A remark on the $H^\infty$-calculus

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Abstract

If $A, B$ are sectorial operators on a Hilbert space with the same domain and range, and if $\|Ax\| \approx \|Bx\|$ and $\|A^{-1}x\| \approx \|B^{-1}x\|$, then it is a result of Auscher, McIntosh and Nahmod that if $A$ has an $H^\infty$--calculus then so does $B$. On an arbitrary Banach space this is true with the additional hypothesis on $B$ that it is almost R-sectorial as was shown by the author, Kunstmann and Weis in a recent preprint. We give an alternative approach to this result.


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1 Introduction

In [1] the authors showed that if $X$ is a Hilbert space and $A, B$ are sectorial operators with the same domain and range and satisfying estimates

$$\|Ax\| \approx \|Bx\| \quad x \in \text{Dom} \,(A)$$

(1.1)

and

$$\|A^{-1}x\| \approx \|B^{-1}x\| \quad x \in \text{Ran} \,(A)$$

(1.2)

then if one of $(A, B)$ admits an $H^\infty$--calculus then so does the other. Results of this type are useful in applications and were studied in [7] for arbitrary Banach spaces. In that paper, a similar result (Theorem 5.1) is proved under the additional hypothesis that $A$ is almost R-sectorial.

In this note we give a rather different approach to this result. We replace the almost R-sectoriality assumption by the technically weaker assumption of almost U-sectoriality, although this is probably not of great significance. However, our approach here is perhaps a little simpler. We also point out

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that some additional assumption is necessary in arbitrary Banach spaces; there are examples of sectorial operators $A, B$ satisfying (1.1) and (1.2) but such that only one has an $H^\infty$—calculus.

It is possible to consider estimates on fractional powers and our results can be extended in this direction (as in [7]); however to keep the exposition simple we will not discuss this point. We also point out that our approach is really based on an interpolation method, known as the Gustavsson-Peetre method [5] (see also [4]); but to avoid certain technicalities we have not made this explicit.

2 U-bounded collections of operators

Let $X$ be a complex Banach space. A family $\mathcal{T}$ of operators $T : X \to X$ is called $U$-bounded if there is a constant $C$ such that if $(x_j)_{j=1}^n \subset X, \ (x_j^*)_{j=1}^n \subset X^*$, $(T_j)_{j=1}^n \subset \mathcal{T},$

$$\sum_{j=1}^n |\langle T_j x_j, x_j^* \rangle| \leq C \sup_{|a_j|=1} \| \sum_{j=1}^n a_j x_j \| \sup_{|a_j|=1} \| \sum_{j=1}^n a_j x_j^* \|.$$ 

The best such constant $C$ is called the $U$-bound for $\mathcal{T}$ and is denoted $U(\mathcal{T})$. This concept was introduced in [8].

We recall that $\mathcal{T}$ is called $R$-bounded if there is a constant $C$ such that if $(x_j)_{j=1}^n \subset X, \ (T_j)_{j=1}^n \subset \mathcal{T},$

$$(\mathbb{E} \| \sum_{j=1}^n \epsilon_j T x_j \| ^2)^{1/2} \leq C (\mathbb{E} \| \sum_{j=1}^n \epsilon_j x_j \| ^2)^{1/2}.$$ 

Here $(\epsilon_j)_{j=1}^n$ is a sequence of independent Rademachers. The best such constant $C$ is called the $R$-bound for $\mathcal{T}$ and is denoted $R(\mathcal{T})$. An $R$-bounded family is automatically $U$-bounded [8].

We will need the following elementary property:

**Proposition 2.1.** Suppose $F : (0, \infty) \to \mathcal{L}(X)$ is a continuous function and that $\mathcal{T} = \{ F(t) : 0 < t < \infty \}$ is $U$-bounded with $U$-bound $U(F)$. Suppose $g \in L_1(\mathbb{R}, dt/t)$. Then the family of operators

$$G(s) = \int_0^{\infty} g(st) F(t) \frac{dt}{t} \quad 0 < s < \infty$$

is $U$-bounded with constant at most $U(F) \int_0^{\infty} |g(t)| dt/t$. 

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Proof. Suppose \((x_j)_{j=1}^n \subset X, (x_j^*)_{j=1}^n \subset X^*\) with
\[
\sup_{|a_j|=1} \| \sum_{j=1}^n a_j x_j \|, \sup_{|a_j|=1} \| \sum_{j=1}^n a_j x_j^* \| \leq 1.
\]

Then for \(s_1, \ldots, s_n \in \mathbb{R}\) we have
\[
\sum_{j=1}^n |\langle G(s_j)x_j, x_j^* \rangle| \leq \sum_{j=1}^n \int_0^\infty |g(t)| |\langle F(s_j^{-1}t)x_j, x_j^* \rangle| \frac{dt}{t}
\]
\[
\leq U(F) \int_0^\infty |g(t)| \frac{dt}{t}.
\]

\(\square\)

3 Sectorial operators

Let \(X\) be a complex Banach space and let \(A\) be a closed operator on \(X\). \(A\) is called sectorial if \(A\) has dense domain \(\text{Dom} (A)\) and dense range \(\text{Ran} (A) = \text{Dom} (A^{-1})\) and for some \(0 < \varphi < \pi\) the resolvent \((\lambda - A)^{-1}\) is bounded for \(|\arg \lambda| \geq \varphi\) and satisfies the estimate
\[
\sup_{|\arg \lambda| \geq \varphi} \| \lambda(\lambda - A)^{-1} \| < \infty.
\]

The infimum of such angles \(\varphi\) is denoted \(\omega(A)\).

Let \(\Sigma_\varphi\) be the open sector \(\{z \neq 0 : |\arg z| < \varphi\}\). If \(f \in H^\infty(\Sigma_\varphi)\) we say that \(f \in H_0^\infty(\Sigma_\varphi)\) if there exists \(\delta > 0\) such that \(|f(z)| \leq C \max(|z|^\delta, |z|^{-\delta})\).

For \(f \in H_0^\infty(\Sigma_\varphi)\) where \(\varphi > \omega(A)\) we can define \(f(A)\) by a contour integral, which converges as a Bochner integral in \(\mathcal{L}(X)\).

\[
f(A) = \frac{1}{2\pi i} \int_{\Gamma_\nu} f(\lambda)(\lambda - A)^{-1} d\lambda
\]

where \(\Gamma_\nu\) is the contour \(\{|t|e^{-i\nu x} : -\infty < \nu < \infty\}\) and \(\omega(A) < \nu < \varphi\). We can then estimate \(\|f(A)\|\) by
\[
\|f(A)\| \leq C_\varphi \int_{\Gamma_\nu} |f(\lambda)| \frac{|d\lambda|}{|\lambda|}.
\]
If we have a stronger estimate
\[ \|f(A)\| \leq C\|f\|_{H^\infty(\Sigma_\varphi)} \quad f \in H^\infty_0(\Sigma_\varphi) \]
then we say that \( A \) has an \( H^\infty(\Sigma_\varphi) \)-calculus; in this case we may extend the functional calculus to define \( f(A) \) for every \( f \in H^\infty(\Sigma_\varphi) \). The infimum of all such angles \( \varphi \) is denoted by \( \omega_H(A) \).

We will need a criterion for the existence of an \( H^\infty \)-calculus. It will be convenient to use the notation \( f_\lambda(z) = f(\lambda z) \) and to let \( u(z) = z(1 + z)^{-2} \) so that \( u \in H^\infty_0(\Sigma_\varphi) \) for all \( \varphi < \pi \). The following criterion goes back to [2] and [3]. A simple proof is given in [10].

**Proposition 3.1.** Let \( A \) be a sectorial operator and suppose \( 0 < \varphi < \pi \). Then the following are equivalent:

(i) There is a constant \( C \) so that
\[
\int_0^\infty |\langle u_\mu(tA)x, x^* \rangle| \frac{dt}{t} \leq C\|x\|\|x^*\| \quad |\arg \mu| = \varphi, \ x \in X, x^* \in X^*.
\]

(ii) \( A \) has an \( H^\infty \)-calculus with \( \omega_H(A) \leq \pi - \varphi \).

**Remark.** (i) is equivalent by the Maximum Modulus Principle to
\[
\int_0^\infty |\langle u_\mu(tA)x, x^* \rangle| \frac{dt}{t} \leq C\|x\|\|x^*\| \quad |\arg \mu| \leq \varphi, \ x \in X, x^* \in X^*.
\]

If \( A \) is sectorial we can define a closed operator \( A^\ast \) on \( X^* \) by \( A^\ast x^* = x^* \circ A \) with domain \( \text{Dom} (A^\ast) \) consisting of all \( x^* \) such that \( x \to x^*(Ax) \) extends to a bounded linear functional on \( X \). Then \( A^\ast \) need not be sectorial since it need not have dense domain or range. Note that
\[
\|A^\ast x^*\| = \sup_{\|A^{-1}x\| \leq 1} |\langle x, x^* \rangle| \quad x^* \in \text{Dom} (A^\ast)
\]
and
\[
\|(A^\ast)^{-1}x\| = \sup_{\|Ax\| \leq 1} |\langle x, x^* \rangle| \quad x^* \in \text{Ran} (A^\ast).
\]

Thus if \( A \) and \( B \) are sectorial operators satisfying (1.1) and (1.2) they will also satisfy \( \text{Dom} (A^\ast) = \text{Dom} (B^\ast), \text{Ran} (A^\ast) = \text{Ran} (B^\ast) \) and
\[
\|A^\ast x^*\| \approx \|B^\ast x^*\| \quad x^* \in \text{Dom} (A^\ast) \quad (3.1)
\]
and
\[ \|(A^*)^{-1}x^*\| \approx \|(B^*)^{-1}x^*\| \quad x^* \in \text{Ran} (A^*) \quad (3.2) \]

If \( A \) is a sectorial operator and \( \varphi > \omega(A) \) we shall that \( f \in H_0^\infty(\Sigma_\varphi) \) is U-bounded (respectively R-bounded) for \( A \) if the family of operators \( \{ f(tA) : 0 < t < \infty \} \) is a U-bounded (respectively R-bounded) collection.

**Proposition 3.2.** Suppose \( A \) has an \( H^\infty \)-calculus and that \( \varphi > \omega_H(A) \). Then for any \( f \in H_0^\infty(\Sigma_\varphi) \) we have that \( f \) is R-bounded (and thus U-bounded) for \( A \).

**Proof.** Suppose \( \omega(A) < \psi < \varphi \). Then the map \( \lambda \to f(\lambda A) \) is analytic on \( \Sigma_{\varphi-\psi} \) and extends continuously to the boundary. The operators \( \{ f(2^k t e^{\pm i(\varphi-\psi)} A) \}_{k \in \mathbb{Z}} \) are R-bounded (uniformly in \( 0 < t < \infty \)) by Theorem 3.3 of [8] and the result follows by Lemma 3.4 of the same paper. \( \square \)

Suppose \( A \) is a sectorial operator on \( X \) and \( \varphi > \omega(A) \). We will say that \( A \) is almost \( U \)-sectorial (respectively almost \( R \)-sectorial) if there is an angle \( \varphi \) such that the set of operators \( \{ \lambda A R(\lambda, A)^2 : |\arg \lambda| \geq \varphi \} \) is U-bounded (respectively R-bounded). If we define \( u(z) = z(1+z)^{-2} \) this implies that the functions \( u_{\lambda}(z) = u(\lambda z) \) are uniformly U-bounded (respectively uniformly R-bounded) for \( |\arg \lambda| \leq \pi - \varphi \). The infimum of such angles is denoted \( \tilde{\omega}_U(A) \).

By Lemma 3.4 of [8] this definition is equivalent to
\[ \tilde{\omega}_U(A) = \pi - \sup \{ \theta : u_{e^{\pm i\theta}} \text{ is U-bounded} \} \]
or, respectively
\[ \tilde{\omega}_R(A) = \pi - \sup \{ \theta : u_{e^{\pm i\theta}} \text{ is R-bounded} \}. \]

**Proposition 3.3.** Suppose \( A \) admits an \( H^\infty \)-calculus. Then \( A \) is almost \( R \)-sectorial (and hence almost \( U \)-sectorial) and \( \omega_U(A) \leq \tilde{\omega}_R(A) \leq \omega_H(A) \).

**Proof.** This follows from Proposition 3.2. \( \square \)

**Lemma 3.1.** Suppose \( A \) is almost \( U \)-sectorial and \( \varphi > \nu > \tilde{\omega}_U(A) \). Then there is a constant \( C = C(\varphi) \) so that if \( f \in H_0^\infty(\Sigma_\varphi) \) then \( f \) is U-bounded for \( A \) with U-bound
\[ U(f) \leq C \int_{\Gamma_\varphi} |f(\lambda)| \frac{|d\lambda|}{|\lambda|}. \]
Proof. Fix $\varphi > \psi > \nu > \omega_U(A)$. We may write $f(tA)$ in the form

$$f(tA) = \frac{1}{2\pi i} \int_{\Gamma_{\psi}} f(t\lambda)\lambda^{-1/2}A_{1/2}(\lambda - A)^{-1}d\lambda.$$ 

Therefore the result follows from Lemma 2.1 once we show that the two families of operators $\{h(e^{\pm i\theta}tA) : 0 < t < \infty\}$ are U-bounded where $\theta = \pi - \psi$ and $h(z) = z^{1/2}(1+z)^{-1}$.

Consider

$$g(z) = -i \log \frac{1 + iz^{1/2}}{1 - iz^{1/2}} - \pi \frac{z}{1 + z} \quad |\arg z| < \pi.$$ 

Then $g \in H_0^\infty(\Sigma_{\psi})$. Furthermore

$$g'(z) = z^{-1/2}(1+z)^{-1} - \pi(1+z)^{-2}.$$ 

Hence $g_{e^{\pm i\theta}} \in H_0^\infty(\Sigma_{\psi})$. For convenience we consider the case of $+\theta$. Thus if

$$T_t = \frac{1}{2\pi i} \int_{\Gamma_{\nu}} g(te^{i\theta}\lambda)A(\lambda - A)^{-2}d\lambda$$

the family of operators $\{T_t : 0 < t < \infty\}$ is U-bounded, again by Lemma 2.1. Now integration by parts shows that

$$T_t = \frac{te^{i\theta}}{2\pi i} \int_{\Gamma_{\nu}} ((te^{i\theta}\lambda)^{-1/2}(1 + te^{i\theta}\lambda)^{-1} - \pi(1 + te^{i\theta}\lambda)^{-2})\lambda(\lambda - A)^{-1}d\lambda$$

$$= \frac{1}{2\pi i} \int_{\Gamma_{\nu}} (h(te^{i\theta}\lambda) - \pi u(te^{i\theta}\lambda)) (\lambda - A)^{-1}d\lambda$$

$$= h(te^{i\theta}A) - \pi u(te^{i\theta}A).$$

Thus it follows that the family $\{h(te^{i\theta}A) : 0 < t < \infty\}$ is U-bounded. 

\[\square\]

4 The main results

If $A$ is sectorial then the space $\text{Dom}(A) \cap \text{Ran}(A)$ is a Banach space (densely) embedded into $X$ under the norm $\|Ax\| + \|A^{-1}x\| + \|x\|$; similarly $\text{Dom}(A^*) \cap \text{Ran}(A^*)$ is a Banach space embedded into $X^*$ under the norm $\|A^*x^*\| + \|(A^*)^{-1}x^*\| + \|x^*\|$.
Theorem 4.1. Suppose $A$ is a sectorial operator. In order that $A$ have an $H^\infty$-calculus with $\omega_H(A) = \varphi$ it is necessary and sufficient that:

(i) $A$ is almost $U$-sectorial with $\hat{\omega}_U(A) = \varphi$.

(ii) There exists a constant $C_1$ so that for each $x \in X$ there is a continuous function $\xi : (0, \infty) \to \text{Dom}(A) \cap \text{Ran}(A)$ such that

$$\| \sum_{k=-N}^{N} a_k 2^{jk} t^j A^j \xi(2^k t) \| \leq C_1 \|x\|, \quad j = -1, 0, 1, \quad |a_k| \leq 1, \quad N = 1, 2, \ldots, \quad 0 < t < \infty$$

and

$$\langle x, x^* \rangle = \int_0^\infty \langle \xi(t), x^* \rangle \frac{dt}{t} \quad x^* \in X^*.$$

(iii) There exists a constant $C_2$ so that for each $x^* \in X^*$ there is a continuous function $\xi^* : (0, \infty) \to \text{Dom}(A^*) \cap \text{Ran}(A^*)$ such that

$$\| \sum_{k=-N}^{N} a_k 2^{jk} t^j (A^j)^* \xi^*(2^k t) \| \leq C_2 \|x^*\|, \quad j = -1, 0, 1, \quad |a_k| \leq 1, \quad N = 1, 2, \ldots, \quad 0 < t < \infty$$

and

$$\langle x, x^* \rangle = \int_0^\infty \langle x, \xi^*(t) \rangle \frac{dt}{t} \quad x \in X.$$

Proof. Let us assume (i), (ii) and (iii). Suppose $|\theta| < \pi - \varphi$ and $\|x\| \leq 1$, $\|x^*\| \leq 1$. Let $\xi(t), \xi^*(t)$ be chosen according to (ii) and (iii). We define

$$\tilde{\xi}(t) = t A \xi(t) + t^{-1} A^{-1} \xi(t) + 2 \xi(t), \quad \tilde{\xi}^*(t) = t A^* \xi^*(t) + t^{-1} A^* \xi^*(t) + 2 \xi^*(t).$$

Thus we have

$$\| \sum_{k=-N}^{N} a_k 2^{jk} \tilde{\xi}(2^k t) \| \leq 3C_1, \quad j = -1, 0, 1, \quad |a_k| \leq 1, \quad N = 1, 2, \ldots, \quad 0 < t < \infty$$

and

$$\| \sum_{k=-N}^{N} a_k 2^{jk} \tilde{\xi}^*(2^k t) \| \leq 3C_2, \quad j = -1, 0, 1, \quad |a_k| \leq 1, \quad N = 1, 2, \ldots, \quad 0 < t < \infty.$$

Note that $\tilde{\xi} : (0, \infty) \to X$ and $\tilde{\xi}^* : (0, \infty) \to X^*$ are both continuous and

$$\xi(t) = u(t A) \tilde{\xi}(t) \quad 0 < t < \infty$$

$$\xi^*(t) = (u(t A))^* \tilde{\xi}^*(t) \quad 0 < t < \infty.$$
If \( \pi - |\arg \mu| > \nu > \varphi \) we have
\[
\int_0^\infty | < u_\mu(rA)x, x^* > | \frac{dr}{r} \leq \int_0^\infty \int_0^\infty \int_0^\infty |(u_\mu(rA)\xi(s), \xi^*(t))| \frac{dt}{t} \frac{ds}{s} \frac{dr}{r}
\]
\[
= \int_0^\infty \int_0^\infty \int_0^\infty |(u_\mu(rtA)\xi(st), \xi^*(t))| \frac{dt}{t} \frac{ds}{s} \frac{dr}{r}
\]
For fixed \( r, s \)
\[
\int_0^\infty |(u_\mu(rtA)\xi(st), \xi^*(t))| \frac{dt}{t} = \int_0^\infty |(u_\mu(rtA)u(stA)\tilde{\xi}(st), (u(tA)^*\tilde{\xi}^*(t))| \frac{dt}{t}
\]
\[
= \int_1^2 \sum_{j \in \mathbb{Z}} |(u_{r\mu}(2^j tA)u_s(2^j tA)u(2^j tA)\tilde{\xi}(s 2^j t), \tilde{\xi}^*(2^j t))| \frac{dt}{t}
\]
\[
\leq 9C_1 C_2 U(u_{r\mu}u_s)
\]
\[
\leq C \int_{\Gamma \nu} |u(r\mu\lambda)u(s\lambda)u(\lambda)| \frac{|d\lambda|}{|\lambda|},
\]
where \( C \) is constant independent of \( x, x^* \). Integrating over \( r, s \) gives:
\[
\int_0^\infty | < u_\mu(rA)x, x^* > | \frac{dr}{r} \leq C \left( \int_{\Gamma \nu} |u_\mu(\lambda)| \frac{|d\lambda|}{|\lambda|} \right) \left( \int_{\Gamma \nu} |u(\lambda)| \frac{|d\lambda|}{|\lambda|} \right)^2.
\]
This estimate shows, by Proposition 3.1, that \( A \) has an \( H^\infty \)-calculus with \( \omega_H(A) \leq \varphi \). Since \( \tilde{\omega}_U(A) \leq \omega_H(A) \) by Proposition 3.3 we have equality.

To complete the proof we show that if \( A \) has an \( \tilde{H}^\infty \)-calculus then (i), (ii) and (iii) hold and that \( \tilde{\omega}_U(A) \leq \omega_H(A) \).

To show (ii) and (iii) we observe that
\[
12 \int_0^\infty (u(tz))^2 \frac{dt}{t} = 1.
\]
Note that \( z^ju(z)^2 \in H_0^\infty(\Sigma \varphi) \) for \( j = -1, 0, 1 \). It follows easily if \( x \in X \) and \( x^* \in X^* \) then
\[
\xi(t) = 12u(tA)^2x, \quad \xi^*(t) = 12(u(tA)^2)^*x^*
\]
give the required functions.

For (i) observe that \( \tilde{\omega}_U(A) \leq \omega_H(A) \) but the first part of the proof shows equality.

\[\square\]
Theorem 4.2. Suppose $A$ and $B$ are sectorial operators such that $\text{Dom} (A) = \text{Dom} (B)$, $\text{Ran} (A) = \text{Ran} (B)$ and for a suitable constant $C$ we have

$$C^{-1}\|Ax\| \leq \|Bx\| \leq C\|Ax\| \quad x \in \text{Dom} (A)$$

and

$$C^{-1}\|A^{-1}x\| \leq \|B^{-1}x\| \leq C\|A^{-1}x\| \quad x \in \text{Ran} (A).$$

Suppose $A$ has an $H^\infty$-calculus. Then the following are equivalent:

(i) $B$ has an $H^\infty$-calculus with $\omega_B(B) = \varphi$.

(ii) $B$ is almost $U$-sectorial and $\hat{\omega}_U(B) = \varphi$.

Proof. This is now immediate from Theorem 4.1 using (3.1) and (3.2). \qed

If $X$ is a Hilbert space then the assumption that $B$ is almost $U$-sectorial is redundant and this reduces to the result of Auscher, McIntosh and Nahmod [1]. However, in general this assumption cannot be eliminated. It suffices to take a sectorial operator $A$ with an $H^\infty$-calculus with $\omega_B(A) > \omega(A)$. Such examples exist [6]; in fact examples are known on subspaces of $L_p$ when $1 < p < 2$ [9]. Now fix $\theta$ with $\pi - \omega_B(A) < \theta < \pi - \omega(A)$. Thus $e^{\pm it}A$ are sectorial with $\omega(e^{\pm it}A) \leq \omega(A)+\pi-\theta$. However if both have an $H^\infty$-calculus we would deduce that for a suitable constant $C$

$$\int_0^\infty |\langle u(t e^{\pm it}A)x, x^* \rangle| \frac{dt}{t} \leq C\|x\|\|x^*\| \quad x \in X, \ x^* \in X^*$$

which would imply that $\omega_B(A) \leq \pi - \theta$. This contradiction implies that at least one of $e^{\pm it}A$ fails to have an $H^\infty$-calculus. However if $B = e^{\pm it}A$ then (1.1) and (1.2) are trivially satisfied.

References


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