On mixed plane curves of polar degree 1

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Abstract. Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a mixed strongly polar homogeneous polynomial of 3 variables $\mathbf{z}=(z_1,z_2,z_3)$. It defines a Riemann surface $V:=\{[\mathbf{z}]\in\mathbb{P}^2\,|\,f(\mathbf{z},\bar{\mathbf{z}})=0\}$ in the complex projective space $\mathbb{P}^2$. We will show that for an arbitrary given $g\ge 0$, there exists a mixed polar homogeneous polynomial with polar degree 1 which defines a projective surface of genus g. For the construction, we introduce a new type of weighted homogeneous polynomials which we call *polar weighted homogeneous polynomials of twisted join type.*

1. Introduction

Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a strongly polar homogeneous mixed polynomial of n-variables $\mathbf{z}=(z_1,\dots,z_n)\in\mathbb{C}^n$ with polar degree q and radial degree d. Recall that a strongly polar homogeneous polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ satisfies the equality ([**O3**]):

$$f((t,\rho)\circ\mathbf{z},\overline{(t,\rho)\circ\mathbf{z}})=t^d\rho^q f(\mathbf{z},\bar{\mathbf{z}}),\quad (t,\rho)\in\mathbb{R}^+\times S^1. \tag{1.1}$$

Here $(t,\rho)\circ\mathbf{z}$ is defined by the usual action $(t,\rho)\circ\mathbf{z}=(t\rho z_1,\dots,t\rho z_n)$. Let $\tilde{V}$ be the mixed affine hypersurface

$$\tilde{V}=f^{-1}(0)=\{\mathbf{z}\in\mathbb{C}^n\,|\,f(\mathbf{z},\bar{\mathbf{z}})=0\}.$$

We assume that $\tilde{V}$ has an isolated singularity at the origin. Let $f:\mathbb{C}^n\setminus\tilde{V}\to\mathbb{C}^*$ be the global Milnor fibration defined by f and let F be the fiber. Namely F is the hypersurface $f^{-1}(1)\subset\mathbb{C}^n$. The monodromy map $h:F\to F$ is defined by

$$h(\mathbf{z})=(\eta z_1,\dots,\eta z_n),\quad \eta=\exp(\frac{2\pi\, i}{q}).$$

We consider the smooth projective hypersurface V defined by

$$V=\{[\mathbf{z}]\in\mathbb{P}^{n-1}\,|\,f(\mathbf{z},\bar{\mathbf{z}})=0\}.$$

By (1.1), if $\mathbf{z}\in f^{-1}(0)$ and $\mathbf{z}'$ is in the same $\mathbb{R}^+\times S^1$ orbit of $\mathbf{z}$, then $\mathbf{z}'\in f^{-1}(0)$. Thus the hypersurface $V=\{[\mathbf{z}]\in\mathbb{P}^{n-1}|f(\mathbf{z})=0\}$ is well-defined. Consider the quotient map $\pi:\mathbb{C}^n\setminus\{O\}\to\mathbb{P}^{n-1}$ and its restriction to the Milnor fiber $\pi: F\to\mathbb{P}^{n-1}\setminus V$. This is a q-cyclic covering map. In the previous paper [**O3**], we have shown that $\tilde{V}$ and V has canonical orientations and the following key assertion is proved:

Theorem 1.1. (Theorem 11, [**O3**]) *The embedding degree of V is equal to the polar degree q.*

First we observe that

Proposition 1.2. *The Euler characteristics satisfy the following equalities.*

(1) $\chi(F) = q\,\chi(\mathbb{P}^{n-1} \setminus V)$.
(2) $\chi(\mathbb{P}^{n-1} \setminus V) = n - \chi(V)$ *and* $\chi(V) = n - \chi(F)/q$.
(3) *The following sequence is exact.*

$$1 \to \pi_1(F) \xrightarrow{\pi_\sharp} \pi_1(\mathbb{P}^{n-1} \setminus V) \to \mathbb{Z}/q\mathbb{Z} \to 1.$$

Corollary 1.3. *If $q = 1$, the projection $\pi : F \to \mathbb{P}^{n-1} \setminus V$ is a diffeomorphism.*

Corollary 1.4. *Assume that $n = 3$. Then the genus $g(V)$ of V is given by the formula:*

$$g(V) = \frac{1}{2}\left(\frac{\chi(F)}{q} - 1\right)$$

The monodromy map $h : F \to F$ gives free $\mathbb{Z}/q\mathbb{Z}$ action on F. Thus using the periodic monodromy argument in [**M**], we get

Proposition 1.5. *The zeta function of the monodromy $h : F \to F$ is given by*

$$\zeta(t) = (1 - t^q)^{-\chi(F)/q}.$$

In particular, if $q = 1$, $h = \mathrm{id}_F$ and $\zeta(t) = (1 - t)^{-\chi(F)}$.

1.1. Projective mixed curves. Let C be a smooth C^∞ surface embedded in $\mathbb{P}^2$ and let g be the genus of C and let q be the embedding degree of C. It is known that the following inequality is satisfied.

$$g \geq \frac{(q-1)(q-2)}{2}.$$

This was first conjectured by R. Thom and it has been proved by many people. For example see Kronheimer-Mrowka, [**KM**]. We are interested to present C as a mixed algebraic curve in the smallest embedding degree q of a Riemann surface of a given genus g as a mixed algebraic curve. (So we are not interested in the embedding with $q = 0$.) In our previous paper, we have used the join type construction starting from a strongly polar homogeneous polynomial of two variables $f(z_1, z_2, \bar{z}_1, \bar{z}_2)$ of polar degree q and radial degree $q + 2r$ and we considered

$$g(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) = f(z_1, z_2, \bar{z}_1, \bar{z}_2) + z_3^{q+r} \bar{z}_3^r.$$

Using such a polynomial, we have shown that there exists a mixed curve of a given genus g with the embedding degree 2 ([**O3**]). Note that if degree $q = 1$, the join theorem ([**Mol**]) says that the Euler number of the Milnor fiber of g is 1 (i.e., the Milnor number is 0) and thus we only get genus 0. Thus to get a mixed curve of polar degree 1 and the genus arbitrary large, we have to find another type of polynomials. This is the reason we introduce *polar weighted homogeneous polynomials of twisted join type* (See §3). For example, in the above setting, we consider the polynomial:

$$g'(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) = f(z_1, z_2, \bar{z}_1, \bar{z}_2) + \bar{z}_2 z_3^{q+r} \bar{z}_3^{r-1}.$$

Using polynomials of this type, we will show that *there exists a mixed surface with the polar degree $q = 1$ for any g (Theorem 3.3, Corollary 3.4).*

This paper is a continuation of our paper [**O3**] (see also [**O4, O2**]) and we use the same notations as those we have used previously.

2. Mixed projective curves

Let $\mathcal{M}(q + 2r, q; n)$ be the space of strongly polar homogeneous polynomials of n-variables $z_1, \ldots, z_n$ with polar degree q and radial degree $q + 2r$.

2.1. Important mixed affine curves. We consider the following mixed strongly polar homogeneous polynomial of two variables:

$$h_{q,r,j}(\mathbf{w},\bar{\mathbf{w}}) = (w_1^{q+j}\bar{w}_1^j + w_2^{q+j}\bar{w}_2^j)(w_1^{r-j} - \alpha w_2^{r-j})(\bar{w}_1^{r-j} - \beta\bar{w}_2^{r-j}), \quad r \geq j \geq 0$$

with $\alpha, \beta \in \mathbb{C}^*$ generic. This polynomial plays a key role for the construction. Note that $h_{q,r,j}$ is a strongly polar homogeneous polynomial with radial degree $q + 2r$ and polar degree q respectively i.e., $h_{q,r,j} \in \mathcal{M}(q + 2r, q; 2)$. Then the Milnor fiber $H_{q,r,j} := h_{q,r,j}^{-1}(1)$ of $h_{q,r,j}$ is connected. The Euler characteristic of $\chi(H^*_{q,r,j})$ (where $H^*_{q,r,j} = H_{q,r,j} \cap \mathbb{C}^{*2}$) is given by

$$\chi(H^*_{q,r,j}) = -r_{q,r,j} \times q \quad \text{and} \quad \chi(H_{q,r,j}) = -r_{q,r,j}\, q + 2q$$

where $r_{q,r,j}$ is the link component number of the mixed curve $C = h_{q,r,j}^{-1}(0)$. Note that the link component number $r_{q,r,j}$ is given by $r_{q,r,j} = q + 2(r - j)$ by Lemma 64, [**O4**]. Thus

Proposition 2.1.

$$\chi(H_{q,r,j}) \;=\; -q\,((q-2) \;+\; 2\,(r-j))$$

2.2. Join type polynomials. We consider the following strongly polar homogeneous polynomial of join type.

$$f_{q,r,j}(\mathbf{z},\bar{\mathbf{z}}) = h_{q,r,j}(\mathbf{w},\bar{\mathbf{w}}) + z_3^{q+r}\bar{z}_3^r, \quad \mathbf{w} = (z_1, z_2)$$

The the Milnor fiber $F_{q,r,j} = f_{q,r,j}^{-1}(1)$ of $f_{q,r,j}$ is connected. By the Join theorem (Cisneros-Molina [**Mol**]), $F_{q,r,j}$ is a simply connected 2-dimensional CW-complex so that

$$\begin{aligned}\chi(F_{q,r,j}) &= -(q-1)\chi(H_{q,r,j}) + q \\ &= q(q-1)(q-2) + 2q(q-1)(r-j) + q.\end{aligned}$$

Let $C_{q,r,j}$ be the projective curve defined by $\{f_{q,r,j}(\mathbf{z},\bar{\mathbf{z}}) = 0\}$ in $\mathbb{P}^2$. By Corollary 1.4, the genus $g(C_{q,r,j})$ of $C_{q,r,j}$ is given by

$$g(C_{q,r,j}) = \frac{(q-1)(q-2)}{2} + (q-1)(r-j) \geq \frac{(q-1)(q-2)}{2}.$$

For $q = 2$, we get

$$g(C_{2,r,j}) = (r-j) \geq 0.$$

Thus this shows that for arbitrary $g \geq 0$, the mixed curve $C_{2,g+j,j}$ is a curve of genus g and the embedding degree 2. Note that $g(C_{1,r,j}) = 0$. Thus $q = 1$ gives only rational curves, as is already mentioned in 1.1.

3. Twisted join type polynomial

In this section, we introduce a new class of mixed polar weighted polynomials which we use to construct curves with embedded degree 1. Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a polar weighted homogeneous polynomial of n-variables $\mathbf{z} = (z_1,\dots,z_n)$. Let $Q = {}^t(q_1,\dots,q_n)$, $P = {}^t(p_1,\dots,p_n)$ be the radial and polar weight respectively and let d, q be the radial and polar degree respectively. For simplicity, we call that $Q' = {}^t(q_1/d,\dots,q_n/d)$ and $P' = {}^t(p_1/q,\dots,p_n/q)$ *the normalized radial weights* and *the normalized polar weights* respectively. Consider the mixed polynomial of $(n+1)$-variables:

$$g(\mathbf{z},\bar{\mathbf{z}},w,\bar{w}) \;=\; f(\mathbf{z},\bar{\mathbf{z}}) + \bar{z}_n w^a\bar{w}^b, \quad a > b.$$

Consider the rational numbers $\bar{q}_{n+1}$, $\bar{p}_{n+1}$ satisfying

$$\frac{q_n}{d} + (a+b)\,\bar{q}_{n+1} = 1, \quad -\frac{p_n}{q} + (a-b)\,\bar{p}_{n+1} = 1.$$

We assume that $q_n < d$ so that $\bar{q}_{n+1}, \bar{p}_{n+1}$ are positive rational numbers. The polynomial g is a polar weighted homogeneous polynomial with the normalized radial and polar weights $\widetilde{Q'} = {}^t(q_1/d, \ldots, q_n/d, \bar{q}_{n+1})$ and $\widetilde{P'} = {}^t(p_1/q, \ldots, p_n/q, \bar{p}_{n+1})$ respectively. The radial and polar degree of g are given by $\mathrm{lcm}(d, denom(\bar{q}_{n+1}))$ and $\mathrm{lcm}(q, denom(\bar{p}_{n+1}))$ where $denom(x)$ is the denominator of $x \in \mathbb{Q}$. We call g *a twisted join of* $f(\mathbf{z}, \bar{\mathbf{z}})$ *and* $\bar{z}_n w^a \bar{w}^b$. We say that g is a polar weighted homogeneous polynomial of *twisted join type*. A twisted join type polynomial behaves differently than the simple join type, as we will see below.

We recall that $f(\mathbf{z}, \bar{\mathbf{z}})$ is called to be *1-convenient* if the restriction of f to each coordinate hyperplane $f_i := f|_{\{z_i=0\}}$ is non-trivial for $i = 1, \ldots, n$ ([**O1**])

Lemma 3.1. *Assume that $n \geq 2$ and f is 1-convenient. Then*

$$\phi_\sharp : \pi_1((\mathbb{C}^*)^n \setminus F_f^*) \cong \mathbb{Z}^n \times \mathbb{Z}$$

is an isomorphism where ϕ is the canonical mapping $\phi : (\mathbb{C}^*)^n \setminus F_f^* \to (\mathbb{C}^*)^n \times (\mathbb{C} \setminus \{1\})$ defined by $\phi(\mathbf{z}) = (\mathbf{z}, f(\mathbf{z}, \bar{\mathbf{z}}))$ and $F_f^* := f^{-1}(1) \cap (\mathbb{C}^*)^n$.

Proof. Let us use the notations:

$$D_\delta := \{\eta \in \mathbb{C} \,|\, |\eta| \leq \delta\}, \quad S_\delta(1) = \{\eta \in \mathbb{C} \,|\, |\eta - 1| = \delta\}.$$

Denote by $\hat{f}$ the restriction of f to $(\mathbb{C}^*)^n$. The fact that the mapping $\hat{f} : (\mathbb{C}^*)^n \setminus f^{-1}(0) \to \mathbb{C}^*$ is a fibration and the inclusion $D_{1-\varepsilon} \cup S_\varepsilon(1) \hookrightarrow \mathbb{C} \setminus \{1\}$ is a deformation retract implies the following inclusion is also a deformation retract:

$$\iota : \hat{f}^{-1}(D_{1-\varepsilon}) \cup \hat{f}^{-1}(S_\varepsilon(1)) \subset (\mathbb{C}^*)^n \setminus F_f^*, \quad 0 < \varepsilon \ll 1.$$

On the other hand, $\hat{f}^{-1}(S_\varepsilon(1)) \cong \hat{f}^{-1}(1-\varepsilon) \times S_\varepsilon(1) \cong F_f^* \times S_\varepsilon(1)$ and $\pi_1(\hat{f}^{-1}(S_\varepsilon(1))) \cong \pi_1(F_f^*) \times \mathbb{Z}$. The 1-convenience of f implies the homomorphism $i_\sharp : \pi_1(F_f^*) \to \pi_1((\mathbb{C}^*)^n)$ is surjective. Moreover $\hat{f}^{-1}(D_{1-\varepsilon})$ is homotopic to $(\mathbb{C}^*)^n$, as $D_{1-\varepsilon} \hookrightarrow \mathbb{C}$ is a deformation retract. Thus the assertion follows from the van Kampen lemma, applied to the decomposition

$$\hat{f}^{-1}(D_{1-\varepsilon} \cup S_\varepsilon(1)) = \hat{f}^{-1}(D_{1-\varepsilon}) \cup \hat{f}^{-1}(S_\varepsilon(1)),$$
$$\hat{f}^{-1}(D_{1-\varepsilon}) \cap \hat{f}^{-1}(S_\varepsilon(1)) = \hat{f}^{-1}(1-\varepsilon) \cong F_f^*.$$

□

Put $F_{f_n} := f_n^{-1}(1) = F_f \cap \{z_n = 0\} \subset \mathbb{C}^{n-1}$ with $f_n := f|_{\mathbb{C}^n \cap \{z_n=0\}}$.

Theorem 3.2. *Assume that $n \geq 2$ and f is 1-convenient and $g(\mathbf{z}, \bar{\mathbf{z}}, w, \bar{w})$ is a twisted join polynomial as above. Then*

1. *the Milnor fiber of g, $F_g = g^{-1}(1)$, is simply connected.*
2. *The Euler characteristic of F_g is given by the formula:*

$$\chi(F_g) = -(a-b-1)\chi(F_f) + (a-b)\chi(F_{f_n}).$$

PROOF. Consider $F_g^* := F_g \cap (\mathbb{C}^*)^{n+1}$ and the projection map $\pi : F_g^* \to (\mathbb{C}^*)^n$ defined by $(\mathbf{z}, w) \mapsto \mathbf{z}$. Then the image of F_g^* by π is $(\mathbb{C}^*)^n \setminus F_f^*$ and $\pi : F_g^* \to (\mathbb{C}^*)^n \setminus F_f^*$ gives an $(a-b)$-cyclic covering. In fact the fiber $\pi^{-1}(\mathbf{z})$ is given as

$$\pi^{-1}(\mathbf{z}) = \{(\mathbf{z}, w) \mid w^a \bar{w}^b = \frac{1 - f(\mathbf{z}, \bar{\mathbf{z}})}{\bar{z}_n}\}$$

Therefore

$$\pi_1((\mathbb{C}^*)^n \setminus F_f^*)/\pi_\sharp(\pi_1(F_g^*)) \cong \mathbb{Z}/(a-b)\mathbb{Z}.$$

By Lemma 3.1, we see that $\pi_1((\mathbb{C}^*)^n \setminus F_f^*) \cong \mathbb{Z}^{n+1}$ and any subgroup of $\mathbb{Z}^{n+1}$ with a finite index is a free abelian group of the same rank $n+1$. Therefore $\pi_1(F_g^*) \cong \mathbb{Z}^{n+1}$. Note that $g(\mathbf{z}, \bar{\mathbf{z}}, w, \bar{w})$ is 1-convenient. Thus taking normal slice of each smooth divisor $z_i = 0$ in F_g, we see that

$$\iota_\sharp : \pi_1(F_g^*) \to \pi_1((\mathbb{C}^*)^{n+1})$$

is surjective. Consider the inclusion map $\iota : F_g^* \to (\mathbb{C}^*)^{n+1}$. If $\iota_\sharp$ is not injective, $\pi_1((\mathbb{C}^*)^{n+1}) \cong \pi_1(F_g^*)/\operatorname{Ker} \iota_\sharp$ can not be a free abelian group of rank $n+1$. Thus $\iota_\sharp : \pi_1(F_g^*) \to \pi_1((\mathbb{C}^*)^{n+1})$ is an isomorphism. Note that the canonical generators of $\pi_1((\mathbb{C}^*)^{n+1})$ are given by the lassos for the coordinate divisors $\{z_i = 0\}$, $i = 1, \dots, n+1$. We can take explicit generators by the loops

$$\omega_i : \ S^1 \to (\mathbb{C}^*)^{n+1},\ t \mapsto (b_1, \dots, b_{i-1}, \varepsilon \exp(2\pi t i), \dots, b_{n+1})$$

with $i = 1, \dots, n+1$ and $b_1, \dots, b_{n+1}$ are non-zero constants. Thus we can take a lasso ω_i' for the divisor $\{z_i = 0\} \subset F_g$ represented by the boundary loop ∂D_i of a small smooth normal disk D_i at a smooth point of the divisor $\{z_i = 0\}$. Clearly we have $[\omega_i'] \mapsto [\omega_i]$. Here $[\omega_i']$ and $[\omega_i]$ are the corresponding homotopy classes. As $\iota_\sharp$ is an isomorphism, $\{[\omega_i'] \mid i = 1, \dots, n+1\}$ are generators of $\pi_1(F_g^*)$. On the other hand, the inclusion $F_g^* \to F_g$ gives a surjection on their fundamental groups and $[\omega_i'] \mapsto 0 \in \pi_1(F_g)$. This implies that $\pi_1(F_g)$ is trivial.

For the proof of the assertion (2), we apply the additivity of the Euler characteristic to the union $F_g = F_g^{*\{n\}} \cup F_{g_n}$ where $F_g^{*\{n\}} := F_g \cap \{z_n \neq 0\}$ and $F_{g_n} := F_g \cap \{z_n = 0\}$. Note that $F_{g_n} \cong F_{f_n} \times \mathbb{C}$. Put $\mathbb{C}^{*\{n\}} = \mathbb{C}^n \cap \{z_n \neq 0\}$ and $F_f^{*\{n\}} = F_f \cap \{z_n \neq 0\}$. In the following, we consider the projection $\pi_n : \mathbb{C}^{n+1} \to \mathbb{C}^n$ defined by $\pi_n(\mathbf{z}, w) = \mathbf{z}$. Note that $\pi_n^{-1}(F_f) = F_f \times \mathbb{C}$ and $F_g^{*\{n\}} \cap \pi_n^{-1}(F_f) = \{(\mathbf{z}, 0) \mid \mathbf{z} \in F_f^{*\{n\}}\}$.

$$\begin{aligned}
\chi(F_g^{*\{n\}}) &= \chi(F_g^{*\{n\}} \setminus \pi_n^{-1}(F_f)) + \chi(F_g^{*\{n\}} \cap \pi_n^{-1}(F_f)) \\
&= (a-b)\chi(\mathbb{C}^{*\{n\}} \setminus F_f^{*\{n\}}) + \chi(F_f^{*\{n\}}) \\
&= -(a-b-1)\chi(F_f^{*\{n\}}) \\
\chi(F_{g_n}) &= \chi(F_{f_n} \times \mathbb{C}) = \chi(F_{f_n}).
\end{aligned}$$

The last equality follows from $F_{g_n} = F_{f_n} \times \mathbb{C}$. To complete the proof, we use the additivity of the Euler characteristic which gives the equality

$$\chi(F_f) = \chi(F_f^{*\{n\}}) + \chi(F_{f_n}).$$

□

3.1. Construction of a family of mixed curves with polar degree q. Now we are ready to construct a key family of mixed curves with embedding degree q. Recall the polynomial:

$$h_{q,r,j}(\mathbf{w},\bar{\mathbf{w}}) := (z_1^{q+j}\bar{z}_1^j + z_2^{q+j}\bar{z}_2^j)(z_1^{r-j} - \alpha z_2^{r-j})(\bar{z}_1^{r-j} - \beta\bar{z}_2^{r-j}), \quad \mathbf{w} = (z_1, z_2).$$

$h_{q,r,j}(\mathbf{w},\bar{\mathbf{w}})$ is 1-convenient strongly polar homogeneous polynomial with the radial degree $q+r$ and the polar degree q respectively. The constants α, β are generic. For this, it suffices to assume that $|\alpha|, |\beta| \neq 0, 1$ and $|\alpha| \neq |\beta|$. Consider the twisted join polynomial of 3 variables z_1, z_2, z_3:

$$s_{q,r,j}(\mathbf{z},\bar{\mathbf{z}}) = h_{q,r,j}(\mathbf{w},\bar{\mathbf{w}}) + \bar{z}_2 z_3^{q+r}\bar{z}_3^{r-1}, \quad \mathbf{z} = (z_1, z_2, z_3).$$

Let $F_{q,r,j} = s_{q,r,j}^{-1}(1) \subset \mathbb{C}^3$ be the Milnor fiber and let $S_{q,r,j} \subset \mathbb{P}^2$ be the corresponding mixed projective curve:

$$S_{q,r,j} = \{[\mathbf{z}] \in \mathbb{P}^2 \mid s_{q,r,j}(\mathbf{z},\bar{\mathbf{z}}) = 0\}.$$

Note that $S_{q,r,j}$ is a smooth mixed curve. The following describes the topology of $F_{q,r,j}$ and $S_{q,r,j}$.

Theorem 3.3. (1) *The Euler characteristic of the Milnor fiber $F_{q,r,j}$ is given by:*

$$\chi(F_{q,r,j}) = q(q^2 - q + 1 + 2(r-j)).$$

(2) *The genus of $S_{q,r,j}$ is given by:*

$$g(S_{q,r,j}) = \frac{q(q-1)}{2} + (r-j)$$

Proof. Let $H_{q,r,j} = h_{q,r,j}^{-1}(1)$. Then by Proposition 2.1,

$$\chi(H_{q,r,j}) = -q(q - 2 + 2(r-j))$$
$$\chi(H_{q,r,j} \cap \{z_2 = 0\}) = q$$

and the assertion follows from Theorem 3.2. □

3.2. Mixed curves with polar degree 1. We consider the case $q = 1, j = 0$:

$$\begin{cases} h(\mathbf{w},\bar{\mathbf{w}}) & := (z_1 + z_2)(z_1^r - \alpha z_2^r)(\bar{z}_1^r - \beta\bar{z}_2^r) \\ f_r(\mathbf{z},\bar{\mathbf{z}}) & := h(\mathbf{w},\bar{\mathbf{w}}) + \bar{z}_2 z_3^{r+1}\bar{z}_3^{r-1} \\ S_r & := \{[\mathbf{z}] \in \mathbb{P}^2 \mid f_r(\mathbf{z},\bar{\mathbf{z}}) = 0\}. \end{cases}$$

Corollary 3.4. *Let S_r be the mixed curve as above. Then the embedding degree of S_r is 1 and the genus of S_r is r.*

Proof. Let $F_r = f_r^{-1}(1)$ be the Milnor fiber of f_r. By Theorem 3.2, we have $\chi(F_r) = 2r + 1$. Thus by Corollary 1.4, the assertion follows immediately. □

Remark 3.5. $h(\mathbf{w},\bar{\mathbf{w}})$ can be replaced by $(z_1^{r+1} - z_2^{r+1})(\bar{z}_1 - \beta\bar{z}_2^r)$ without changing the topology.

4. Further embeddings of smooth curves

Consider a smooth curve $C \subset \mathbb{P}^2$ with genus g. If C is a complex algebraic curve of degree q, they are related by the Plücker formula $g = \frac{(q-1)(q-2)}{2}$. In particular, q is the positive integer root of $x^2 - 3x + 2 - 2g = 0$. Thus for a given $g \geq 1$, q is unique if it exists. In this section, we consider this problem in the category of mixed projective curves. Consider the family of mixed curves.

$$S_{q,r,1} : h_{q,r,1}(\mathbf{w}, \bar{\mathbf{w}}) + \bar{z}_2 z_3^{q+r} \bar{z}_3^{r-1}$$

We have shown that the genus g is given as follows.

$$g = \frac{q(q-1)}{2} + r - 1.$$

Assume that g is fixed and we consider the possible degree q. We can solve as

$$r = g - \frac{q(q-1)}{2} + 1.$$

This shows that

THEOREM 4.1. *For a given $g > 0$ and q which satisfies the inequality*

$$g \geq \frac{q(q-1)}{2},$$

the mixed curve $S_{q,r,1}$ with $r = g - \frac{q(q-1)}{2} + 1$ has genus g and degree q.

REMARK 4.2. Assume that

$$(\sharp) \quad \frac{q(q-1)}{2} \geq g \geq \frac{(q-1)(q-2)}{2}.$$

For the construction of a curve with $\{g, q\}$ satisfying $(\sharp)$, we can not use the surface $S_{q,r,1}$. If $g - \frac{(q-1)(q-2)}{2} \equiv 0 \mod q-1$, we can use the mixed curve $C_{q,r,1}$. If $g \not\equiv \frac{(q-1)(q-2)}{2} \mod q-1$, we do not know if such an embedding exists.

5. Mixed polar weighted polynomial with polar degree 1 of n variables

Let us consider mixed polar weighted homogeneous polynomials of n variables with polar degree 1. They have the following strong property:

THEOREM 5.1. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a polar weighted homogeneous polynomial of degree 1 of radial weight $(q_1, \dots, q_n; d)$ and polar weight $(p_1, \dots, p_n; 1)$. Then the Milnor fibration $\varphi = f/|f| : S^{2n-1} \setminus K \to S^1$ with $K = f^{-1}(0) \cap S^{2n-1}$ is trivial. In fact, the explicit diffeomorphism is given using the one-parameter family of diffeomorphisms of the monodromy flows $h_\theta : F \to F_\theta$ with $\theta \in \mathbb{R}$ and $F_\theta := \varphi^{-1}(\exp(i\theta))$ and*

$$h_\theta(\mathbf{z}) = \exp(i\theta) \circ \mathbf{z}$$

where $\rho \circ \mathbf{z} = (\rho^{p_1} z_1, \dots, \rho^{p_n} z_n)$ and $\rho \in S^1$. Note that $h_{2\pi} = \mathrm{id}$. The trivialization of the fibration is given by the diffeomorphism $\psi : F \times S^1 \to S^{2n-1} \setminus K$ which is defined by

$$\psi(\mathbf{z}, \exp(i\theta)) = h_\theta(\mathbf{z})$$

Observe that the trivialization is not an extension of the trivialization of the normal bundle of K in S^{2n-1}.

Corollary 5.2. *Let $f(\mathbf{w})$, $\mathbf{w} = (z_1, z_2)$ be a polar weighted homogeneous polynomial with polar degree 1. Then the link $K := f^{-1}(0) \cap S^3$ is trivially fibered over the circle. Thus we have*

$$\pi_1(S^3 \setminus K) \cong \mathbb{Z} \times \pi_1(F)$$

where F is the Milnor fiber.

Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a polar weighted homogeneous polynomial of n variables. On the topology of the hypersurface $F = f^{-1}(1)$, we propose the following basic question. *Is the homological (or homotopical) dimension of F is $n-1$ under a certain condition (say mixed non-degeneracy)?*

We say that $f(\mathbf{z}, \bar{\mathbf{z}})$ satisfies *the homological dimension property* if the assertion is satisfied for $F = f^{-1}(1)$. There are several cases in which the assertion is true.

(1) Simplicial type: Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a simplicial type polar weighted homogeneous polynomial. Then the homological dimension of F is at most $n-1$. This follows from Theorem 10, [**O1**].

(2) (Join type) Assume that $f(\mathbf{z}, \bar{\mathbf{z}}) = h(\mathbf{w}, \bar{\mathbf{w}}) + k(\mathbf{u}, \bar{\mathbf{u}})$ where $\mathbf{w} = (w_1, \dots, w_m)$, $\mathbf{u} = (u_1, \dots, u_\ell)$ and $\mathbf{z} = (\mathbf{w}, \mathbf{u})$. Assume that $h(\mathbf{w}, \bar{\mathbf{w}})$, $k(\mathbf{u}, \bar{\mathbf{u}})$ are polar weighted homogeneous polynomials which satisfies the homological dimension property. Then f also satisfies the property. This follows from the Join theorem by Cisneros Molino [**Mol**].

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