Symmetry via Lie algebra cohomology

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ABSTRACT. The Killing operator on a Riemannian manifold is a linear differential operator on vector fields whose kernel provides the infinitesimal Riemannian symmetries. The Killing operator is best understood in terms of its prolongation, which entails some simple tensor identities. These simple identities can be viewed as arising from the vanishing of certain Lie algebra cohomologies. The point is that this case provides a model for other more complicated operators similarly concerned with symmetry.

1. Disclaimer

The results in this article are not widely known but are implicitly already contained in [**BCEG, CD, CSS**], for example. The object of this short exposition is to introduce the method, by means of familiar examples, to a wider audience.

2. Notation

The notation in this article follows the standard index conventions of differential geometry. Precisely, we shall follow Penrose's abstract index notation [**PR**] in which tensors are systematically adorned with indices to specify their type. For example, vector fields are denoted with an upper index X^a whilst 2-forms have 2 lower indices ω_{ab}. The natural contraction between them is denoted by repeating an index $X^a \omega_{ab}$ in accordance with the Einstein summation convention. Round brackets are used to denote symmetrisation over the indices they enclose whilst square brackets are used to denote skewing, e.g.

$$\psi_{[abc]d} = \tfrac{1}{6}[\psi_{abcd} + \psi_{bcad} + \psi_{cabd} - \psi_{bacd} - \psi_{acbd} - \psi_{cbad}].$$

3. The Levi-Civita connection

Suppose g_{ab} is a Riemannian metric. The Levi-Civita connection ∇_a associated with g_{ab} is characterised by the following well-known properties

- ∇_a is torsion-free,
- $\nabla_a g_{bc} = 0$.

Its existence and uniqueness boils down to a tensor identity as follows. Choose D_a, any torsion-free connection. Any other must be of the form

$$\nabla_a \phi_b = D_a \phi_b - \Gamma_{ab}{}^c \phi_c$$

for some tensor $\Gamma_{ab}{}^c = \Gamma_{(ab)}{}^c$ and then $\nabla_a g_{bc} = 0$ if and only if

$$0 = D_a g_{bc} - \Gamma_{ab}{}^d g_{dc} - \Gamma_{ac}{}^d g_{bd} = D_a g_{bc} - \Gamma_{abc} - \Gamma_{acb},$$

where we are using the metric g_{ab} to 'lower indices' in the usual fashion. These are two conditions on Γ_{abc}, namely

$$\Gamma_{[ab]c} = 0 \qquad \text{and} \qquad \Gamma_{a(bc)} = \tfrac{1}{2} D_a g_{bc}$$

that always have a unique solution. To see this, note that the general solution of the second equation has the form

$$\Gamma_{abc} = \tfrac{1}{2} D_a g_{bc} - K_{abc}, \quad \text{where } K_{abc} = K_{a[bc]}.$$

Having done this, the first equation reads

$$K_{[ab]c} = \tfrac{1}{2} D_{[a} g_{b]c},$$

which always has a unique solution owing to the tensor isomorphism

$$
\boxed{
\begin{array}{ccc}
\Lambda^1 \otimes \Lambda^2 & \xrightarrow{\;\cong\;} & \Lambda^2 \otimes \Lambda^1 \\
K_{abc} = K_{a[bc]} & \longmapsto & K_{[ab]c}\,,
\end{array}
}
\tag{3.1}
$$

where Λ^p denotes the bundle of p-forms. This isomorphism is typical of the tensor identities to be explained in this article by means of Lie algebra cohomology.

4. The Killing operator

A vector field X^a on a Riemannian manifold with metric g_{ab} is said to be a *Killing field* if and only if $\mathcal{L}_X g_{ab} = 0$, where $\mathcal{L}_X$ is the Lie derivative along X^a. The geometric interpretation of Lie derivative means that the flow of X^a is an isometry. Thus, a Killing field is an *infinitesimal symmetry* in the context of Riemannian geometry.

It is useful to regard the Killing equation $\mathcal{L}_X g_{ab} = 0$ as a linear partial differential equation on the vector field X^a as follows. For any torsion-free connection ∇_a,

$$\mathcal{L}_X \phi_b = X^a \nabla_a \phi_b + \phi_a \nabla_b X^a$$

so, if we use the Levi-Civita connection for g_{ab}, then

$$
\begin{aligned}
\mathcal{L}_X g_{bc} &= X^a \nabla_a g_{bc} + g_{ac} \nabla_b X^a + g_{ba} \nabla_c X^a \\
&= \nabla_b X_c + \nabla_c X_b.
\end{aligned}
$$

Hence, the Killing fields X^a make up the kernel of the *Killing operator*:–

$$
\begin{array}{ccccc}
\text{Tangent bundle} & \xrightarrow{\;\cong\;} & \Lambda^1 & \longrightarrow & \odot^2 \Lambda^1 \\
X^a & \longmapsto & X_a & \longmapsto & \nabla_{(a} X_{b)}\,.
\end{array}
$$

5. Prolongation of the Killing operator

For any torsion-free connection ∇_a, the equation $\nabla_{(a} X_{b)} = 0$ may be understood as follows. Certainly, we may rewrite it as

$$\nabla_a X_b = K_{ab}, \quad \text{where } K_{ab} \text{ is skew.} \tag{5.1}$$

In this case $\nabla_{[a} K_{bc]} = 0$, a condition which we may rewrite as

$$\nabla_a K_{bc} = \nabla_c K_{ba} - \nabla_b K_{ca}$$

and substitute from (5.1) to conclude, as a differential consequence, that

$$\nabla_a K_{bc} = \nabla_c \nabla_b X_a - \nabla_b \nabla_c X_a = R_{bc}{}^d{}_a X_d,$$

where $R_{ab}{}^c{}_d$ is the curvature of ∇_a characterised by

$$[\nabla_a \nabla_b - \nabla_b \nabla_a] X^c = R_{ab}{}^c{}_d X^d.$$

Therefore,

$$\nabla_{(a} X_{b)} = 0 \iff \boxed{\begin{array}{rcl} \nabla_a X_b &=& K_{ab} \\ \nabla_a K_{bc} &=& R_{bc}{}^d{}_a X_d \end{array}}$$

In other words, Killing fields are in 1–1 correspondence with covariant constant sections of the vector bundle $\mathbb{T} \equiv \Lambda^1 \oplus \Lambda^2$ equipped with the connection

$$\mathbb{T} \ni \begin{bmatrix} X_b \\ K_{bc} \end{bmatrix} \overset{\nabla_a}{\longmapsto} \begin{bmatrix} \nabla_a X_b - K_{ab} \\ \nabla_a K_{bc} - R_{bc}{}^d{}_a X_d \end{bmatrix} \in \Lambda^1 \otimes \mathbb{T}. \tag{5.2}$$

At this point, we may use the standard theory of vector bundles with connection to investigate Killing fields. In particular, it is immediately clear that the Killing fields on a connected manifold form a vector space whose dimension is bounded by the rank of $\mathbb{T}$, namely $n(n+1)/2$.

6. The Killing operator in flat space

Be that as it may, suppose ask only about the Killing operator on flat space. It is easily verified in this case that the connection (5.2) is flat (and, in fact, the same is true on any constant curvature space). Therefore, we may couple the de Rham sequence with (5.2) to obtain a locally exact complex

$$\mathbb{T} \xrightarrow{\nabla} \Lambda^1 \otimes \mathbb{T} \xrightarrow{\nabla} \Lambda^2 \otimes \mathbb{T} \xrightarrow{\nabla} \Lambda^3 \otimes \mathbb{T} \xrightarrow{\nabla} \cdots$$

and, at this point, the isomorphism (3.1) re-emerges! Specifically, in the absence of the curvature term (5.2) may be written as

$$\begin{bmatrix} X_b \\ K_{bc} \end{bmatrix} \overset{\nabla_a}{\longmapsto} \begin{bmatrix} \nabla_a X_b \\ \nabla_a K_{bc} \end{bmatrix} - \partial \begin{bmatrix} X_b \\ K_{bc} \end{bmatrix}, \text{ where } \partial \begin{bmatrix} X_b \\ K_{bc} \end{bmatrix} = \begin{bmatrix} K_{ab} \\ 0 \end{bmatrix}.$$

The homomorphism $\partial : \mathbb{T} \to \Lambda^1 \otimes \mathbb{T}$ induces $\partial : \Lambda^p \otimes \mathbb{T} \to \Lambda^{p+1} \otimes \mathbb{T}$ by $\partial(\omega \otimes X) = \omega \wedge \partial X$ and we obtain a complex

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{T} & \overset{\partial}{\longrightarrow} & \Lambda^1 \otimes \mathbb{T} & \overset{\partial}{\longrightarrow} & \Lambda^2 \otimes \mathbb{T} & \overset{\partial}{\longrightarrow} & \Lambda^3 \otimes \mathbb{T} & \overset{\partial}{\longrightarrow} & \cdots \\
& & \| & & \| & & \| & & \| & & \\
& & \Lambda^1 & & \Lambda^1 \otimes \Lambda^1 & & \Lambda^2 \otimes \Lambda^1 & & \Lambda^3 \otimes \Lambda^1 & & \cdots \\
& & \oplus & \nearrow & \oplus & \boxed{\text{NB}} \nearrow & \oplus & \nearrow & \oplus & \nearrow & \\
& & \Lambda^2 & & \Lambda^1 \otimes \Lambda^2 & & \Lambda^2 \otimes \Lambda^2 & & \Lambda^3 \otimes \Lambda^2 & & \cdots
\end{array} \tag{6.1}$$

in which $\partial : \Lambda^1 \otimes \mathbb{T} \to \Lambda^2 \otimes \mathbb{T}$ is carried by the isomorphism (3.1). More generally, we can ask about the cohomology of the complex $(\Lambda^\bullet \otimes \mathbb{T}, \partial)$ and conclude, by inspection, that

$$\begin{array}{rcl}
H^0(\Lambda^\bullet \otimes \mathbb{T}, \partial) &=& \{X_a\} \\
H^1(\Lambda^\bullet \otimes \mathbb{T}, \partial) &=& \{X_{ab} = X_{(ab)}\} \\
H^2(\Lambda^\bullet \otimes \mathbb{T}, \partial) &=& \{K_{abcd} = K_{[ab][cd]} \text{ s.t. } K_{[abc]d} = 0\} \\
H^3(\Lambda^\bullet \otimes \mathbb{T}, \partial) &=& \{K_{abcde} = K_{[abc][de]} \text{ s.t. } K_{[abcd]e} = 0\} \\
H^4(\Lambda^\bullet \otimes \mathbb{T}, \partial) &=& \{K_{abcdef} = K_{[abcd][ef]} \text{ s.t. } K_{[abcde]f} = 0\} \\
& \vdots & \vdots \quad \vdots
\end{array}$$

recognising that each of these bundles is an irreducible tensor bundle, which we may write as Young diagrams [**FH**]

$$H^0 = \square \quad H^1 = \square\square \quad H^2 = \begin{array}{c}\square\square\\\square\square\end{array} \quad H^3 = \begin{array}{c}\square\square\\\square\square\\\square\end{array} \quad H^4 = \begin{array}{c}\square\square\\\square\\\square\end{array} \quad \cdots \tag{6.2}$$

Readers may notice that $H^2(\Lambda^\bullet \otimes \mathbb{T}, \partial)$ is the natural location for the Riemann curvature tensor and that $H^3(\Lambda^\bullet \otimes \mathbb{T}, \partial)$ is the natural location for the Bianchi identity. These observations are more fully explained in [**E**]. Here, suffice it to observe that a simple diagram chase on (6.1) reveals a locally exact complex

$$\square \xrightarrow{\nabla} \square\square \xrightarrow{\nabla^{(2)}} \begin{array}{c}\square\square\\\square\square\end{array} \xrightarrow{\nabla} \begin{array}{c}\square\square\\\square\square\\\square\end{array} \xrightarrow{\nabla} \begin{array}{c}\square\square\\\square\\\square\end{array} \xrightarrow{\nabla} \cdots \tag{6.3}$$

and, in particular, an identification of the range of the Killing operator in flat space as follows.

Theorem *Suppose U is an open subset of $\mathbb{R}^n$ with $H^1(U, \mathbb{R}) = 0$. Then a symmetric tensor ω_{ab} on U is of the form $\nabla_{(a}X_{b)}$ for some X_a on U if and only if*

$$\nabla_a\nabla_c\omega_{bd} - \nabla_b\nabla_c\omega_{ad} - \nabla_a\nabla_d\omega_{bc} + \nabla_b\nabla_d\omega_{ac} = 0.$$

7. Higher Killing operators

So far, we have not seen any Lie algebra cohomology, although it is lurking in the background. The identifications (6.2) can be obtained by elementary means. As soon as we consider more complicated operators, however, then the corresponding identifications are not so obvious. A *Killing tensor* of valence ℓ is a symmetric tensor field $X_{bc\cdots de}$ with ℓ indices annihilated by the higher Killing operator

$$X_{bc\cdots de} \mapsto \nabla_{(a}X_{bc\cdots de)}.$$

Killing tensors induce conserved quantities along geodesics and arise naturally in the theory of separation of variables. The higher Killing operators may be prolonged along the lines explained in §5. The details are more complicated and this is where Lie algebra cohomology comes to the fore. Without going into details, the prolonged bundle

$$\mathbb{T} = \Lambda^1 \oplus \Lambda^2 = \square \oplus \begin{array}{c}\square\\\square\end{array}$$

that we saw in §5 should be replaced by

$$\mathbb{T}^\ell = \mathbb{T}_0^\ell \oplus \cdots \oplus \mathbb{T}_\ell^\ell = \underbrace{\square\square\square\cdots\square}_{\ell \text{ boxes}} \oplus \begin{array}{c}\square\square\square\cdots\square\\\square\end{array} \oplus \begin{array}{c}\square\square\square\cdots\square\\\square\square\end{array} \oplus \cdots \oplus \begin{array}{c}\square\square\square\cdots\square\\\square\square\cdots\square\end{array},$$

realised as

$$\begin{bmatrix} X_{bc\cdots de} = X_{(bc\cdots de)} \\ K_{pbc\cdots de} = K_{p(bc\cdots de)} \text{ s.t. } K_{(pbc\cdots de)} = 0 \\ K'_{pqbc\cdots de} = K'_{(pq)(bc\cdots de)} \text{ s.t. } K'_{p(qbc\cdots de)} = 0 \\ K''_{pqrbc\cdots de} = K''_{(pqr)(bc\cdots de)} \text{ s.t. } K''_{pq(rbc\cdots de)} = 0 \\ \vdots \\ K''^{\cdots\prime}_{pq\cdots rsbc\cdots de} = K''^{\cdots\prime}_{(pq\cdots rs)(bc\cdots de)} \text{ s.t. } K''^{\cdots\prime}_{pq\cdots r(sbc\cdots de)} = 0 \end{bmatrix}$$

with $\partial : \mathbb{T}^\ell \to \Lambda^1 \otimes \mathbb{T}^\ell$ defined by

$$\partial \begin{bmatrix} X_{bc\cdots de} \\ K_{pbc\cdots de} \\ K'_{pqbc\cdots de} \\ K''_{pqrbc\cdots de} \\ \vdots \\ K''^{\cdots\prime}_{pq\cdots rsbc\cdots de} \end{bmatrix} = \begin{bmatrix} K_{abc\cdots de} \\ K'_{apbc\cdots de} \\ K''_{apqbc\cdots de} \\ \vdots \\ K''^{\cdots\prime}_{ap\cdots qrbc\cdots de} \\ 0 \end{bmatrix}.$$

The identifications generalising (6.2) are as follows.

$$H^0(\Lambda^\bullet \otimes \mathbb{T}^\ell, \partial) = \underbrace{\boxed{\cdots}}_{\ell \text{ boxes}} \qquad H^1(\Lambda^\bullet \otimes \mathbb{T}^\ell, \partial) = \underbrace{\boxed{\cdots}}_{\ell+1 \text{ boxes}} \qquad (7.1)$$

and

$$H^2(\Lambda^\bullet \otimes \mathbb{T}^\ell, \partial) = \underbrace{\boxed{\cdots}}_{\ell+1 \text{ boxes}} \qquad H^p(\Lambda^\bullet \otimes \mathbb{T}^\ell, \partial) = \boxed{\cdots} \quad \updownarrow p \quad \text{for } p \geq 2. \qquad (7.2)$$

The locally exact complex generalising (6.3) is

$$\boxed{\cdots} \xrightarrow{\nabla} \boxed{\cdots} \xrightarrow{\nabla^{(\ell+1)}} \boxed{\cdots} \xrightarrow{\nabla} \boxed{\cdots} \xrightarrow{\nabla} \cdots \qquad (7.3)$$

where the first operator is the higher Killing operator. It is a special case of the Bernstein-Gelfand-Gelfand resolution [**CD, CSS**].

8. Tensor identities

Be that as it may, the identifications of $H^p(\Lambda^\bullet \otimes \mathbb{T}^\ell, \partial)$ claimed in the previous section are not so easy and entail some tricky tensor identities. The natural generalisation of (3.1), for example, follows by writing out the complex

$$0 \to \mathbb{T}^\ell \xrightarrow{\partial} \Lambda^1 \otimes \mathbb{T}^\ell \xrightarrow{\partial} \Lambda^2 \otimes \mathbb{T}^\ell \xrightarrow{\partial} \Lambda^3 \otimes \mathbb{T}^\ell \xrightarrow{\partial} \Lambda^4 \otimes \mathbb{T}^\ell \xrightarrow{\partial} \cdots$$

as in (6.1) and pinning down the locations of the cohomologies

$$
\begin{array}{ccccccccccc}
0 & & \boxed{\mathbb{T}^\ell_0} & & \boxed{\Lambda^1 \otimes \mathbb{T}^\ell_0} & & \Lambda^2 \otimes \mathbb{T}^\ell_0 & & \Lambda^3 \otimes \mathbb{T}^\ell_0 & \\
\oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow \\
0 & & \mathbb{T}^\ell_1 & & \Lambda^1 \otimes \mathbb{T}^\ell_1 & & \Lambda^2 \otimes \mathbb{T}^\ell_1 & & \Lambda^3 \otimes \mathbb{T}^\ell_1 & \\
\oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow \\
0 & & \mathbb{T}^\ell_{\ell-2} & & \Lambda^1 \otimes \mathbb{T}^\ell_{\ell-2} & & \Lambda^2 \otimes \mathbb{T}^\ell_{\ell-2} & & \Lambda^3 \otimes \mathbb{T}^\ell_{\ell-2} & \\
\oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow \\
0 & & \mathbb{T}^\ell_{\ell-1} & & \Lambda^1 \otimes \mathbb{T}^\ell_{\ell-1} & & \Lambda^2 \otimes \mathbb{T}^\ell_{\ell-1} & & \Lambda^3 \otimes \mathbb{T}^\ell_{\ell-1} & \\
\oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow \\
0 & & \mathbb{T}^\ell_\ell & & \Lambda^1 \otimes \mathbb{T}^\ell_\ell & & \boxed{\Lambda^2 \otimes \mathbb{T}^\ell_\ell} & & \boxed{\Lambda^3 \otimes \mathbb{T}^\ell_\ell} &
\end{array}
$$

simply by the number of boxes involved to deduce that

$$0 \to \Lambda^1 \otimes \underbrace{\boxed{\cdots}}_{\ell} \xrightarrow{\partial} \Lambda^2 \otimes \boxed{\cdots} \xrightarrow{\partial} \Lambda^3 \otimes \boxed{\cdots} \xrightarrow{\partial}$$
$$\cdots \xrightarrow{\partial} \Lambda^\ell \otimes \boxed{\cdots} \xrightarrow{\partial} \Lambda^{\ell+1} \otimes \boxed{\cdots} \to 0$$

is exact. Already the injectivity of the first homomorphism gives useful information regarding the higher Killing operator. Specifically it says that

$$\left.\begin{array}{l} K_{apq\cdots rsbc\cdots de} = K_{a(pq\cdots rs)(bc\cdots de)} \\ \overset{\leftarrow \ell \rightarrow}{} \overset{\leftarrow \ell \rightarrow}{} K_{apq\cdots r(sbc\cdots de)} = 0 \\ K_{[ap]q\cdots rsbc\cdots de} = 0 \end{array}\right\} \quad \Rightarrow \quad K_{bpq\cdots rsbc\cdots de} = 0.$$

In the flat case, if $X_{bc\cdots de}$ is a Killing tensor of valence ℓ, it follows immediately from the Killing equation $\nabla_{(a} X_{bc\cdots de)} = 0$, that

$$K_{apq\cdots rsbd\cdots de} \equiv \underbrace{\nabla_a \nabla_p \nabla_q \cdots \nabla_r \nabla_s}_{\ell+1} X_{bc\cdots de}$$

satisfies exactly these symmetries and hence vanishes. In other words, the Killing tensors of valence ℓ on $\mathbb{R}^n$ are polynomial of degree at most ℓ. More generally, prolongation in the curved case implies that the Killing tensors of valence ℓ near any point are determined by their ℓ-jet at that point.

9. Lie algebra cohomology

It remains to explain where (7.1) and (7.2) come from and the answer is a special case of Kostant's generalised Bott-Borel-Weil Theorem [**K**], which we now explain. The special case we need involves only the cohomology of an Abelian Lie algebra but for Kostant's results to apply it is important that this Abelian Lie algebra be contained inside a semisimple Lie algebra in a particular way. Specifically, let

$$\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R}) = \{(n+1) \times (n+1) \text{ matrices } X \text{ s.t. } \text{trace}(X) = 0\}$$

and write $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, comprising matrices of the form

$$\begin{pmatrix} \begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & 0 & \\ * & & & \end{array} \end{pmatrix} \quad \begin{pmatrix} \begin{array}{c|ccc} * & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \end{pmatrix} \quad \begin{pmatrix} \begin{array}{c|ccc} 0 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{array} \end{pmatrix},$$

respectively. Suppose $\mathbb{V}$ is an irreducible tensor representation of $\mathfrak{g}$. It restricts to a representation of the Abelian subalgebra $\mathfrak{g}_{-1}$. Kostant's theorem computes the Lie algebra cohomology $H^p(\mathfrak{g}_{-1}, \mathbb{V})$. Explicitly, this means that the cohomology of the complex of $\mathfrak{g}_0$-modules

$$0 \to \mathbb{V} \xrightarrow{\partial} (\mathfrak{g}_{-1})^* \otimes \mathbb{V} \xrightarrow{\partial} \Lambda^2(\mathfrak{g}_{-1})^* \otimes \mathbb{V} \xrightarrow{\partial} \Lambda^3(\mathfrak{g}_{-1})^* \otimes \mathbb{V} \xrightarrow{\partial} \cdots$$

is computed as a $\mathfrak{g}_0$-module, where $\partial : \mathbb{V} \to (\mathfrak{g}_{-1})^* \otimes \mathbb{V}$ is defined by the action of $\mathfrak{g}_{-1}$ on $\mathbb{V}$. To state the result, we need a notation for the irreducible representations of $\mathfrak{sl}(n+1, \mathbb{R})$ and for this we follow [**BE**] writing, for example,

$$\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{1}{\bullet} \quad \text{and} \quad \overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}$$

for the defining representation $\mathbb{R}^{n+1}$ and its dual $(\mathbb{R}^{n+1})^*$, respectively. In particular, Kostant's theorem yields

$$H^0(\mathfrak{g}_{-1}, \overset{0}{\bullet}\!-\!\overset{\ell}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}) = \overset{0}{\times}\!-\!\overset{\ell}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{0}{\bullet}$$

where, again, we are following the [**BE**] to denote $\mathfrak{g}_0$ and its irreducible representations. More generally,

$$H^1(\mathfrak{g}_{-1}, \overset{0}{\bullet}\!-\!\overset{\ell}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) \;=\; \overset{-2}{\times}\!-\!\overset{\ell+1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$$

$$H^2(\mathfrak{g}_{-1}, \overset{0}{\bullet}\!-\!\overset{\ell}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) \;=\; \overset{-\ell-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{\ell+1}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$$

$$H^3(\mathfrak{g}_{-1}, \overset{0}{\bullet}\!-\!\overset{\ell}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) \;=\; \overset{-\ell-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{\ell}{\bullet}\!-\!\overset{1}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$H^{n-1}(\mathfrak{g}_{-1}, \overset{0}{\bullet}\!-\!\overset{\ell}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) \;=\; \overset{-\ell-n}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{\ell}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{1}{\bullet}$$

$$H^n(\mathfrak{g}_{-1}, \overset{0}{\bullet}\!-\!\overset{\ell}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) \;=\; \overset{-\ell-n-1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{\ell}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$$

where the right hand side follows the affine action of the Weyl group as explained in [**BE**]. For our purposes, the crossed node can be dropped, viewing the results as irreducible tensor representations of $\mathfrak{sl}(n,\mathbb{R})$. As tensor identities for $\mathfrak{sl}(n,\mathbb{R})$, they are exactly what we need induce (7.1) and (7.2) on a manifold.

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