Commutator estimates in the operator $L^p$-spaces.

Denis Potapov and Fyodor Sukochev

Abstract

We consider commutator estimates in non-commutative (operator) $L^p$-spaces associated with general semi-finite von Neumann algebra. We discuss the difficulties which appear when one considers commutators with an unbounded operator in non-commutative $L^p$-spaces with $p \neq \infty$. We explain those difficulties using the example of the classical differentiation operator. MSC (2000): 46L52, 47B47. Received 31 July 2006 / Accepted 2 November 2006.

1 Introduction

Let us consider the spaces $L^p := L^p(\mathbb{R})$, $1 \leq p \leq \infty$, i.e. the spaces of all Lebesgue measurable functions with integrable $p$-th power, if $1 \leq p < \infty$ and which are essentially bounded, if $p = \infty$.

Let us fix a Lipschitz function $f : \mathbb{R} \mapsto \mathbb{C}$, i.e. a function for which there exists a constant $c_f > 0$, such that

$$|f(t_1) - f(t_2)| \leq c_f |t_1 - t_2|, \quad t_1, t_2 \in \mathbb{R}.$$

Let us take $x \in L^\infty$. We denote by $\frac{1}{i} \frac{dx}{dt}$ (or $x'$) the derivative of $x$, taken in the sense of tempered distributions. Let us recall that the chain rule says that, for every Lipschitz function $f$,

$$\frac{1}{i} \frac{d}{dt}(f(x)) = f'(x) \cdot \frac{1}{i} \frac{dx}{dt},$$

where $f'$ is the derivative of the tempered distribution $f$. If $\frac{1}{i} \frac{dx}{dt} \in L^p$ for some $1 \leq p \leq \infty$, then the latter identity implies that $\frac{1}{i} \frac{d}{dt}(f(x)) \in L^p$ as well.
and
\[ \left\| \frac{1}{i} \frac{d}{dt} (f(x)) \right\|_{L^p} \leq c_f \left\| \frac{1}{i} \frac{dx}{dt} \right\|_{L^p}, \]
where \( c_f \) is the Lipschitz constant of the function \( f \). The latter relation may serve as a criterion for a function \( f \) to be Lipschitz. Indeed, let us introduce the following definition.

A function \( f : \mathbb{R} \mapsto \mathbb{C} \) is called \( p \)-Lipschitz, for some \( 1 \leq p \leq \infty \), if and only if there is a constant \( c_{f,p} \) such that
\[ \left\| \frac{1}{i} \frac{d}{dt} (f(x)) \right\|_{L^p} \leq c_{f,p} \left\| \frac{1}{i} \frac{dx}{dt} \right\|_{L^p} \quad \text{(1.2)} \]
for every \( x \in L^\infty \) such that \( \frac{1}{i} \frac{dx}{dt} \in L^p. \)

In the classical (function) case we have the following result.

**Theorem 1.1.** Let \( f : \mathbb{R} \mapsto \mathbb{C} \) be a function. The following statements are equivalent:

a. the function \( f \) is Lipschitz;

b. the function \( f \) is \( p \)-Lipschitz, for some \( 1 \leq p \leq \infty \);

c. the function \( f \) is \( p \)-Lipschitz, for every \( 1 \leq p \leq \infty \).

**Proof.** The proof uses a standard argument based on integration by parts and using an approximation identity. We leave details to the reader. \( \square \)

We now introduce the class of \( p \)-Lipschitz functions in the general (operator) setting.

Let \( \mathcal{M} \) be a semi-finite von Neumann algebra acting on a Hilbert space \( \mathcal{H} \) and equipped with normal semi-finite faithful (n.s.f.) trace \( \tau \). We denote the operator norm by \( \| \cdot \| \). Let \( \tilde{\mathcal{M}} \) stands for the collection of all \( \tau \)-measurable operators, i.e. the collection of all linear operators \( x : \mathcal{D}(x) \mapsto \mathcal{H} \) affiliated.

---

1The latter inequality supposed to be read as follows. If \( x \in L^\infty \) and the derivative \( \frac{1}{i} \frac{dx}{dt} \) is a function in \( L^p \), then the composition \( f(x) \) is a tempered distribution such that the derivative \( \frac{1}{i} \frac{d}{dt} (f(x)) \) is a function in \( L^p \) and the inequality (1.2) holds.
with $\mathcal{M}$ such that for every $\epsilon > 0$ there is a projection $p_\epsilon \in \mathcal{M}$ with $\tau(1-p_\epsilon) < \epsilon$ and $p_\epsilon(\mathcal{H}) \subseteq \mathcal{D}(x)$. The class $\tilde{\mathcal{M}}$ is a $*$-algebra. Furthermore, there is a topology on the algebra $\tilde{\mathcal{M}}$, which is called the measure topology. This topology is defined by the collection of neighborhoods of the origin $\{N_{\epsilon,\delta}\}_{\epsilon,\delta > 0}$, where $N_{\epsilon,\delta}$ consists of all linear operators $x : \mathcal{D}(x) \mapsto \mathcal{H}$ affiliated with $\mathcal{M}$ such that there is a projection $p_\epsilon \in \mathcal{M}$ for which $\tau(1-p_\epsilon) < \epsilon$ and $\|xp\| \leq \delta$. The class $\tilde{\mathcal{M}}$ equipped with the measure topology is a complete topological algebra. We refer the reader to [19, 12, 15] for more details.

We now construct the non-commutative $L^p$-spaces $\mathcal{L}^p := L^p(\mathcal{M}, \tau)$, $1 \leq p \leq \infty$, see [10] and references therein. Indeed, the space $\mathcal{L}^p$, is defined by

$$\mathcal{L}^p := \{x \in \tilde{\mathcal{M}} : \|x\|_{\mathcal{L}^p} < \infty\}$$

where

$$\|x\|_{\mathcal{L}^p} := \tau\left((x^*x)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \quad \text{when } p < \infty,$$

$$\|x\|_{\mathcal{L}^\infty} := \|x\|, \quad x \in \tilde{\mathcal{M}}.$$  

The spaces $\mathcal{L}^p$ resemble their classical counterparts. The spaces $\mathcal{L}^\infty$ coincides with $\mathcal{M}$ and the space $\mathcal{L}^1$ is the predual of the algebra $\mathcal{M}$. Furthermore, the Hölder inequality is valid in the spaces $\mathcal{L}^p$, that is

$$\|xy\|_{\mathcal{L}^p} \leq \|x\|_{\mathcal{L}^p} \|y\|_{\mathcal{L}^q}, \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{s}, \quad 1 \leq p, q, s \leq \infty. \quad (1.3)$$

**Remark 1.1.** Let us mention two basic examples of the above construction.

a. The algebra of all complex $n \times n$-matrices acting on the sequence space $\ell^2_n$ which is usually denoted by $B(\ell^2_n)$ equipped with the standard trace $Tr$, $n \in \mathbb{N}$. The algebra of $\tau$-measurable operators coincides with $B(\ell^2_n)$ in this case. The space $\mathcal{L}^p$, $1 \leq p \leq \infty$ consists of all $n \times n$-matrices and the norm $\| \cdot \|_{\mathcal{L}^p}$ is given by the $p$-th Schatten-von Neumann norm, i.e. $\|x\|_{\mathcal{L}^p} = \|s(x)\|_{\ell^p}$, where $s(x)$ is the sequence of singular values of the operator $x$ counted with multiplicities, see [13],

113
b. The algebra \( M = L^\infty \) acting on the space \( L^2 \), where every function \( x \in L^\infty \) is considered as a multiplication operator, i.e.

\[
x(\xi) := x \cdot \xi, \quad \xi \in L^2.
\]

The trace \( \tau \) on the algebra \( L^\infty \) is given by Lebesgue integration. The algebra \( \tilde{M} \) consists of all Lebesgue measurable functions which are bounded except on a set of finite measure. The spaces \( L^p \) turn into the classical \( L^p \)-spaces \( L^p(\mathbb{R}) \).

Let us fix a linear self-adjoint operator \( D : \mathcal{D}(D) \mapsto \mathcal{H} \) (not necessary affiliated with \( M \)) such that

(D1) \( e^{itD} x e^{-itD} \in L^\infty \), whenever \( x \in L^\infty \), \( t \in \mathbb{R} \);

(D2) \( \tau(e^{itD} x e^{-itD}) = \tau(x) \), whenever \( x \in L^1 \cap L^\infty \).

Let us recall that the subspace \( \mathcal{D} \subseteq \mathcal{D}(D) \) is called a core of the operator \( D \) if and only if the closure \( \overline{(D|_{\mathcal{D}})} \) coincides with \( D \).

**Definition 1.1.** Let \( x \in M \). We say that the commutator \( [D, x] \) is defined and belongs to \( L^p \), for some \( 1 \leq p \leq \infty \) if and only if there is a core \( \mathcal{D} \subseteq \mathcal{D}(D) \) of the operator \( D \) such that \( x(\mathcal{D}) \subseteq \mathcal{D}(D) \) and the operator \( D x - x D \), initially defined on \( \mathcal{D} \), is closable, in which case the closure \( \overline{D x - x D} \) belongs to \( L^p \). In this case, the symbol \( [D, x] \) stands for the closure \( \overline{D x - x D} \).

In the case \( p = \infty \), we have the following observation.

**Lemma 1.1** ([5, Proposition 3.2.55]). Let \( D : \mathcal{D}(D) \mapsto \mathcal{H} \) be a self-adjoint linear operator and \( x \in M \). If \( [D, x] \) is bounded, then \( x(\mathcal{D}(D)) \subseteq \mathcal{D}(D) \).

The relation \( x(\mathcal{D}(D)) \subseteq \mathcal{D}(D) \) in the cases \( 1 \leq p < \infty \) may fail as it is shown in the example with the differentiation operator below. On the other hand, the weaker relation \( x(\mathcal{D}) \subseteq \mathcal{D}(D) \) for some core \( \mathcal{D} \subseteq \mathcal{D}(D) \) is much easier to attack and, more importantly, is sufficient for the applications we study; see Theorems 3.2, 3.3 and 3.4.

By analogy with the beginning of the section, we introduce the following definition.
**Definition 1.2.** A function $f : \mathbb{R} \to \mathbb{C}$ is called $p$-Lipschitz for some $1 \leq p \leq \infty$ (with respect to the couple $(\mathcal{M}, \tau)$ and the operator $D$) if and only if there is a constant $c_{f,p}$ such that $[D, f(x)] \in \mathcal{L}^p$ and

$$
\| [D, f(x)] \|_{\mathcal{L}^p} \leq c_{f,p} \| [D, x] \|_{\mathcal{L}^p},
$$

for every $x = x^* \in \mathcal{M}$ such that $[D, x] \in \mathcal{L}^p$.

The present note is concerned with the following problem.

**Problem 1.1.** Which the function $f : \mathbb{R} \to \mathbb{C}$ is $p$-Lipschitz?

Similar problems have been under considerable investigation over a long period. We refer the reader to the works [7, 14, 1, 2, 3, 4, 10, 8, 20, 17].

In this note, we shall show some sufficient criteria for a function to be $p$-Lipschitz stated in terms of (scalar) smoothness properties of this function. The main results, Theorems 3.2, 3.3 and 3.4, are essentially proved in [16]. The purpose of the present note is to give an additional insight in the matter and explain some interesting points about the construction of commutators in the non-commutative $L^p$-spaces with respect to atomless algebras using the example of the classical differentiation operator.

## 2 Commutators with the differentiation operator $\frac{1}{i} \frac{d}{dt}$

In the present section, we fix $\mathcal{M} = L^\infty$ (see Remark 1.1) and $\tau(\cdot) = \int (\cdot) \, dt$. Let us consider the operator $D := \frac{1}{i} \frac{d}{dt} : \mathscr{D}(D) \to L^2$ with the domain given by

$$
\mathscr{D}(D) := \left\{ \xi \in L^2 : \frac{1}{i} \frac{d\xi}{dt} \in L^2 \right\}.
$$

The operator $D$ is self-adjoint and the unitary group $\{e^{itD}\}_{t \in \mathbb{R}}$ is given by the translations, i.e.

$$
e^{itD}(\xi)(s) = \xi(s + t), \quad s \in \mathbb{R}.
$$

(2.1)
Consequently,
\[
(e^{itD}xe^{-itD}\xi)(s) = (xe^{-itD}\xi)(s + t) = x(s + t)(e^{-itD}\xi)(s + t) = x(s + t)\xi(s), \quad \xi \in L^2, \; t, s \in \mathbb{R}.
\]

Therefore, for every \( x \in L^\infty \), the operator \( e^{itD}xe^{-itD} \) is a multiplication operator on \( L^2 \) induced by the translated function \( x(\cdot + t) \in L^\infty \). The latter readily yields the fact that the operator \( D \) satisfies (D1)–(D2).

Let \( x \in L^\infty \) be such that \( [D, x] \in L^p \), \( 1 \leq p \leq \infty \). By Definition 1.1, there is a core \( \mathcal{D} \subseteq \mathcal{D}(D) \) such that \( x(\mathcal{D}) \subseteq \mathcal{D}(D) \) and
\[
(Dx - xD)(\xi) = \frac{1}{i} \frac{d}{dt}(x \cdot \xi) - x \cdot \frac{1}{i} \frac{d\xi}{dt} = \frac{1}{i} \frac{dx}{dt} \cdot \xi, \quad \xi \in \mathcal{D}. \tag{2.2}
\]
Thus, if the derivative \( \frac{1}{i} \frac{dx}{dt} \) is a function, then the operator \( Dx - xD \) acts as a multiplication operator on \( \mathcal{D} \). Clearly, \( Dx - xD \) is closable and the closure \( \overline{Dx - xD} \in L^p \) if and only if \( \frac{1}{i} \frac{dx}{dt} \in L^p \).

In other words, by Definition 1.1, the operator \( [D, x] \) belongs to \( L^p \), \( 1 \leq p \leq \infty \), for a given \( x \in L^\infty \) if and only if there is a core \( \mathcal{D} \subseteq \mathcal{D}(D) \) such that
\[
x(\mathcal{D}) \subseteq \mathcal{D}(D) \quad \text{and} \quad \frac{1}{i} \frac{dx}{dt} \in L^p. \tag{2.3}
\]

Furthermore, let us note that the inclusion \( x(\mathcal{D}) \subseteq \mathcal{D}(D) \) means that for every function \( \xi \in \mathcal{D} \), the function \( x \cdot \xi \) is differentiable and
\[
\frac{1}{i} \frac{d}{dt}(x \cdot \xi) \in L^2. \tag{2.4}
\]
Since \( x \cdot \frac{1}{i} \frac{d\xi}{dt} \in L^2 \), for every \( \xi \in \mathcal{D}(D) \), \( x \in L^\infty \), it follows from the last identity in (2.2) that (2.4) is equivalent to \( \frac{1}{i} \frac{dx}{dt} \cdot \xi \in L^2 \). The latter means that, if \( \mathcal{D} \subseteq \mathcal{D}(D) \) is a core, then
\[
x(\mathcal{D}) \subseteq \mathcal{D}(D) \iff \frac{1}{i} \frac{dx}{dt}(\mathcal{D}) \subseteq L^2. \tag{2.5}
\]
Thus, we can restate (2.3) as \( [D, x] \in L^p \), \( 1 \leq p \leq \infty \) for a given \( x \in L^\infty \) if and only if there exists a core \( \mathcal{D} \subseteq \mathcal{D}(D) \) such that
\[
\frac{1}{i} \frac{dx}{dt}(\mathcal{D}) \subseteq L^2 \quad \text{and} \quad \frac{1}{i} \frac{dx}{dt} \in L^p. \tag{2.6}
\]
Thus, in general, a verification of the statement \([D, x] \in L^p, 1 \leq p < \infty\) consists of two steps whose nature is quite different. A verification of the condition \(\frac{1}{i} \frac{dx}{dt} \in L^p\) is carried out in the literature almost exclusively via methods related to Banach space geometry (Schur multipliers, double operator integrals, vector-valued Fourier multipliers [9, 6, 11, 10]). However, the first condition in (2.6) has an operator-theoretical nature and does not correspond to the methods listed above. We outline an approach to this problem when \(D = \frac{1}{i} \frac{dx}{dt}\).

Let us first consider \([D, x] \in L^p\) when \(2 \leq p < \infty\). We shall show that in the present setting, the required core \(\mathcal{D}\) appears very naturally due to the fact that the underlying Hilbert space \(L^2\) possesses the additional Banach structure induced by the \(L^p\)-scale. Indeed, let us set

\[
\mathcal{D} := \mathcal{D}(D) \cap L^q, \quad \text{where} \quad \frac{1}{2} = \frac{1}{p} + \frac{1}{q}. \tag{2.7}
\]

Clearly, the Hölder inequality implies that (2.6) holds for the subset \(\mathcal{D}\) and any \(x \in L^\infty\) such that \(\frac{1}{i} \frac{dx}{dt} \in L^p\). We shall verify that \(\mathcal{D}\) is a core of \(D\) in Theorem 3.3 below. What we would like to emphasize is that the core \(\mathcal{D}\) is found purely by a Banach space construction. Thus, we see that in the case \(2 \leq p < \infty\), we have

\[
[D, x] \in L^p \iff \frac{1}{i} \frac{dx}{dt} \in L^p.
\]

Finally, we comment on the case \(1 \leq p < 2\). Here, the problem of finding the core \(\mathcal{D}\) satisfying the first condition in (2.6) cannot be resolved by a purely Banach space approach as in (2.7) above. Indeed, let \(C(\mathbb{R})\) be the class of all continuous functions on \(\mathbb{R}\). We note that \(\mathcal{D}(D) \subseteq C(\mathbb{R})\), [18, Theorem 2, p. 124]. If we now consider the function \(x \in L^\infty\) such that

\[
\frac{1}{i} \frac{dx}{dt} \in L^p, \quad \text{but} \quad \frac{1}{i} \frac{dx}{dt} \not\in L^2_{loc},
\]

then

\[
\frac{1}{i} \frac{dx}{dt} \cdot \xi \not\in L^2, \quad \text{for every} \quad \xi \in \mathcal{D}(D), \xi \not\equiv 0.
\]
That means that despite the fact that the derivative $\frac{1}{i} \frac{dx}{dt}$ exists in the sense of tempered distributions and belongs to $L^p$, there is no core such that the commutator $[D, x]$ may be defined according to Definition 1.1.

3 Main result

As we have seen in the example with the operator $D = \frac{1}{i} \frac{d}{dt}$, a meaningful resolution of Problem 1.1 requires locating a core $\mathcal{D}$ of the operator $D$ satisfying the first condition in (2.5). As we indicated in that example, a possible candidate on the role of such $\mathcal{D}$ is the space

$$\mathcal{D}(D) \cap \mathcal{L}^1 \cap \mathcal{L}^\infty.$$ 

Unfortunately, in general, the domain $\mathcal{D}(D) \subseteq \mathcal{H}$ may have an empty intersection with the space $\mathcal{L}^1 \cap \mathcal{L}^\infty$. We shall show below that this is not the case when $\mathcal{M}$ is taken in the left regular representation (see Theorem 3.3).

3.1 The left regular representation

Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with n.s.f. trace $\tau$ and let $\mathcal{L}^p := L^p(\mathcal{M}, \tau)$, $1 \leq p \leq \infty$ be the corresponding non-commutative $L^p$-spaces.

Let us consider the mapping $L : \mathcal{M} \mapsto B(\mathcal{L}^2)$, given by $L(x) := L_x$, $x \in \mathcal{M}$, where the operator $L_x \in B(\mathcal{L}^2)$ is given by

$$L_x(\xi) := x \cdot \xi, \quad \xi \in \mathcal{L}^2.$$ 

The image $\mathcal{M}_L := L(\mathcal{M})$ is a von Neumann algebra acting on $\mathcal{L}^2$. The mapping $L$ is a $*$-isomorphism between the algebras $\mathcal{M}$ and $\mathcal{M}_L$. The algebra $\mathcal{M}_L$ is equipped with n.s.f trace $\tau_L := \tau \circ L^{-1}$. With this definition of $\tau_L$, the mapping $L$ becomes a trace preserving $*$-isomorphism. Consequently, it extends to a $*$-homeomorphism between topological $*$-algebras $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_L := (\mathcal{M}_L)^\sim$. We shall denote the latter extension by $L$ also. Alternatively, the mapping $L : \tilde{\mathcal{M}} \mapsto \tilde{\mathcal{M}}_L$ is given by $L(x) = L_x$, where $L_x : \mathcal{D}(L_x) \mapsto \mathcal{L}^2$ is an
operator given by
\[ \mathcal{D}(L_x) = \{ \xi \in \mathcal{L}^2 : x \cdot \xi \in \mathcal{L}^2 \} \text{ and } L_x(\xi) = x \cdot \xi, \quad \xi \in \mathcal{D}(L_x). \]

Since the mapping \( L : \tilde{\mathcal{M}} \mapsto \tilde{\mathcal{M}}_L \) is trace preserving, its restriction to the space \( \mathcal{L}^p \) becomes an isometry between the spaces \( \mathcal{L}^p \) and \( \mathcal{L}^p_L := L^p(\mathcal{M}_L, \tau_L) \), for every \( 1 \leq p \leq \infty \).

### 3.1.1 Approximation of the commutator \([D, x]\)

In the present section we shall consider the construction of an approximation of the commutator \([D, x]\) by means of the corresponding unitary group \( \{ e^{itD} \}_{t \in \mathbb{R}} \).

For illustration, let us again consider the example of the differentiation operator. If \( x \in L^\infty(\mathbb{R}) \) and \( D = \frac{1}{i} \frac{d}{dt} \), then we have the well known relations

\[
x(t + s) - x(s) = i \int_0^t \frac{1}{i} \frac{d}{dt} (s + \tau) d\tau, \quad t, s \in \mathbb{R}, \quad (3.1)
\]

\[
\frac{1}{i} \frac{d}{dt} (s) = \lim_{t \to 0} \frac{x(s + t) - x(s)}{it}. \quad (3.2)
\]

An operator version of (3.1) and (3.2), in the case \( p = \infty \) may be found in [5, Section 3.2.5]

**Theorem 3.1.** Let \( D : \mathcal{D}(D) \mapsto \mathcal{H} \) be a self-adjoint linear operator, satisfying (D1)–(D2) and let \( x \in \mathcal{M} \). If \([D, x] \in \mathcal{L}^\infty \), then

a. \( e^{itD}x e^{-itD} - x = i \int_0^t e^{isD}[D, x] e^{-isD} ds, \quad t \in \mathbb{R}; \)

b. \( \left\| \frac{e^{itD}x e^{-itD} - x}{t} \right\| \leq \|[D, x]\| \mathcal{L}^\infty; \)

c. \( \lim_{t \to 0} \frac{e^{itD}x e^{-itD} - x}{t} = i[D, x]; \)

where the integral and the limit converge with respect to the weak operator topology.
The natural framework to deal with the commutator \([D, x] \in \mathcal{L}^p\) when \(p < \infty\) is the setting of the left regular representation. Thus, from now on, we consider the algebra \(\mathcal{M}_L\) with the n.s.f. trace \(\tau_L\). We denote by \(\mathcal{L}^p_L := L^p(\mathcal{M}_L, \tau_L)\), \(1 \leq p \leq \infty\) the corresponding non-commutative \(L^p\)-space.

We shall discuss the extension of Theorem 3.1 to the spaces \(\mathcal{L}^p_L\), \(1 \leq p < \infty\).

To explain the next step, let us note that the proof of Theorem 3.1 crucially depends on the fact that the domain \(\mathcal{D}(D)\) where the commutator \([D, x]\), initially defined, according to Definition 1.1 and Lemma 1.1, is invariant with respect to the group \(\{e^{itD}\}_{t \in \mathbb{R}}\). On the other hand, the core \(\mathcal{D}\) in Definition 1.1 lacks this invariance when \(p < \infty\). We now extend Definition 1.1.

**Definition 3.1.** Let \(x \in \mathcal{M}_L\) and let \(D : \mathcal{D}(D) \mapsto \mathcal{L}^2\) be a linear self-adjoint operator. We shall say that the commutator \([D, x]\) is defined and belongs to \(\mathcal{L}^p_L\), for some \(1 \leq p \leq \infty\) if and only if

a. there is a core \(\mathcal{D} \subseteq \mathcal{L}^1 \cap \mathcal{L}^\infty\) of the operator \(D\) such that \(e^{itD}(\mathcal{D}) \subseteq \mathcal{D}\), for every \(t \in \mathbb{R}\), and \(x(\mathcal{D}) \subseteq \mathcal{D}(D)\);

b. the operator \(Dx - xD\), initially defined on \(\mathcal{D}\), is closable;

c. the closure \((Dx - xD)\) belongs to \(\mathcal{L}^p\). In this case, the symbol \([D, x]\) stands for the closure \((Dx - xD)\).

The next result provides an extension of Theorem 3.1 over the spaces \(\mathcal{L}^p_L\), \(1 \leq p < \infty\).

**Theorem 3.2.** Let \(D : \mathcal{D}(D) \mapsto \mathcal{L}^2\) be a self-adjoint linear operator, satisfying (D1)–(D2) and let \(x \in \mathcal{M}_L\). If \([D, x] \in \mathcal{L}^p_L\), for some \(1 \leq p < \infty\), then

\[
\begin{align*}
a. & \quad e^{itD}xe^{-itD} - x = i \int_0^t e^{isD}[D, x]e^{-isD} ds, t \in \mathbb{R}; \\
b. & \quad \left\| \frac{e^{itD}xe^{-itD} - x}{t} \right\|_{\mathcal{L}^p_L} \leq \|[D, x]\|_{\mathcal{L}^p_L};
\end{align*}
\]
c. \[ \lim_{t \to 0} \frac{e^{itD}xe^{-itD} - x}{t} = i[D, x]; \]

where the integral and the limit converge with respect to the norm topology in \( L^p_L \).

### 3.1.2 Commutator estimates

Let us recall that we have fixed the pair \((M, \tau)\) and we consider the left regular representation \((M_L, \tau_L)\). Let \( D : \mathcal{D}(D) \mapsto L^2 \) be a linear self-adjoint operator satisfying (D1)–(D2).

Let us again consider the subspace

\[ \mathcal{D}_0(D) := \mathcal{D}(D) \cap L^1 \cap L^\infty \subseteq L^2. \quad (3.3) \]

Unfortunately, in general case when the operator \( D \) is not affiliated with the algebra \( M_L \), there is no hope to expect that the latter subspace will be a core of the operator \( D \). To single out the class of operators \( D \) for which the subspace \( \mathcal{D}_0(D) \) is a core let us introduce the assumption

(D3) the unitary group \( \{e^{itD}\}_{t \in \mathbb{R}} \) is a \( \sigma(L^1 \cap L^\infty, L^1 + L^\infty) \)-continuous group of contractions in the space \( L^1 \cap L^\infty \).

If \( D = \frac{1}{i} \frac{d}{dt} \), then the assumption (D3) is clearly satisfied, since \( \{e^{itD}\}_{t \in \mathbb{R}} \) is a group of translations, see (2.1). Also, if \( D \) is affiliated with \( M_L \), then (D3) holds, due to the fact that \( e^{itD} = L(u_t) \), for every \( t \in \mathbb{R} \), where \( \{u_t\}_{t \in \mathbb{R}} \subseteq M \) is a group of unitaries.

**Theorem 3.3.** If \( D : \mathcal{D}(D) \mapsto L^2 \) is a linear self-adjoint operator satisfying (D1)–(D3), then the subspace \( \mathcal{D}_0(D) \) is a core of the operator \( D \).

To state the main result, let us first recall that a Borel function \( f : \mathbb{R} \mapsto \mathbb{C} \) is called of bounded \( \beta \)-variation, \( 1 \leq \beta < \infty \) if and only if

\[ \|f\|_{V_\beta} := \sup \left[ \sum_{j=-\infty}^{+\infty} |f(t_j) - f(t_{j+1})|^\beta \right]^{\frac{1}{\beta}} < \infty, \quad (3.4) \]
where the supremum is taken over all possible increasing two-sided sequences \( \{t_j\}_{j=-\infty}^{+\infty} \subseteq \mathbb{R} \). \( V_\beta \) will stand for the class of all functions of bounded \( \beta \)-variation, \( 1 \leq \beta < \infty \). The class \( V_\beta \) is equipped with the norm \( \| \cdot \|_{V_\beta} \) defined in (3.4). We also define \( V_\infty \) to be the collection of all bounded Borel functions equipped with the uniform norm.

Let us next state the main result of the text. Its proof consists of a combination of the technique developed in [8] with the approach explained above. In the special case \( \mathcal{M} = B(\mathcal{H}) \), the result which follows gives an alternative (and simpler) proof of [4, Example III]. Let us note that the result distinguishes two different cases \( p < 2 \) and \( p \geq 2 \) as discussed in the example of Section 2.

**Theorem 3.4.** Let \( D : \mathcal{D}(D) \mapsto L^2 \) be a linear self-adjoint operator satisfying \((D1)-(D3)\) and let \( x = x^* \in \mathcal{M}_L \). Let a function \( f : \mathbb{R} \mapsto \mathbb{C} \) be such that \( f' \in V_\beta \) for some \( 1 \leq \beta \leq \infty \).

a. For every \( 2 \leq p < \frac{2\beta}{\beta - 1} \) there is a constant \( c'_p \) such that if \( [D, x] \in \mathcal{L}_p^\beta \), then \( [D, f(x)] \in \mathcal{L}_p^\beta \) and

\[
\|[D, f(x)]\|_{\mathcal{L}_p^\beta} \leq c'_p \|f'\|_{V_\beta} \|[D, x]\|_{\mathcal{L}_p^\beta}.
\]

b. For every \( \frac{2\beta}{\beta + 1} < p < 2 \) there is a constant \( c''_p \) such that if \( [D, x] \in \mathcal{L}_p^\beta \cap \mathcal{L}_2^\beta \), then \( [D, f(x)] \in \mathcal{L}_p^\beta \cap \mathcal{L}_2^\beta \) and

\[
\|[D, f(x)]\|_{\mathcal{L}_p^\beta} \leq c''_p \|f'\|_{V_\beta} \|[D, x]\|_{\mathcal{L}_p^\beta}.
\]

Now we state the answer to Problem 1.1 in the setting of the left regular representation.

**Theorem 3.5.** Any function \( f : \mathbb{R} \mapsto \mathbb{C} \) such that \( f' \in V_\beta \), for some \( 1 \leq \beta \leq \infty \) is \( p \)-Lipschitz for every \( 2 \leq p < \frac{2\beta}{\beta - 1} \), with respect to any operator \( D : \mathcal{D}(D) \mapsto L^2 \) and every semi-finite von Neumann algebra \((\mathcal{M}_L, \tau_L)\).
References


**Denis Potapov**, School of Informatics and Engineering, Flinders University of South Australia, Bedford Park, 5042, Adelaide, SA, Australia. denis.potapov@flinders.edu.au

**Fyodor Sukochev**, School of Informatics and Engineering, Flinders University of South Australia, Bedford Park, 5042, Adelaide, SA, Australia. sukochev@infoeng.flinders.edu.au