given in Section 2. The theorem is then proved in Section 3.

**Theorem 1.1.** *Let $X, Y$ be UMD spaces. Suppose that $T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N, \mathcal{L}(X,Y))$ is in the class $\mathrm{RCZO}_\nu$ for some $\nu \in (0,1]$, satisfies the weak R-boundedness property, and $T(1) = 0$.*

*(a) $T$ extends to a bounded linear operator from $\dot{H}^s_p(X)$ to $\dot{H}^s_p(Y)$ for each $s \in (0,\nu)$ and each $p \in (1,\infty)$.*

*(b) If in addition $T' \in \mathrm{RCZO}_\nu$ and $T'(1) = 0$, then $T$ extends to a bounded linear operator from $\dot{H}^s_p(X)$ to $\dot{H}^s_p(Y)$ for each $|s| < \nu$ and each $p \in (1,\infty)$.*

Theorem 1.1 contains the $L^p$ result by Hytönen and Weis from [13] as a special case ($s = 0$). However, we use a weaker version of the weak R-boundedness property, which slightly improves their theorem.

# 2 Definitions and Notations

Throughout this paper $X$ and $Y$ are complex Banach spaces. The space $\mathcal{L}(X,Y)$ of bounded linear operators from $X$ to $Y$ is endowed with the uniform operator topology. $X' = \mathcal{L}(X,\mathbb{C})$ denotes the dual space of $X$. All our (possibly vector-valued) functions and distributions will be defined on $\mathbb{R}^N$ for a fixed positive integer $N$. Therefore the various function spaces $E(\mathbb{R}^N, X)$ in this paper are denoted simply by $E(X)$. For example we write $L_p(X)$ for the Bochner-Lebesgue space $L_p(\mathbb{R}^N, X)$, $p \in [1,\infty]$, equipped with its usual norm.

$\mathbb{N} := \{0, 1, 2, \dots\}$ is the set of all nonnegative integers. The conjugate exponent $p'$ of $p \in [1,\infty]$ is given by $\frac{1}{p} + \frac{1}{p'} = 1$.

We write $\mathcal{D}(\mathbb{R}^N, X)$ for the space of all compactly supported smooth functions with values in $X$. The Schwartz class $\mathcal{S}(\mathbb{R}^N, X)$ is the space of all $X$-valued rapidly decreasing smooth functions, endowed with its usual topology. For $\mathcal{D}(\mathbb{R}^N,\mathbb{C})$ and $\mathcal{S}(\mathbb{R}^N,\mathbb{C})$ we also write $\mathcal{D}(\mathbb{R}^N)$ and $\mathcal{S}(\mathbb{R}^N)$ respectively. The space $\mathcal{S}'(\mathbb{R}^N, X)$ of all $X$-valued tempered distributions is defined as the space of all continuous linear operators from $\mathcal{S}(\mathbb{R}^N)$ to $X$. The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is defined by $(\mathcal{F}\varphi)(u) = \widehat{\varphi}(u) = \int_{\mathbb{R}^N} e^{-iuv}\varphi(v)dv$.

Let $\mathcal{Z}(\mathbb{R}^N, X)$ be the space of all Schwartz functions $\varphi \in \mathcal{S}(\mathbb{R}^N, X)$ such that $D^\alpha \widehat{\varphi}(0) = 0$ for all multiindices $\alpha \in \mathbb{N}^N$. Then $\mathcal{Z}(\mathbb{R}^N, X)$ is a closed subspace of $\mathcal{S}(\mathbb{R}^N, X)$. If $\mathcal{Z}'(\mathbb{R}^N, X)$ denotes the space of all continuous linear operators from $\mathcal{Z}(\mathbb{R}^N) = \mathcal{Z}(\mathbb{R}^N, \mathbb{C})$ to $X$, then $\mathcal{S}'(\mathbb{R}^N, X)/\mathcal{P}(\mathbb{R}^N, X)$ and $\mathcal{Z}'(\mathbb{R}^N, X)$ are isomorphic (cf. [22, 5.1.2] and [14, Section 7]). Here $\mathcal{P}(\mathbb{R}^N, X)$ stands for the space of all polynomials on $\mathbb{R}^N$ with coefficients in $X$.

## Riesz potential spaces

Let $p \in (1, \infty)$ and $s \in \mathbb{R}$. The *Riesz potential spaces* $\dot{H}^s_p(X) = \dot{H}^s_p(\mathbb{R}^n, X)$ is the space consisting of all $f \in \mathcal{Z}'(\mathbb{R}^N, X)$ such that

$$\|f\|_{\dot{H}^s_p(X)} = \|\mathcal{F}^{-1}(|\cdot|^s \widehat{f}(\cdot))\|_{L_p(X)}$$

is finite. Note that $\mathcal{F}^{-1}(|\cdot|^s \widehat{f}(\cdot))$ maps $\mathcal{Z}'(\mathbb{R}^N, X)$ onto itself and that $\|\cdot\|_{\dot{H}^s_p(X)}$ is a norm on $\dot{H}^s_p(X)$.

Let $\widehat{\phi} \in \mathcal{D}(\mathbb{R}^N)$ be radial, equal to 1 in $\overline{B}(0,1)$, and supported in $\overline{B}(0,2)$. Let $\widehat{\varphi} = \widehat{\phi} - \widehat{\phi}(2\cdot)$ and $\widehat{\varphi}_j = \widehat{\varphi}(2^j\cdot)$, $j \in \mathbb{Z}$. It follows from the results in [16] that, if $p \in (1, \infty)$ and $X$ is a UMD space (for a definition see below), then

$$\|f\|_{\dot{F}^{s,p}_2(X)} := \left(\int_0^1 \Big\|\sum_{j\in\mathbb{Z}} r_j(t) 2^{-js} f * \varphi_j\Big\|^2_{L_p(X)} dt\right)^{1/2}.$$

is an equivalent norm on $\dot{H}^s_p(X)$. Here $(r_j)$ is some sequence of distinct Rademacher functions.

## The operator $T$

We consider a continuous linear operator

$$T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N, \mathcal{L}(X, Y)).$$

$T$ can be identified with the continuous bilinear form

$$\mathcal{S}(\mathbb{R}^N) \times \mathcal{S}(\mathbb{R}^N) \to \mathcal{L}(X, Y), \qquad (\varphi, \psi) \mapsto (T\varphi)(\psi).$$

In place of $(T\varphi)\psi$ we also use the notation $\langle\psi, T\varphi\rangle$. To $T$ we assign an "adjoint" operator

$$T' : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N, \mathcal{L}(Y', X')), \qquad \langle\psi, T'\varphi\rangle := \langle\varphi, T\psi\rangle',$$

where the latter $'$ designates the usual Banach adjoint of an operator in $\mathcal{L}(X, Y)$.

From $T$ we derive a linear mapping $\widetilde{T} : \mathcal{S}(\mathbb{R}^N) \otimes X \to \mathcal{S}'(\mathbb{R}^N, Y)$ : for $x \in X$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^N)$, we let

$$\langle\psi, \widetilde{T}[\varphi \otimes x]\rangle := \langle\psi, T\varphi\rangle x \in Y.$$

This makes sense, since $\langle\psi, T\varphi\rangle \in \mathcal{L}(X, Y)$. So $\widetilde{T}[\varphi \otimes x]$ is a $Y$-valued tempered distribution and $\widetilde{T}$ is well-defined on $\mathcal{S}(\mathbb{R}^N) \times X$. Now we extend $\widetilde{T}$ to $\mathcal{S}(\mathbb{R}^N) \otimes X$ by linearity. In the following we will not distinguish between $T$ and $\widetilde{T}$.

## The associated kernel

Suppose now that $K : \{(u, v) \in \mathbb{R}^N \times \mathbb{R}^N : u \neq v\} \to \mathcal{L}(X, Y)$ is continuous. We say that $T$ is *associated with* $K$ if

$$\langle\varphi, T\phi\rangle = \int_{\mathbb{R}^N} \varphi(u) \int_{\mathbb{R}^N} K(u, v)\phi(v)dv\, du \tag{2.1}$$

holds for all $\varphi, \phi \in \mathcal{D}(\mathbb{R}^N)$ with $\operatorname{supp}\varphi \cap \operatorname{supp}\phi = \emptyset$. This means that, for each $\phi \in \mathcal{D}(\mathbb{R}^N)$, the distribution $T\phi$ agrees almost everywhere on the complement of $\operatorname{supp}\phi$ with the continuous function $\int_{\mathbb{R}^N} K(\cdot, v)\phi(v)dv$, defined on the complement of $\operatorname{supp}\phi$. It is clear from (2.1) that $T'$ is associated to $K'$ given by $K'(u, v) = K(v, u)'$ for $u \neq v$.

Now we assume that $K$ satisfies the standard estimates

($\mathrm{SE}_0$) $\sup\{|u - v|^N \|K(u, v)\| : u \neq v\} < \infty$,

($\mathrm{SE}_\nu$) $\sup\left\{|u - v|^{N+\nu} \dfrac{\|K(u, v) - K(u_0, v)\|}{|u - u_0|^\nu} : |u - v| > 2|u - u_0| > 0\right\} < \infty$.

for some $\nu \in (0, 1]$. We say that $T \in \mathrm{CZO}_\nu$ if $T$ is associated with $K$ satisfying ($\mathrm{SE}_0$) and ($\mathrm{SE}_\nu$). Note that $T \in \mathrm{CZO}_\nu$ does *not* emply that $T' \in \mathrm{CZO}_\nu$.

## Definition of $T(1)$

The action of $T \in \mathrm{CZO}_\nu$ is not a priori defined on the constant function $1 \notin \mathcal{S}(\mathbb{R}^N)$, but we can make sense of the notion $T(1)$: We will define $T(1)$ as a linear operator acting on

$$\mathcal{D}^0(\mathbb{R}^N) := \{\varphi \in \mathcal{D}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \varphi(u)du = 0\}.$$

For doing this we first observe that, if $\varphi \in \mathcal{D}^0(\mathbb{R}^N)$, the distribution $T'\varphi$ agrees with an integrable function on the exterior of any neighborhood of $\mathrm{supp}\varphi$. Now choose $\psi \in \mathcal{D}(\mathbb{R}^N)$ such that $\psi \equiv 1$ in a neighborhood of $\mathrm{supp}\varphi$ and define

$$\big\langle 1, T'\varphi \big\rangle := \big\langle \psi, T'\varphi \big\rangle + \int_{\mathbb{R}^N} (1 - \psi(u))(T'\varphi)(u)du.$$

Here the first term is given by the usual pairing between test functions and distributions and the second term exists because $T'\varphi$ is integrable on the support of $1 - \psi$. One can show that the value of $\big\langle 1, T'\varphi \big\rangle$ is independent of the actual choice of $\psi$. Now we make the natural definition $\big\langle \varphi, T(1) \big\rangle := \big\langle 1, T'\varphi \big\rangle' \big|_X \in \mathcal{L}(X, Y)$.

## The weak boundedness property

The closed ball with center $u \in \mathbb{R}^N$ and radius $r > 0$ is denoted by $\overline{B}(u, r)$. We say that $\varphi$ is a *normalized bump function* associated with the unit ball $\overline{B}(0, 1)$ if $\varphi \in \mathcal{D}(\mathbb{R}^N)$ with $\mathrm{supp}\varphi \subseteq \overline{B}(0, 1)$ and $\|D^\alpha \varphi\|_{L_\infty} \leq 1$ for all $|\alpha| \leq M$, where $M$ is a large fixed number. $\phi$ is a normalized bump function associated with the ball $\overline{B}(u, r)$ if $\phi(\cdot) = r^{-N}\varphi(r^{-1}(\cdot - u))$, where $\varphi$ is a normalized bump function associated with the unit ball.

The operator $T$ is said to have

- the *weak boundedness property* provided that, for every pair of normalized bump functions $\varphi, \phi$ associated with any ball $\overline{B}(u, r)$ we have $\|\big\langle \phi, T\varphi \big\rangle\| \leq Cr^{-N}$.
- the *weak boundedness property at 0* provided that, for every pair of normalized bump functions $\varphi, \phi$ associated with any ball $\overline{B}(0, r)$ we have $\|\big\langle \phi, T\varphi \big\rangle\| \leq Cr^{-N}$.

The following lemma is a refinement of an auxiliary result in [13, Sect. 2]. For a proof see [15].

**Lemma 2.1.** *Let $k \in \mathbb{N}$, $a > 0$, $w \in \mathbb{R}^N$, and let $\varphi, \phi \in \mathcal{D}^0(\mathbb{R}^N)$ be normalized bump functions associated with $\overline{B}(0,a)$ and $\overline{B}(w, 2^k a)$ respectively. Let $T \in \mathrm{CZO}_\nu$ satisfy the weak boundedness property at 0.*
*(a) There is a constant $C_1 < \infty$ such that*

$$\big\|\langle \phi, T\varphi\rangle\big\| \leq C_1 \frac{1+k}{(a2^k)^N}\left(1 + \frac{|w|}{a2^k}\right)^{-N-\nu}.$$

*(b) If in addition $T(1) = 0$, then there are constants $C_2 < \infty$ and $\delta > 0$ such that*

$$\big\|\langle \varphi, T\phi\rangle\big\| \leq C_2 (a2^k)^{-N-\nu}\left(1 + \frac{|w|}{a2^k}\right)^{-N-\delta}.$$

## Notions from Banach space theory

For our result on Riesz potential spaces we will restrict ourselves to Banach spaces that have a certain geometric property, namely the property of Unconditional Martingale Differences (UMD). There are several equivalent definitions for this property (see [2, p.141-142] and the references given there). Here is one of them:

**Definition 2.1.** A Banach space $X$ is a *UMD space* if and only if the Hilbert transform

$$(\mathcal{H}f)(u) = \mathrm{PV} - \int \frac{f(v)}{u-v} dv, \qquad f \in \mathcal{S}(\mathbb{R}, X),$$

extends to a bounded linear operator on $L_p(\mathbb{R}, X)$ for some (and thus for each) $p \in (1, \infty)$.

*Remark* 2.1. (a) It is clear from the definition that each Hilbert space is a UMD space. The dual space and each closed subspace of a UMD space is a UMD space. A UMD space $X$ always has a uniformly convex renorming [1] and therefore is super-reflexive [7]. In particular, $\ell_1$ is not finitely representable in $X$. Hence $X$ is B-convex [6, Theorem 13.6].
(b) Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space. If $X$ is a UMD space and $p \in (1, \infty)$, then $L_p(\Omega, \mu, X)$ is also a UMD space [2, p.145].

Next we recall the notion of R-boundedness.

**Definition 2.2.** Let $X, Y$ be Banach spaces. A set of operators $\tau \subseteq \mathcal{L}(X,Y)$ is called *R-bounded* if there is a constant $C < \infty$ such that for all $m \in \mathbb{N}$, all $T_1, \dots, T_m \in \tau$ and all $x_1, \dots x_m \in X$ we have that

$$\Big\|\sum_{k=1}^{m} r_k T_k x_k\Big\|_{L_2([0,1],Y)} \le C\Big\|\sum_{k=1}^{m} r_k x_k\Big\|_{L_2([0,1],X)}, \tag{2.2}$$

where $r_k$ is the $k$-th Rademacher function on $[0,1]$. The infimum over all $C$ such that (2.2) holds is called the *R-bound* of $\tau$ and is denoted by $\mathcal{R}(\tau)$.

It is clear from the definition that R-boundedness implies uniform boundedness. But in general the notion of R-boundedness is stronger than that of uniform boundedness. In fact, G. Pisier proved that every bounded subset of $\mathcal{L}(X)$ is R-bounded if and only if $X$ is isomorphic to a Hilbert space (cf. [3]). But R-boundedness is equivalent to uniform boundedness between the so called Rademacher spaces, which we define now.

**Definition 2.3.** For a Banach space $X$, the *Rademacher space* $\mathrm{Rad}X$ is the closure in $L_2([0,1],X)$ of the subspace of all finite linear combinations $\sum_{k=1}^{m} r_k x_k$, where $r_k$ are Rademacher functions and $x_k$ are elements of $X$.

For $T_1, \dots, T_m \in \mathcal{L}(X,Y)$, the operator $[T_k]_{k=1}^m : \mathrm{Rad}X \to \mathrm{Rad}Y$ is defined by

$$[T_k]_{k=1}^m : \sum_{k=1}^{m} r_k x_k \mapsto \sum_{k=1}^{m} r_k T_k x_k.$$

With this definition it is immediate that $\tau \subseteq \mathcal{L}(X,Y)$ is R-bounded if and only if $\{[T_k]_{k=1}^m : m \in \mathbb{Z}_+,\ T_1, \dots, T_m \in \tau\}$ is a bounded subset of $\mathcal{L}(\mathrm{Rad}X, \mathrm{Rad}Y)$.

**Proposition 2.1.** *[17, Proposition 3.5] Let $\tau$ be a R-bounded subset of $\mathcal{L}(X,Y)$. If $X$ is B-convex, then $\{T' : T \in \tau\} \subseteq \mathcal{L}(Y', X')$ is R-bounded.*

In particular, if $X$ and $Y$ are UMD spaces, then $\tau \subseteq \mathcal{L}(X,Y)$ is R-bounded if and only if $\{T' : T \in \tau\} \subseteq \mathcal{L}(Y', X')$ is R-bounded.

## The class $\mathrm{RCZO}_\nu$

For a kernel $K : \{(u,v) \in \mathbb{R}^N \times \mathbb{R}^N : u \neq v\} \to \mathcal{L}(X,Y)$ and a real number $\nu \in (0,1]$ we consider the standard R-estimates

(SRE$_0$) $\mathcal{R}\big(\{|u-v|^N\|K(u,v)\| : u \neq v\}\big) < \infty$,

(SRE$_\nu$) $\mathcal{R}\Bigg(\Bigg\{|u-v|^{N+\nu}\dfrac{\|K(u,v)-K(u_0,v)\|}{|u-u_0|^\nu} : |u-v| > 2|u-u_0| > 0\Bigg\}\Bigg) < \infty.$

We say that $T \in \mathrm{RCZO}_\nu$ if $T$ is associated with $K$ satisfying (SRE$_0$) and (SRE$_\nu$).

It is clear from the definition that the class $\mathrm{RCZO}_\nu$ is contained in $\mathrm{CZO}_\nu$. If $X, Y$ are Hilbert spaces then the two classes coincide.

## The weak R-boundedness property

For $r > 0$ and $w \in \mathbb{R}^N$ we define the dilation and translation operators on $\mathcal{S}(\mathbb{R}^N)$ by

$$\delta_r\varphi = r^{-N/2}\varphi(r^{-1}\cdot), \qquad \tau_w\varphi = \varphi(\cdot - w).$$

Moreover we define the continuous linear operator $T^r_w : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N, \mathcal{L}(X,Y))$ by

$$\big\langle \phi, T^r_w\varphi\big\rangle = \big\langle \tau_w\delta_r\phi, T[\tau_w\delta_r\varphi]\big\rangle, \qquad \varphi, \phi \in \mathcal{S}(\mathbb{R}^N).$$

With this definition we can reformulate the weak boundedness property as follows: The operator $T$ has the weak boundedness property if and only if for all normalized bump functions $\phi, \varphi$ associated with the unit ball, the set $\big\{\big\langle \phi, T^r_w\varphi\big\rangle : w \in \mathbb{R}, r > 0\big\}$ is bounded.

The operator $T$ is said to have the *weak R-boundedness property* provided that there is a constant $C$ such that, for every pair of normalized bump functions $\varphi, \phi$ associated with the unit ball, we have that $\sup_{v\in\mathbb{R}^N} \mathcal{R}\big(\big\{\big\langle \phi, T^{2^j}_v\varphi\big\rangle : j \in \mathbb{Z}\big\}\big) \le C$.

If $T$ is a singular integral operator with associated kernel $K$ and $\phi, \varphi$ are test functions with disjoint support, then

$$\begin{aligned}\big\langle \phi, T^r_w\varphi\big\rangle &= r^{-N}\big\langle \phi(\tfrac{\cdot-w}{r}), T[\varphi(\tfrac{\cdot-w}{r})]\big\rangle = r^{-N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(\tfrac{u-w}{r})K(u,v)\varphi(\tfrac{v-w}{r})dv\,du\\ &= r^N\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} \phi(u)K(ru+w, rv+w)\varphi(v)dv\,du.\end{aligned}$$

Therefore $T^r_w$ is also a singular integral operator associated with the kernel $K^r_w$ given by

$$K^r_w(u,v) = r^N K(ru+w, rv+w).$$

**Corollary 2.1.** *Let $a > 0$, $k \in \mathbb{N}$, $w \in \mathbb{R}^N$, and let $\varphi, \phi \in \mathcal{D}^0(\mathbb{R}^N)$ be normalized bump functions associated with $\overline{B}(0,a)$ and $\overline{B}(w, 2^k a)$ respectively. Let $T \in \mathrm{RCZO}_\nu$ satisfy the weak R-boundedness property.*
*(a) There is a constant $C_1 < \infty$ such that for all $v \in \mathbb{R}^N$*

$$\mathcal{R}\big(\big\{\big\langle \phi, T_v^{2^j} \varphi \big\rangle : j \in \mathbb{Z}\big\}\big) \leq C_1 \frac{1+k}{(a2^k)^N} \left(1 + \frac{|w|}{a2^k}\right)^{-N-\nu}.$$

*(b) If in addition $T(1) = 0$, then there are constants $C_2 < \infty$ and $\delta > 0$ such that for all $v \in \mathbb{R}^N$,*

$$\mathcal{R}\big(\big\{\big\langle \varphi, T_v^{2^j} \phi \big\rangle : j \in \mathbb{Z}\big\}\big) \leq C_2 (a2^k)^{-N-\nu} \left(1 + \frac{|w|}{a2^k}\right)^{-N-\delta}.$$

*Proof.* Let $v \in \mathbb{R}^n$ be fixed and $\tau_v = \{T_v^{2^j} : j \in \mathbb{Z}\}$. For $(S_k)_{k=1}^m \subseteq \tau_v$ consider the continuous linear operator

$$[S_k] : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N, \mathcal{L}(\mathrm{Rad}X, \mathrm{Rad}Y))$$

defined by

$$\big\langle \varphi, [S_k]\phi \big\rangle = \big[\big\langle \varphi, S_k \phi \big\rangle\big], \qquad \varphi, \phi \in \mathcal{S}(\mathbb{R}^N).$$

Then $[S_k]$ is in $\mathrm{CZO}_\nu$, with constant not depending on $v$ or the choice of the finite sequence $(S_k)$. Moreover, $[S_k]$ satisfies the weak boundedness property at 0, also with independent constant. Finally, if $T(1) = 0$, then also $[S_k](1) = 0$. So we can apply Lemma 2.1. □

# 3 Proof of Theorem 1.1

In the proof of Theorem 1.1, we proceed in a similar way as in [13].

We will decompose our operator $T$ into parts we can handle using Corollary 2.1. For this we use a resolution of unity from [13, Sect. 2]: Take $\Phi \in \mathcal{D}^0(\mathbb{R}^N)$ and $\Psi \in \mathcal{S}(\mathbb{R}^N)$ such that $\Phi$ is radial and real-valued, both $\widehat{\Phi}$ and $\widehat{\Psi}$ are non-negative, $\widehat{\Phi}(u) \geq 1$ for $\frac{1}{2} \leq |u| \leq 2$, $\widehat{\Psi}$ is supported in $\{\frac{1}{2} \leq |u| \leq 2\}$, and

$$\sum_{j \in \mathbb{Z}} \widehat{\Phi}(2^j u) \widehat{\Psi}(2^j u) = 1 \qquad \text{for all } u \in \mathbb{R}^N \setminus \{0\}.$$

We write $\Phi_j(u) := 2^{-Nj}\Phi(2^{-j}u)$, $\Psi_j(u) := 2^{-Nj}\Psi(2^{-j}u)$ and $P_j f := \Phi_j * f$, $Q_j f = \Psi_j * f$ for $f \in \mathcal{S}'(\mathbb{R}^N, X)$.

For $f \in \mathcal{S}(\mathbb{R}^N)$ and $j, k \in \mathbb{Z}$,

$$\begin{aligned}(P_j T P_k f)(u) &= (\Phi_j * T[\Phi_k * f])(u) = \left\langle \Phi_j(u - \cdot), T\left[\int_{\mathbb{R}} \Phi_k(\cdot - v) f(v) dv\right]\right\rangle \\ &= \int_{\mathbb{R}} \big\langle \Phi_j(u - \cdot), T[\Phi_k(\cdot - v)]\big\rangle f(v) dv = \int_{\mathbb{R}} K_{j,k}(u, v) f(v) dv\end{aligned}$$

where $K_{j,k}(u, v) := \big\langle \Phi_j(\cdot - u), T[\Phi_k(\cdot - v)]\big\rangle \in \mathcal{L}(X, Y)$. (Recall that $\Phi$ is radial.) We will consider the operators $T_{j,k}$ associated with the kernels $K_{j,k}$:

$$T_{j+k,j} f = \int_{\mathbb{R}^N} K_{j+k,j}(u, v) f(v) dv, \qquad f \in \mathcal{S}(\mathbb{R}^N, X).$$

One can show that $T_{j,k}$ can be extended to bounded linear operators from $L_p(X)$ to $L_p(Y)$ for all $p \in [1, \infty]$ [15].

*Proof.* First we consider the case that $k \in \mathbb{N}$. Then for $g \in \mathcal{S}(\mathbb{R}^N) \otimes Y'$ and $f \in \mathcal{S}(\mathbb{R}^N) \otimes X$, we can estimate

$$\begin{aligned}\left|\sum_{j\in\mathbb{Z}} \big\langle g, Q_{j+k} T_{j+k,j} Q_j f\big\rangle\right| &= \left|\sum_{j\in\mathbb{Z}} \big\langle 2^{(j+k)s} (T_{j+k,j})' Q_{j+k} g, 2^{-(j+k)s} Q_j f\big\rangle\right| \\ &= \left|\int_0^1 \left\langle \sum_{j\in\mathbb{Z}} r_j(t) 2^{(j+k)s} (T_{j+k,j})' Q_{j+k} g, \sum_{l\in\mathbb{Z}} r_l(t) 2^{-(l+k)s} Q_l f\right\rangle dt\right| \\ &\le \left(\int_0^1 \left\|\sum_{j\in\mathbb{Z}} r_j(t) 2^{(j+k)s} (T_{j+k,j})' Q_{j+k} g\right\|_{L_{p'}(X')}^{p'} dt\right)^{1/p'} \\ &\qquad \times \left(\int_0^1 \left\|\sum_{l\in\mathbb{Z}} r_l(t) 2^{-(l+k)s} Q_l f\right\|_{L_p(X)}^{p} dt\right)^{1/p}.\end{aligned}$$

The second factor is bounded by $C 2^{-ks} \|f\|_{\dot{H}^s_p}$ by Kahane's inequality and results from [16]. To estimate the first factor, we observe that

$$\begin{aligned}([T_{j+k,j}]' Q_{j+k} g)(v) &= \int_{\mathbb{R}^N} [K_{j+k,j}(u, v)]' (Q_{j+k} g)(u) du \\ &= \int_{\mathbb{R}^N} 2^{jN} [K_{j+k,j}(v + 2^j u, v)]' (Q_{j+k} g)(v + 2^j u) du.\end{aligned}$$

Since

$$2^{jN}[K_{j+k,j}(v+2^j u, v)]' = 2^{jN}\big\langle \Phi_{j+k}(\cdot - v - 2^j u), T[\Phi_j(\cdot - v)]\big\rangle',$$

and $\Phi_{j+k}(\cdot - v - 2^j u) = \tau_v \delta_{2^j} \tau_u \delta_{2^k} \Phi$, Corollary 2.1 (a) is applicable and yields (cf. Proposition 2.1)

$$\sup_{v\in\mathbb{R}^N} \mathcal{R}\big(\big\{2^{jN}[K_{j+k,j}(v+2^j u, v)]' : j \in \mathbb{Z}\big\}\big) \le C\frac{1+k}{(a2^k)^N}\left(1+\frac{|u|}{a2^k}\right)^{-N-\nu}.$$

On the other hand, by a result of Bourgain ( [10, Lemma 3.5]),

$$\left(\int_0^1 \left\|\sum_{j\in\mathbb{Z}} r_j(t) 2^{(j+k)s}(Q_{j+k}g)(\cdot + 2^j u)\right\|_{L_{p'}(Y')}^{p'} dt\right)^{1/p'} \le C\ln(2+2^{-k}|u|)\|g\|_{\dot{H}^s_p(Y')}.$$

So

$$\begin{aligned}
&\left(\int_0^1 \left\|\sum_{j\in\mathbb{Z}} r_j(t) 2^{(j+k)s}(T_{j+k,j})' Q_{j+k} g\right\|_{L_{p'}(X')}^{p'} dt\right)^{1/p'}\\
\le& C\frac{1+k}{(a2^k)^N}\int_{\mathbb{R}^N}\left(1+\frac{|u|}{a2^k}\right)^{-N-\nu} \ln(2+2^{-k}|u|)\, du\, \|g\|_{\dot{H}^{-s}_{p'}(Y')}\\
=& C(1+k)\int_{\mathbb{R}^N}(1+|u|)^{-N-\nu}\ln(2+a|u|)\, du\, \|g\|_{\dot{H}^{-s}_{p'}(Y')}.
\end{aligned}$$

Now let $-k \in \mathbb{N}$. Then for $g \in \mathcal{S}(\mathbb{R}^N)\otimes Y'$ and $f \in \mathcal{S}(\mathbb{R}^N)\otimes X$, we can estimate

$$\begin{aligned}
\left|\sum_{j\in\mathbb{Z}}\big\langle g, Q_{j+k}T_{j+k,j}Q_j f\big\rangle\right| &= \left|\sum_{j\in\mathbb{Z}}\big\langle 2^{(j+k)s}Q_{j+k}g, 2^{-(j+k)s}T_{j+k,j}Q_j f\big\rangle\right|\\
&\le \left(\int_0^1\left\|\sum_{j\in\mathbb{Z}} r_j(t)2^{(j+k)s}Q_{j+k}g\right\|_{L_{p'}(Y')}^{p'} dt\right)^{1/p'}\\
&\qquad\times\left(\int_0^1\left\|\sum_{l\in\mathbb{Z}} r_l(t)2^{-(l+k)s}T_{j+k,j}Q_l f\right\|_{L_p(Y)}^{p} dt\right)^{1/p}.
\end{aligned}$$

Now we proceed in a similar way. We use Corollary 2.1 (b) to show that

$$\sup_{u\in\mathbb{R}^N}\mathcal{R}(2^{(j+k)N}K_{j+k,j}(u, u+2^{j+k}v) : j \in \mathbb{Z}) \le C_2(a2^{|k|})^{-N-\nu}\left(1+\frac{|v|}{a2^{|k|}}\right)^{-N-\delta}.$$

(Observe that $\Phi_j(\cdot - u - 2^j v) = \tau_u \delta_{2^{j+k}} \tau_v \delta_{2^{-k}} \Phi$ and $\tau_v \delta_{2^{-k}} \Phi \sim \overline{B}(v, 2^{|k|}a)$.) Then, using again Bourgain's result, we obtain

$$\begin{aligned}
&\left( \int_0^1 \left\| \sum_{j \in \mathbb{Z}} r_j(t) 2^{(j+k)s} T_{j+k,j} Q_j f \right\|_{L_p(Y)}^p dt \right)^{1/p} \\
\leq & C (a 2^{|k|})^{-N-\nu} 2^{-ks} \int_{\mathbb{R}^N} \left( 1 + \frac{|v|}{a 2^{|k|}} \right)^{-N-\delta} \ln(2 + 2^k |u|) \, du \, \|f\|_{\dot{H}_p^s(X)} \\
= & C (a 2^{|k|})^{-\nu} 2^{-ks} \int_{\mathbb{R}^N} (1 + |v|)^{-N-\delta} \ln(2 + a|u|) \, du \, \|f\|_{\dot{H}_p^s(X)}.
\end{aligned}$$

Finally, putting everything together,

$$\begin{aligned}
\|Tf\|_{\dot{H}_p^s(Y)} &\leq \sum_{k \in \mathbb{Z}} \left\| \sum_{j \in \mathbb{Z}} Q_{j+k} T_{j+k,j} Q_j f \right\|_{\dot{H}_p^s(Y)} \\
&\leq C \sum_{k=-\infty}^{0} 2^{-|k|(n+\nu)} 2^{|k|s} \|f\|_{\dot{H}_p^s(X)} + C \sum_{k=1}^{\infty} 2^{-ks} (1+k) \|f\|_{\dot{H}_p^s(X)} \\
&\leq C \|f\|_{\dot{H}_p^s(X)}.
\end{aligned}$$

□

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