THE HILBERT TRANSFORM ON LIPSCHITZ CURVES†

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Our aim is to prove the following theorems.

THEOREM A. Let γ be a curve in the complex plane parametrized by $x + ih(x)$, $x \in \mathbb{R}$, where h is a real-valued absolutely continuous function with derivative $h' \in L_\infty(\mathbb{R})$. Let H_γ denote the Hilbert transform on γ , defined, for $u \in L_2(\gamma)$, by

$$(H_\gamma u)(x) = \frac{i}{\pi} \text{ p.v.} \int_{-\infty}^{\infty} \frac{\sqrt{1+ih'(x)}\sqrt{1+ih'(y)}}{(x+ih(x)) - (y+ih(y))} u(y)dy \ .$$

Then $(H_\gamma u)(x)$ is defined for almost all $x \in \mathbb{R}$, and $H_\gamma u \in L_2(\mathbb{R})$. Indeed, there exists a constant c , depending only on $\|h'\|_\infty$, such that

$$\|H_\gamma u\|_2 \le c\|u\|_2 \qquad \text{for all} \quad u \in L_2(\mathbb{R}) \ .$$

THEOREM B. Let $f \in L_\infty(\mathbb{R})$, with $\operatorname{Re} f \ge 1$, and let M denote the maximal accretive operator in $L_2(\mathbb{R})$ defined by $Mu = -\frac{d}{dx}(f\frac{du}{dx})$, with domain $\mathcal{D}(M) = \{u \in H^1(\mathbb{R}) \mid f\frac{du}{dx} \in H^1(\mathbb{R})\}$ (where $H^1(\mathbb{R}) = \{u \in L_2(\mathbb{R}) \mid \frac{du}{dx} \in L_2(\mathbb{R})\}$). Then the domain of the square root, $M^{\frac{1}{2}}$, is $H^1(\mathbb{R})$, and, if $u \in H^1(\mathbb{R})$,

$$\frac{1}{6}\rho\|f\|_\infty^{-6}\|\frac{du}{dx}\|_2 \le \|M^{\frac{1}{2}}u\|_2 \le 6\rho\|f\|_\infty^{6}\|\frac{du}{dx}\|_2 \ .$$

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† This is an alternative treatment of results to be published in the Annals of Mathematics.

Moreover, if $f = f_z$ depends analytically on a parameter $z \in \Omega$, where Ω is an open subset of $\mathbb{C}$, and $M = M_z$ is the corresponding operator, then, for all $u \in H^1(\mathbb{R})$, $M_z^{\frac{1}{2}}u$ depends analytically on z.

The first theorem solves a conjecture of Calderon. It has already been proved by Calderon in the case when $\|h'\|_\infty$ is sufficiently small [C2]. The second theorem answers a question of Kato in the special case of ordinary differential operators defined on the whole real line. Both theorems have applications in the study of partial differential operators with non-smooth data. However such applications will not be given here.

Theorem A follows easily once it is shown that, given $b \in L_\infty(\mathbb{R})$, the operators $C_n(b)$ are L_2-bounded, with *norms growing as a power of* n, where

$$(C_n(b)u)(x) = \frac{i}{\pi}\ \text{p.v.}\int_{-\infty}^{\infty} \frac{(g(x)-g(y))^n}{(x-y)^{n+1}}\, u(y)\,dy\ ,$$

g being a function satisfying $g' = b$ a.e. The L_2-boundedness of $C_1(b)$ was first shown by Calderon in [C1], and that of $C_n(b)$ by Coifman and Meyer in [CM1] and [CM2]. However, the methods used in these and other papers have not been strong enough to give the required dependence of $\|C_n(b)\|$ on n. This paper is based on the following formula for $C_n(b)$:

$$C_n(b)u = \frac{i}{\pi}\ \text{p.v.}\int_{-\infty}^{\infty} R_t(BR_t)^n u\frac{dt}{t}\ , \qquad u \in L_2(\mathbb{R})\ ,$$

where $R_t = (I+itD)^{-1}$, $D = \frac{1}{i}\frac{d}{dx}$ and B denotes multiplication by b. We use this formula to show that

$$\|C_n(b)u\|_2 \leq c(1+n)^4\|b\|_\infty^n\|u\|_2\ ,$$

and thus to prove theorem A.

Theorem B is proved somewhat similarly, for the operators $M^{\frac{1}{2}}$ defined there can be written as a sum of operators somewhat similar to $C_n(b)$.

The paper splits naturally into "Hilbert space methods" and "harmonic analysis methods". In section 2 a new concept concerning Hilbert space operators is developed, namely that of a set $\mathcal{B}$ of bounded operators in a Hilbert space K being (q_j)-*compatible* with a given, possibly unbounded, operator A from a Hilbert space H to K. In the case when $q_j = O(j^m)$, $m < \infty$, estimates similar to the one above are obtained when $B \in \mathcal{B}$ and $R_t = (I+itA)^{-1}$. The required results then follow from the "harmonic analysis result" that $M = \{B \in L(L_2(\mathbb{R})) \mid Bu = bu , b \in L_\infty(\mathbb{R})\}$ is (q_j)-compatible with D , with $q_j = c(1+j)$.

Section 10 on the square-root problem can be read independently of §§5-9 on Calderon's problems. Section 4 contains some basic terminology.

1. Preliminaries.

Throughout this paper H and K denote complex Hilbert spaces and A denotes a closed operator from H to K with domain $\mathcal{D}(A)$ dense in H . We denote the range of A by $\mathcal{R}(A)$ and the kernel by $N(A)$. For $t > 0$, operators $Q_t \in L(H,K)$ and $P_t \in L(K)$ are defined by

$$Q_t = tA(I+t^2A^*A)^{-1}$$

and

$$P_t = I - \overline{tQ_tA^*} \qquad \text{(where } \overline{} \text{ denotes closure).}$$

THEOREM 1.1

(a) (i) $\|Q_t\| \leq \frac{1}{2}$;

(ii) $0 \leq (P_t u,u) \leq \|u\|^2$, $u \in K$;

(iii) $\|P_t\| \leq 1$; $\|I-2P_t\| \leq 1$.

(b) (i) $P_t = P_t^* = I - tAQ_t^*$;

(ii) $Q_t = t\overline{P_t A}$; $Q_t^* = tA^*P_t$;

(iii) $P_t(I-P_t) = Q_tQ_t^*$; $(I-2P_t)^2 = I - 4Q_tQ_t^*$.

(c) $t\dfrac{\partial Q_t}{\partial t} = -(I-2P_t)Q_t$; $t\dfrac{\partial P_t}{\partial t} = -2Q_tQ_t^*$;

$$t\frac{\partial}{\partial t}(Q_t^*(I-2P_t)) = -Q_t^* + 8Q_t^*Q_tQ_t^* .$$

(The fact that $Q_t \in L(H,K)$ and $P_t \in L(K)$ follows from the proofs of (a) (i) and (ii). For most of the paper we only use this theorem in the special case when $H = K$ and $A = A^*$. In this case, the proof of (a) is much simpler.)

Proof. A^*A is self-adjoint, $A^*A \geq 0$, $(I+A^*A)^{-1} \in L(H)$ and $R((I+A^*A)^{-1}) = \mathcal{D}(A^*A) \subset \mathcal{D}(A)$, so Q_t is everywhere defined. Also note that $\mathcal{D}((A^*A)^{\frac{1}{2}}) = \mathcal{D}(A)$, and $\|(A^*A)^{\frac{1}{2}}u\| = \|Au\|$ $\forall u \in \mathcal{D}(A)$.

(a) (i) $$\left\|\frac{t\sqrt{A^*A}}{1+t^2A^*A}\right\| \leq \sup_{\lambda\geq 0}\left|\frac{t\lambda}{1+t^2\lambda^2}\right| = \frac{1}{2} .$$

$$\therefore \quad \|Q_t\| = \sup\{\|tA(I+t^2A^*A)^{-1}u\| \mid \|u\| = 1\}$$

$$= \sup\{\|t\sqrt{A^*A}(I+t^2A^*A)^{-1}u\| \mid \|u\| = 1\} \leq \frac{1}{2} .$$

(ii) Let $A = V|A|$ where $|A| = \sqrt{A^*A}$ and V is a partial isometry. Then, if $u \in \mathcal{D}(A^*)$,

$$(P_t u,u) = \|u\|^2 - (t^2 V|A| \frac{1}{(1+t^2|A|^2)} |A|V^* u,u) \quad (\text{as } A^* = |A|V^*)$$

$$= \|u\|^2 - \left(\frac{t^2|A|^2}{(1+t^2|A|^2)} V^*u, V^*u\right) .$$

$$\therefore \quad - \|V^*u\|^2 \le (P_t u,u) - \|u\|^2 \le 0 .$$

$$\therefore \quad 0 \le \|u\|^2 - \|V^*u\|^2 \le (P_t u,u) \le \|u\|^2 .$$

Now $\mathcal{D}(A^*)$ is dense in K , so $0 \le P_t \le 1$.

(iii) This follows from (ii).

(b) (i),(ii) If $u \in \mathcal{D}(A^*A)$, $P_t tAu = (tA - Q_t t^2 A^*A)u = tA\left(I - \frac{t^2A^*A}{1+t^2A^*A}\right)u = Q_t u$.

$$\therefore \quad \overline{P_t tA} = Q_t .$$

The other parts of (i) and (ii) are straightforward.

(iii) $P_t(I-P_t) = P_t tAQ_t^* = Q_tQ_t^*$.

$$(I-2P_t)^2 = I - 4P_t + 4P_t^2 = I - 4P_t(I-P_t)$$

$$= I - 4Q_tQ_t^* .$$

(c) (i) $t\frac{\partial Q_t}{\partial t} = t\frac{\partial}{\partial t}\left(tA \frac{1}{1+t^2A^*A}\right)$

$$= Q_t - 2tA \frac{t^2A^*A}{(1+t^2A^*A)^2}$$

$$= Q_t - 2\left(tA\frac{1}{1+t^2A^*A} tA^*\right)\left(tA\frac{1}{1+t^2A^*A}\right)$$

$$= Q_t - 2(I-P_t)Q_t = -(I-2P_t)Q_t .$$

(ii) $$t\,\frac{\partial P_t}{\partial t} = t\,\frac{\partial}{\partial t}\,(I - tAQ_t^*)$$

$$= -\,tAQ_t^* + tAQ_t(I-2P_t)$$

$$= -\,2tAQ_t^*P_t = -\,2(I-P_t)P_t = -\,2Q_tQ_t^* \,.$$

(iii) $$t\,\frac{\partial}{\partial t}\,(Q_t^*(I-2P_t))$$

$$= -\,Q_t^*(I-2P_t)(I-2P_t) + 4Q_t^*Q_tQ_t^*$$

$$= -\,Q_t^*(I-4Q_tQ_t^*) + 4Q_t^*Q_tQ_t^*$$

$$= -\,Q_t^* + 8Q_t^*Q_tQ_t^* \,.$$ ||

The next lemma will be used often. Note that integrals of operators are always taken in the sense of strong convergence.

<u>LEMMA 1.2</u> Suppose H_j are Hilbert spaces, $j = 1,\ldots,4$, and $T_t \in L(H_1,H_2)$, $Z_t \in L(H_2,H_3)$ and $S_t \in L(H_4,H_3)$ are operators depending continuously (in the strong topology) on t. Suppose

$$\|S_{(.)}\| = \sup\left\{\int_0^\infty \|S_tu\|^2dt \;\middle|\; \|u\| = 1\right\}^{\frac{1}{2}} < \infty \,,$$

and

$$\|T_{(.)}\| < \infty \,.$$

Then

(i) $$\int_0^\infty S_t^*Z_tT_tdt \in L(H_1,H_4) \quad \text{and}$$

$$\left\|\int_0^\infty S_t^*Z_tT_tdt\right\| \le \sup_{t>0}\,\|Z_t\|\|S_{(.)}\|\|T_{(.)}\| \;;$$

(ii) in the case when $Z_t = I$ and $S_t = T_t$,

$$\left\|\int_0^\infty S_t^* S_t\,dt\right\| = \|S_{(.)}\|^2 .$$

Proof

(i) Let $u \in H$, and $0 < t_0 < t_1 < \infty$. Then

$$\left\|\int_{t_0}^{t_1} S_t^* Z_t T_t u\,dt\right\| = \sup_{\|v\|=1} \left|\int_{t_0}^{t_1} (S_t^* Z_t T_t u, v)\,dt\right|$$

$$\leq \sup_{\|v\|=1} \int_{t_0}^{t_1} \|Z_t\|\|T_t u\|\|S_t v\|\,dt$$

$$\leq \sup_{t>0} \|Z_t\|\|S_{(.)}\| \left\{\int_{t_0}^{t_1} \|T_t u\|^2 dt\right\}^{\frac{1}{2}} .$$

$\therefore \displaystyle\int_{\delta_0}^{\delta_1} S_t^* Z_t T_t u\,dt \to 0$ as $\delta_0, \delta_1 \to 0$, and

$\displaystyle\int_{M_0}^{M_1} S_t^* Z_t T_t u\,dt \to 0$ as $M_0, M_1 \to \infty$.

So $\displaystyle\int_0^\infty S_t^* Z_t T_t u\,dt$ exists and

$$\left\|\int_0^\infty S_t^* Z_t T_t u\,dt\right\| \leq \sup_{t>0} \|Z_t\|\|S_{(.)}\|\|T_{(.)}\|\|u\| .$$

(ii) As $\displaystyle\int_0^\infty S_t^* S_t\,dt$ is self-adjoint,

$$\left\|\int_0^\infty S_t^* S_t\,dt\right\| = \sup_{\|u\|=1} \left|\int_0^\infty (S_t^* S_t u, u)\,dt\right|$$

$$= \|S_{(.)}\|^2 . \qquad \|$$

As a first application of this lemma, we present the following result.

LEMMA 1.3

(i) $\quad 2\int_0^\infty Q_t^* Q_t \frac{dt}{t}$ is the orthogonal projection on $N(A)^\perp$;

(ii) $\quad 2\int_0^\infty Q_t Q_t^* \frac{dt}{t}$ is the orthogonal projection on $\overline{R(A)}$;

(iii) $\quad \int_0^\infty \|Q_t u\|^2 \frac{dt}{t} \le \tfrac{1}{2}\|u\|^2 , \quad u \in H$;

(iv) $\quad \int_0^\infty \|Q_t^* u\|^2 \frac{dt}{t} \le \tfrac{1}{2}\|u\|^2 , \quad u \in K$.

Proof

(i) $\quad Q_t^* Q_t = \dfrac{1}{1+t^2A^*A}\, t^2A^*A \,\dfrac{1}{1+t^2A^*A} = \dfrac{t^2A^*A}{(1+t^2A^*A)^2}$

$\therefore \quad 2\int_0^\infty Q_t^* Q_t \frac{dt}{t} = 2\int_0^\infty \frac{t^2A^*A}{(1+t^2A^*A)^2}\frac{dt}{t}$, and this is the orthogonal projection on $N(A)^\perp = N(A^*A)^\perp$.

(ii) If $\quad u \in D(A)$, $\quad Q_tQ_t^*Au = A \dfrac{1}{(1+t^2A^*A)^2}\, t^2A^*Au = A\dfrac{t^2A^*A}{(1+t^2A^*A)^2}\, u$.

$\therefore \quad 2\int_0^\infty Q_tQ_t^*Au \frac{dt}{t} = 2A\int_0^\infty \frac{t^2A^*A}{(1+t^2A^*A)^2}\frac{dt}{t}\, u = Au$.

If $\quad w \in R(A)^\perp = N(A^*)$, $\quad$ then $\quad 2\int_0^\infty Q_tQ_t^*w \frac{dt}{t} = 0$.

So (ii) is proved.

(iii) and (iv) These follow from (i) and (ii) on using part (ii) of lemma 1.2. $\quad\|$

The following technical lemma will be needed later.

<u>LEMMA 1.4</u> Suppose $v(t), u_k(t) \in H$ depend continuously on $t \in (0,\infty)$, $k = 0,1,2,\ldots$, and

(i) for all $0 < \delta < M < \infty$, $\sup_{t\in[\delta,M]} \|\sum_{k=0}^{N} u_k(t) - v(t)\| \to 0 \quad N \to \infty$,

(ii) $\int_0^\infty u_k(t)\,dt$ exists in H for all k, and

(iii) there exists $c_k \ge 0$ such that $\sum_{k=0}^{\infty} c_k < \infty$, and, for all $0 < \delta < M < \infty$, $\|\int_\delta^M u_k(t)\,dt\| \le c_k$.

Then

(a) $V = \sum_{k=0}^{\infty} \int_0^\infty u_k(t)\,dt$ exists in H,

(b) $\|V\| \le \sum_{k=0}^{\infty} \|\int_0^\infty u_k(t)\,dt\| < \infty$, and

(c) $\int_0^\infty v(t)\,dt$ exists in H, and $\int_0^\infty v(t)\,dt = V$.

<u>Proof</u>

(a) By (ii) and (iii), $\sum_{k=0}^{\infty} \|\int_0^\infty u_k(t)\,dt\| \le \sum_{k=0}^{\infty} c_k < \infty$, so

$$V = \sum_{k=0}^{\infty} \int_0^\infty u_k(t)\,dt \quad \text{exists in } H .$$

(b) This is now clear

(c) Let $\varepsilon > 0$. Choose N such that $\sum_{k=N+1}^{\infty} c_k < \varepsilon$. Then

$$\|\sum_{k=0}^{N} \int_0^\infty u_k(t)\,dt - V\| < \varepsilon , \text{ and,}$$

for all $0 < \delta < M < \infty$, $\|\sum_{k=N+1}^{\infty} \int_\delta^M u_k(t)\,dt\| < \varepsilon$, and hence by (i),

$$\left\| \int_\delta^M v(t)\,dt - \int_\delta^M \sum_{k=0}^N u_k(t)\,dt \right\| < \varepsilon \ .$$

Now choose $0 < \delta_0 < M_0 < \infty$ such that

$$\left\| \sum_{k=0}^N \left\{ \int_\delta^M u_k(t)\,dt - \int_0^\infty u_k(t)\,dt \right\} \right\| < \varepsilon \ , \qquad 0 < \delta \le \delta_0 < M_0 \le M < \infty \ ,$$

$$\therefore \quad \left\| \int_\delta^M v(t)\,dt - V \right\| < 3\varepsilon \ , \qquad 0 < \delta \le \delta_0 < M_0 \le M < \infty \ .$$

The result follows. ||

2. <u>Compatibility</u>. Let $\mathcal{B} \subset L(K)$, and let $A : H \to K$ be closed and densely defined. Let $\mathcal{B}_1 = \{\lambda B \mid B \in \mathcal{B} \ , \ \lambda \ge 0, \ \|\lambda B\| \le 1\}$. Let $q = (q_k) = (q_0, q_1, q_2, \ldots)$ where $0 \le q_k \le \infty$, $k = 0,1,\ldots,$ $q_0 = \frac{1}{\sqrt{2}}$.

<u>DEFINITION</u> $\mathcal{B}$ is q-*compatible* with A if,

$$\forall B_1, \ldots B_k \in \mathcal{B}_1 \ , \quad \forall u \in K \ ,$$

$$\int_0^\infty \|Q_t^*(B_k P_t B_{k-1} P_t \ldots B_1 P_t) u\|^2 \frac{dt}{t} \le q_k^2 \|u\|^2 \ .$$

(Lemma 1.3 gives the reason for taking $q_0 = \frac{1}{\sqrt{2}}$.)

<u>THEOREM 2.1</u>

(a) $\mathcal{B}$ is q-compatible with A if and only if, $\forall j,k \ge 1$ such that $q_j, q_k < \infty$, $\forall B_1, \ldots, B_k$, $C_1, \ldots C_j \in \mathcal{B}_1$, $\forall Z_t \in L(H)$ depending continuously on $t \in (0,\infty)$, with $\|Z_t\| \le 1$,

$$\int_0^\infty (P_t C_1^* P_t \ldots C_j^*) Q_t Z_t Q_t^* (B_k P_t \ldots B_1 P_t) \frac{dt}{t} \in L(K)$$

and has norm $\le q_j q_k$.

(b) Let (r_k) be a sequence of positive real numbers, $k = 0,1,\dots,$ with $r_0 = \frac{1}{\sqrt{2}}$, and $q_k = \sum_{j=0}^{k} r_j$. Suppose, $\forall B_1,\dots B_k \in \mathcal{B}_1$, $\exists Z_t \in L(H)$ and $Y_t \in L(K,H)$ depending continuously on $t \in (0,\infty)$, such that $\|Z_t\| \le 1$, $Q_t^* B_k P_t = Z_t Q_t^* + Y_t$, and

$$\int_0^\infty \|Y_t(B_{k-1}P_tB_{k-2}P_t\dots B_1P_t)u\|^2 \frac{dt}{t} \le r_k^2\|u\|^2 \qquad \forall u \in K .$$

Then $\mathcal{B}$ is q-compatible with A.

(c) If $\mathcal{B}$ is q-compatible with A, $q_1 < \infty,\dots,q_k < \infty$, and $B_1,B_2,\dots,B_k \in \mathcal{B}_1$, then

$$\int_0^\infty Q_t^*(B_kP_t\dots B_1P_t)\frac{dt}{t} \in L(K,H)$$

and

$$\left\|\int_0^\infty Q_t^*(B_kP_tB_{k-1}\dots B_1P_t)\frac{dt}{t}\right\| \le \frac{1}{\sqrt{2}}\left(\sum_{j=0}^{k-1} q_j + 4q_k\right) .$$

Moreover, if $0 < \delta < M < \infty$, then

$$\left\|\int_\delta^M Q_t^*(B_kP_t\dots B_1P_t)\frac{dt}{t}\right\| \le 1 + \frac{1}{\sqrt{2}}\left(\sum_{j=0}^{k-1} q_j + 4q_k\right) .$$

(d) If $K = H$, $A = A^*$, $\mathcal{B}$ is q-compatible with A, $i = \sup(j,k)$, $q_1 < \infty,\dots,q_i < \infty$, and $B_1,\dots,B_k,C_1,\dots,C_j \in \mathcal{B}_1$, then

$$\int_0^\infty (P_tC_1^*P_t\dots C_j^*)Q_t(B_kP_t\dots B_1P_t)\frac{dt}{t} \in L(K,H)$$

and has norm $\le q_j \sum_{\ell=0}^{k-1} q_\ell + q_k \sum_{\ell=0}^{j-1} q_\ell + 4q_jq_k$.

Proof

(a) This follows directly from lemma 1.2.

(b) Let $B_1, \dots B_k \in \mathcal{B}_1$, and for each $j \in \{1,2,\dots k\}$ let $Z_{j,t}$ and $Y_{j,t}$ be operators depending continuously on t such that $\|Z_{j,t}\| \le 1$, $Q_t^* B_j P_t = Z_{j,t} Q_t^* + Y_{j,t}$, and

$$\left\|\int_0^\infty Y_{j,t} B_{k-1} P_t \dots B_1 P_t u\right\|^2 \frac{dt}{t} \le r_k^2 \|u\|^2 \qquad \forall u \in K .$$

Then, for $k \ge 1$,

$$Q_t^*(B_k P_t B_{k-1} P_t \dots B_1 P_t)$$

$$= Y_{k,t}(B_{k-1}P_t \dots B_1 P_t) + Z_{k,t} Y_{k-1,t}(B_{k-2}P_t \dots B_1 P_t)$$

$$+ Z_{k,t} Z_{k-1,t} Y_{k-2,t}(B_{k-3}P_t \dots B_1 P_t) + \dots +$$

$$+ Z_{k,t} Z_{k-1,t} \dots Z_{2,t} Y_{1,t} + Z_{k,t} Z_{k-1,t} \dots Z_{1,t} Q_t^* .$$

$$\therefore \quad \|Q_t^*(B_k P_t \dots B_1 P_t)u\|$$

$$\le \|Y_{k,t}(B_{k-1}P_t \dots B_1 P_t)u\| + \|Y_{k-1,t}(B_{k-2}P_t \dots B_1 P_t)u\|$$

$$+ \|Y_{k-2,t}(B_{k-3,t} \dots B_1 P_t)u\| + \dots + \|Y_{1,t}u\| + \|Q_t^* u\| .$$

$$\therefore \quad \int_0^\infty \|Q_t^*(B_k P_t \dots B_1 P_t)u\|^2 \frac{dt}{t} \le (r_k + r_{k-1} + \dots + r_1 + \tfrac{1}{\sqrt{2}})^2 \|u\|^2$$

(applying lemma 1.3 and the triangle inequality for $L_2(0,\infty)$) .

(c) Suppose $\mathcal{B}$ is q-compatible with A , and $B_1, \dots B_k \in \mathcal{B}_1$. We apply part (c) of theorem 1.1.

$$Q_t^*(B_kP_t\dots B_1P_t) = 8Q_t^*Q_tQ_t^* \prod_{j=0}^{k-1} (B_{k-j}P_t)$$

$$- t\frac{\partial}{\partial t}(Q_t^*(I-2P_t)) \prod_{j=0}^{k-1} (B_{k-j}P_t)$$

$$= 8Q_t^*Q_tQ_t^* \prod_{j=0}^{k-1} (B_{k-j}P_t) - t\frac{\partial}{\partial t}\left\{Q_t^*(I-2P_t) \prod_{j=0}^{k-1} (B_{k-j}P_t)\right\}$$

$$+ \sum_{\ell=0}^{k-1} Q_t^*(I-2P_t) \prod_{j=0}^{k-\ell-2} (B_{k-j}P_t)B_{\ell+1}(-2Q_tQ_t^*) \prod_{j=0}^{\ell-1} (B_{\ell-j}P_t)$$

(where $\prod_{j=0}^{-1} S_j$ is defined to equal I for any S_j)

$$= - t\frac{\partial}{\partial t}\left\{Q_t^*(I-2P_t) \prod_{j=0}^{k-1} (B_{k-j}P_t)\right\}$$

$$+ Q_t^*Z_{k,t}Q_t^* \prod_{j=0}^{k-1} (B_{k-j}P_t) + \sum_{\ell=0}^{k-1} Q_t^*Z_{\ell,t}Q_t^* \prod_{j=0}^{\ell-1} (B_{\ell-j}P_t)$$

where $Z_{\ell,t} \in L(H,K)$ depends continuously on t and satisfies $\|Z_{\ell,t}\| \le 1$, $\ell = 0,\dots k-1$, $\|Z_{k,t}\| \le 4$. Therefore if $0 < \delta < M < \infty$,

$$\int_\delta^M Q_t^* \prod_{j=0}^{k-1} (B_{k-j}P_t) \frac{dt}{t} = - Q_M^*(I-2P_M) \prod_{j=0}^{k-1} (B_{k-j}P_M)$$

$$+ Q_\delta^*(I-2P_\delta) \prod_{j=0}^{k-1} (B_{k-j}P_\delta) + \sum_{\ell=0}^{k} \int_\delta^M Q_t^*Z_{\ell,t}Q_t^* \prod_{j=0}^{\ell-1} (B_{\ell-j}P_t) \frac{dt}{t} .$$

The first two terms on the right hand side each have norm $\le \frac{1}{2}$, and converge strongly to zero as $M \to \infty$ and $\delta \to 0$. The last term has norm $\le$

$$\sum_{\ell=0}^{k} \left\|\int_\delta^M Q_t^*Z_{\ell,t}Q_t^* \prod_{j=0}^{\ell-1} (B_{\ell-j}P_t) \frac{dt}{t}\right\| \le \frac{1}{\sqrt{2}} (q_0+q_1+\dots q_{k-1}+4q_k) ,$$

as is seen on applying the triangle inequality and the definition of q-compatibility. Moreover, it converges strongly in $\mathcal{L}(K,H)$. This completes the proof of (c).

(d) The proof is similar to that of (c). ||

REMARKS

1. If $q_0 = \frac{1}{\sqrt{2}}$, $q_j = \infty$, $j \geq 1$, then $\mathcal{L}(K)$ is q-compatible with A .

2. If $q_1 < \infty$, then it is not always true that $\mathcal{L}(K)$ is q-compatible with A . This can be seen, for example, by noting that theorem 6.1 is not valid if there are no compatibility assumptions on B [Mc1,Mc3].

3. If P_t commutes with B for all $B \in \mathcal{B}$ and $t > 0$, then $\mathcal{B}$ is q-compatible with A when $q_k = \frac{1}{\sqrt{2(1+2k)}}$. To see this, first note that $P_tQ_t = Q_tS_t$, where $S_t = (I+t^2A^*A)^{-1}$. So

$$\int_0^\infty (P_tB_1^*P_t\ldots B_k^*)Q_tQ_t^*(B_kP_t\ldots B_1P_t)\,\frac{dt}{t}$$

$$= (B_1^*\ldots B_k^*)\int_0^\infty Q_tS_t^{2k}Q_t^*\,\frac{dt}{t}(B_k\ldots B_1) \; .$$

As in lemma 1.3, it can be shown that

$$\left\|\int_0^\infty Q_tS_t^{2k}Q_t^*\,\frac{dt}{t}\right\| \leq \frac{1}{2(1+2k)} \; .$$

The result follows.

4. The size of the q_k is in some sense a measure of the non-commutativity of P_t with $B \in \mathcal{B}$, or, in other words, of A^*A with A^*BA for $B \in \mathcal{B}$.

5. Parts (c) and (d) of the above theorem are of vital importance. The notion of compatability would be useless without them.

3. The Operator D .

Throughout this section $H = K = L_2(\mathbb{R})$, $A = D = \frac{1}{i}\frac{d}{dx}$, $M = \{B \mid \exists\, b \in L_\infty(\mathbb{R}) \ni Bu = bu \ \forall u \in L_2(\mathbb{R})\} \subset L(H)$. Our aim is to prove the following theorem.

THEOREM 3.1. Let $q = (q_k)$, where $q_k = \frac{1}{\sqrt{2}} + 6k(1 + \frac{1}{e\sqrt{2}})$, $k \geq 0$. Then M is q-compatible with D .

We first prove an identity.

LEMMA 3.2. If $b \in L_\infty(\mathbb{R})$, $f \in L_2(\mathbb{R})$, then

(i) $P_t b, Q_t b \in L_\infty(\mathbb{R})$, $\|P_t b\|_\infty \leq \|b\|_\infty$, $\|Q_t b\|_\infty \leq \|b\|_\infty$, and

(ii) $Q_t(bP_t f) = (Q_t b)(P_t f) - Q_t((Q_t b)(Q_t f)) + P_t((P_t b)(Q_t f))$.

Proof

(i) Let $\phi(x) = \frac{1}{2}e^{-|x|}$, $\psi(x) = \frac{i}{2}e^{-|x|}\operatorname{sgn}(x)$.

Let $\phi_t(x) = \frac{1}{t}\phi(\frac{x}{t})$, $\psi_t(x) = \frac{1}{t}\psi(\frac{x}{t})$.

Then $P_t b = b*\phi_t$, $Q_t b = b*\psi_t$. So $P_t b, Q_t b \in L_\infty$ and $\|P_t b\|_\infty \leq \|\phi_t\|_1 \|b\|_\infty$, $\|Q_t b\|_\infty \leq \|\psi_t\|_1 \|b\|_\infty$. Now $\|\phi_t\|_1 = \|\phi\|_1 = 1$ and $\|\psi_t\|_1 = \|\psi\|_1 = 1$. The result follows.

(ii) On noting (i), we see that it suffices to prove this for $f \in S(\mathbb{R})$. By scaling, we see that it also suffices to prove it for $t = 1$. Write $Q_1 = Q$, $P_1 = P$. Let

$$\beta = (I+D^2)^{-1}b \in L_\infty \subset S'(\mathbb{R}) , \quad g = (I+D^2)^{-1}f \in S(\mathbb{R}) .$$

On operating on both sides of (ii) with $(I+D^2)$, we see that it suffices to prove

$$D(((I+D^2)\beta)g) = (I+D^2)((D\beta)g) - D((D\beta)(Dg)) + \beta(Dg) .$$

i.e. $D(\beta g) + D((D^2\beta)g) = (D\beta)g + D^2((D\beta)g) - D((D\beta)(Dg)) + \beta(Dg)$.

This makes sense in $\mathcal{S}'(\mathbb{R})$, and is easily verified. ||

Proof of Theorem 3.1. We shall apply (b) of theorem 2.1, with $r_j = 6(1+\frac{1}{e\sqrt{2}})$, $j = 1,2,...$ Let $b_1 ... b_k \in L_\infty(\mathbb{R})$ with $\|b_j\|_\infty \le 1$, $j = 1,2,...k$, and let $B_1,...B_k$ be the corresponding multiplication operators. By the preceding lemma

$$Q_t B_k P_t = Z_t Q_t + Y_t ,$$

where

$$Z_t f = P_t((P_t b_k) f) ,$$

and

$$Y_t f = (Q_t b_k) P_t f - Q_t((Q_t b_k)(Q_t f)) .$$

Now $\|P_t b_k\|_\infty \le 1$ (by the preceding lemma), and $\|P_t\| \le 1$, so $\|Z_t\| \le 1$. So what remains to be proved is that, if $u \in L_2(\mathbb{R})$, then

$$\int_0^\infty \|Y_t(B_{k-1}P_t B_{k-2}P_t \dots B_1 P_t)u\|^2 \frac{dt}{t} \le r_k^2 \|u\|^2 .$$

This will follow from the above formula for Y_t once we have shown that, $\forall u \in L_2(\mathbb{R})$,

$$\iint_{\mathbb{R}^2_+} |Q_t B_{k-1} P_t B_{k-2} \dots B_1 P_t u(x)|^2 \, |Q_t b_k(x)|^2 \frac{dxdt}{t}$$

$$\le 16(1+\frac{1}{e\sqrt{2}})^2 \|u\|^2$$

and

$$\iint_{\mathbb{R}^2_+} |P_tB_{k-1}P_tB_{k-2}\ldots B_1P_tu(x)|^2 \; |Q_tb_k(x)|^2 \frac{dxdt}{t}$$

$$16(1 + \frac{1}{e\sqrt{2}})^2\|u\|^2 \; .$$

To prove these estimates, we first prove three lemmas.

<u>LEMMA 3.3</u> Let $b \in L_\infty(\mathbb{R})$ with $\|b\|_\infty \le 1$. Then $d\mu = |Q_tb(x)|^2 \frac{dxdt}{t}$ defines a Carleson measure μ of norm $c_\mu \le (1 + \frac{1}{e\sqrt{2}})^2$ on $\mathbb{R}^2_+$. That is, μ is the completion of a Borel measure on $\mathbb{R}^2_+ = \{(x,t) | t > 0\}$ satisfying, for all $x_0 \in \mathbb{R}$, $d > 0$,

$$\mu(T(x_0,d)) = \iint_{T(x_0,d)} d\mu \le (1 + \frac{1}{e\sqrt{2}})^2 2d \; ,$$

where $T(x_0,d) = \{(x,t) \mid 0 < t + |x-x_0| < d\}$.

<u>LEMMA 3.4</u> If μ is a Carleson measure of norm c_μ on $\mathbb{R}^2_+$, and if g is a μ-measurable function on $\mathbb{R}^2_+$, then $M|g|$ is a Borel measurable function on $\mathbb{R}$, and

$$\iint_{\mathbb{R}^2_+} |g(x,t)|^2 d\mu \le c_\mu \int_{-\infty}^{\infty} (M|g|)^2(x)\,dx \; ,$$

where $M|g|(x) = \sup_{|y-x|\le t} |g(y,t)|$.

<u>LEMMA 3.5</u> For all integers $k \ge 1$ and functions $b_1, b_2, \ldots b_{k-1} \in L_\infty$ with $\|b_j\|_\infty \le 1$, $j = 1,\ldots k-1$,

$$M|P_tB_{k-1}P_tB_{k-2}\ldots B_1P_tu|(x) \le 2u^*(x)$$

and

$$M|Q_tB_{k-1}P_tB_{k-2}\ldots B_1P_tu|(x) \le 2u^*(x)$$

for all $u \in L_2(\mathbb{R})$ and $x \in \mathbb{R}$, where $B_j f = b_j f$, and u^* is the Hardy-Littlewood maximal function defined by

$$u^*(x) = \sup_{d>0} \frac{1}{2d} \int_{x-d}^{x+d} |u(y)|dy .$$

COMPLETION OF PROOF OF THEOREM 3.1 The estimates stated before the lemmas follow directly from the lemmas, and the fact [H] that

$$\|u^*\|_2 \leq 2\|u\|_2 .$$

We now prove these lemmas. We remark that Lemma 3.3 is a result of Fefferman and Stein. However a proof is included here for completeness. Lemma 3.4 is also well-known.

PROOF OF LEMMA 3.3 Let $b = b_1 + b_2$, where

$$b_1(x) = \begin{cases} b(x) , & |x-x_0| \leq 2d \\ 0 , & |x-x_0| > 2d . \end{cases}$$

Define μ_1 and μ_2 using b_1 and b_2 respectively. Then

$$\iint_{T(x_0,d)} d\mu_1 \leq \int_0^\infty \|Q_t b_1\|_2^2$$

$$\leq \tfrac{1}{2} \|b_1\|_2^2 \frac{dt}{t} \qquad \text{(from lemma 1.3)}$$

$$\leq \tfrac{1}{2} \cdot 4d \sup |b_1|^2 \leq 2d .$$

Also

$$\begin{aligned}
|Q_t b_2(x)| &= |\psi_t^* b_2| \\
&= \tfrac{1}{2}\left|\frac{1}{t}\int_{-\infty}^{\infty} e^{-|x-y|/t}\,\mathrm{sgn}(x-y)\,b_2(y)\,dy\right| \\
&\le \frac{1}{2t}\int_{|y-x_0|\ge 2d} e^{-|x-y|/t}\,dy \\
&\le \frac{1}{2t}\int_{|s|\ge d} e^{-|s|/t}\,ds \qquad (s=y-x) \\
&= e^{-d/t}\,.
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \iint_{T(x_0,d)} d\mu_2 &\le 2d\int_0^d e^{-2d/t}\,\frac{dt}{t} \qquad \left(s=\frac{1}{t}\right) \\
&\le 2d^2\int_{1/d}^{\infty} e^{-2ds}\,ds \\
&= de^{-2}\,.
\end{aligned}$$

We conclude that

$$\iint_{T(x_0,d)} d\mu \le 2d\left(1+\frac{1}{e\sqrt{2}}\right)^2 . \qquad \|$$

PROOF OF LEMMA 3.4 If S is a μ-measurable subset of $\mathbb{R}_+^2$, define

$$S' = \{x \in \mathbb{R} \mid \exists\,(y,t) \in S \text{ such that } |x-y| \le t\}\,.$$

Note that S' is a countable disjoint union of intervals I_k and therefore a Borel set, and that $S \subset \bigcup_k T_k$, where $T_k = \{(x,t) \mid [x-t,x+t] \subset I_k\}$. Also note that $\mu(T_k) \le c_\mu \times$ length (I_k) , even if I_k is not open, so that $\mu(S) \le c_\mu m(S')$, where m denotes Lebesgue measure.

If $\alpha \in (0,\infty)$ and $S = \{(x,t) \mid |g(x,t)| > \alpha\}$, then $S' = \{x \in \mathbb{R} \mid M|g|(x) > \alpha\}$. This shows that $M|g|$ is Borel measurable and also that $\mu\{(x,t) \mid |g(x,t)| > \alpha\} \le c_\mu m\{x|M|g|(x) > \alpha\}$. The result follows.

<u>PROOF OF LEMMA 3.5</u> Since $\phi_t \ge 0$ and $|\psi_t| = \phi_t$, we see that if $g,h \in L_2(\mathbb{R})$ and $|g(x)| \le h(x)$ for all x, then

$$|P_t g(x)| \le (P_t|h|)(x)$$

and

$$|Q_t g(x)| \le (P_t|h|)(x) .$$

Consequently,

$$|P_t B_{k-1} P_t \dots P_t B_1 P_t u(x)| \le P_t^k(|u|)(x) ,$$

and

$$|Q_t B_{k-1} P_t \dots P_t B_1 P_t u(x)| \le P_t^k(|u|)(x) .$$

So, it suffices to prove that, for $k = 1,2,\dots,$

$$M(P_t^k(|u|)) \le 2u^* .$$

Let $P_t^{k-1}(|u|) = v$. Then $v \ge 0$.

$$\begin{aligned} M(P_t^k(|u|)) &= M(P_t(v)) \\ &= \sup_{-t\le h\le t} (\phi_{h,t} * v) , \quad \text{where } \phi_{h,t}(x) = \frac{1}{t}\,\phi\left(\frac{x-h}{t}\right) \\ &\le 2 \sup_{t>0} (\theta_t * v) \quad \text{where } \theta_t(x) = \begin{cases} \frac{1}{4t} , & |x| \le t \\ \frac{1}{4t}\, e^{1-|x/t|} , & |x| \ge t . \end{cases} \end{aligned}$$

$$\therefore \quad M(P_t^k(|u|)) \le 2 \sup_{t>0} (\theta_t * \phi_t * \ldots * \phi_t * |u|) .$$

We need one final lemma, stated below, to conclude that

$$M(P_t^k|u|) \le 2|u|^* = 2u^* .$$

This follows directly from part (v) of lemma 3.6, with $g = |u|$, and $f = \theta_t * \phi_t \ldots * \phi_t$. ‖

<u>LEMMA 3.6</u> Let $F = \{f \in L_1(\mathbb{R}) \mid f \ge 0$, f is a bounded even decreasing function, and $\int_{-\infty}^{\infty} f(x)\,dx = 1\}$. Then

(i) $\phi_t, \theta_t \in F$ for all $t \in (0,\infty)$.

(ii) F is a convex subset of $L_1(\mathbb{R})$,

(iii) $F_0 = \mathrm{co}\{\chi_d \mid d > 0\}$ is L_1-dense and L_2-dense in F, where $\chi_d = (2d)^{-1}\chi_{[-d,d]}$, and "co" denotes the convex hull,

(iv) if $f_1, f_2 \in F$, then $f_1 * f_2 \in F$,

(v) if $g \in L_2(\mathbb{R})$ and $f \in F$, then $|(g*f)(x)| \le g^*(x)$, where g^* is defined in lemma 3.5.

<u>Proof</u> Parts (i), (ii) and (iii) are readily verified.

(iv) On noting parts (ii) and (iii), we see that it suffices to prove (iv) when $f = \chi_d$ for some d. This is easily checked.

(v) Let $g \in L_2(\mathbb{R})$. If $f = \chi_d$, then, for all $x \in \mathbb{R}$,

$$|(g*f)(x)| = \frac{1}{2d}\left|\int_{-d}^{d} g(x-y)\,dy\right| \le g^*(x) .$$

Thus the inequality holds when $f \in F_0$. Finally, if $f \in F$, there exists a sequence $f_n \in F_0$ such that $\|f_n - f\|_2 \to 0$. So, if $x \in \mathbb{R}$,

$$|(g*f_n)(x) - (g*f)(x)| = \left|\int g(x-t)(f_n-f)(t)dt\right|$$
$$\leq \|g\|_2^{1/2}\|f_n-f\|_2^{1/2} \to 0 \ .$$

The result follows. ||

4. Facts about operators. By an *operator* T from a Hilbert space H to a Hilbert space K is meant a linear mapping $T : \mathcal{D}(T) \to K$, where the *domain* $\mathcal{D}(T)$ is a linear subspace of H. The *range* (or *image*) of T is denoted by $R(T)$. Let us denote the set of all operators from H to K by $Op(H,K)$, or if $H = K$, by $Op(H)$. If $T \in Op(H,K)$, then $\|T\|$ is the (possibly infinite) number,

$$\|T\| = \sup\{\|Tu\| \mid u \in \mathcal{D}(T), \|u\| = 1\} \ .$$

The set of bounded linear operators from H to K is denoted $L(H,K)$. That is,

$$L(H,K) = \{T \in Op(H,K) \mid \mathcal{D}(T) = H, \|T\| < \infty\} \ .$$

Also

$$L(H) = L(H,H) \ .$$

When new operators are constructed from old, the domains are taken to be the largest for which the construction makes sense. For example,

$$\mathcal{D}(S+T) = \mathcal{D}(S) \cap \mathcal{D}(T) \ ,$$

$$\mathcal{D}(ST) = \{u \in \mathcal{D}(T) \mid Tu \in \mathcal{D}(S)\} \ .$$

If $S(t) \in L(H,K)$ depends continuously on $t \in (0,\infty)$, then

$$\mathcal{D}\left(\int_0^{\infty} S(t)\,dt\right) = \left\{u \in H \;\middle|\; \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\varepsilon}^{N} S(t)u\,dt \text{ exists in } K\right\} .$$

We write $S \subset T$ if $\mathcal{D}(S) \subset \mathcal{D}(T)$ and $Su = Tu$ for all $u \in \mathcal{D}(S)$. So $S = T$ if and only if $S \subset T$ and $T \subset S$. Note that

$$(ST)U = S(TU) .$$

$$S(T+U) \supset ST + SU$$

$$(S+T)U = SU + TU$$

$$S - S \subset 0 .$$

Some care needs to be exercise owing to the fact that the above relationships are not all equalities.

5. <u>Facts about commutators</u>. If $G \in L(H)$, then the derivation $\delta_G : \mathcal{O}p(H) \to \mathcal{O}p(H)$ is defined by

$$\delta_G(S) = -i(GS-SG) .$$

Note that $\mathcal{D}(\delta_G(S)) = \mathcal{D}(GS) \cap \mathcal{D}(SG)$.

The following properties can be verified in turn. In them, $S,T,G \in \mathcal{O}p(H)$, $\delta = \delta_G$, and $\lambda \neq 0$.

i) $$\delta(S+T) \supset \delta(S) + \delta(T)$$

ii) $$\delta(ST) \supset S\delta(T) + \delta(S)T$$

$$S\delta(T) \supset \delta(ST) - \delta(S)T$$

$$\delta(S)T \supset \delta(ST) - S\delta(T)$$

iii) $$\delta(\lambda S+I) = \lambda\delta(S)$$

iv) If $(\lambda S+I)^{-1} \in L(H)$, then

$$\delta((\lambda S+I)^{-1}) \supset -\lambda(\lambda S+I)^{-1}\delta(S)(\lambda S+I)^{-1} ,$$

and

$$\delta(S(\lambda S+I)^{-1}) \supset (\lambda S+I)^{-1}\delta(S)(\lambda S+I)^{-1} .$$

Moreover, if $G(\mathcal{D}(S)) \subset \mathcal{D}(S)$, then these inclusions can be replaced by equality.

v) If $S(t) \in L(H)$ depends continuously on $t \in (0,\infty)$, then

$$\delta\left(\int_0^\infty S(t)\right)dt \subset \int_0^\infty \delta(S(t))dt .$$

The proofs are straightforward. For example, the first member of (ii) is proved as follows.

$$\begin{aligned} S\delta(T) + \delta(S)T &= -iS(GT-TG) - i(GS-SG)T \\ &= -iS(GT-TG) - iGST + iSGT \\ &\subset -iS(GT-TG-GT) - iGST \\ &\subset -iS(0-TG) - iGST \\ &= \delta(ST) . \end{aligned}$$

To prove (iv), note that

$$(\lambda S+I)^{-1}(\lambda S+I) = I .$$

$$\begin{aligned} \therefore\ \delta((\lambda S+I)^{-1})(\lambda S+I) &\supset \delta(I) - (\lambda S+I)^{-1}\delta(\lambda S+I) \\ &= 0|_{\mathcal{D}(G)} - \lambda(\lambda S+I)^{-1}\delta(S) \\ &= -\lambda(\lambda S+I)^{-1}\delta(S) . \end{aligned}$$

$$\therefore\ \delta((\lambda S+I)^{-1}) \supset -\lambda(\lambda S+I)^{-1}\delta(S)(\lambda S+I)^{-1} .$$

$$\therefore \quad \delta(S(\lambda S+I)^{-1}) = \frac{1}{\lambda}\,\delta(\{(\lambda S+I) - I\}(\lambda S+I)^{-1})$$
$$= \frac{1}{\lambda}\,\delta(I-(\lambda S+I)^{-1})$$
$$= -\frac{1}{\lambda}\,\delta((\lambda S+I)^{-1})$$
$$\supset (\lambda S+I)^{-1}\delta(S)(\lambda S+I)^{-1} \ .$$

6. Bounds on commutators. If A is a self-adjoint operator in H, then $|A| = (A^2)^{\frac{1}{2}}$. Note that

$$|A| = \frac{2}{\pi}\int_0^\infty A^2(I+t^2A^2)^{-1}dt$$
$$= \frac{1}{\pi}\ \text{p.v.}\int_{-\infty}^{\infty} iA(I+itA)^{-1}\,\frac{dt}{t} \ .$$

Let us first prove the following theorem.

THEOREM 6.1 Suppose A is a self-adjoint operator in H, and $G \in L(H)$. Suppose also that $G(\mathcal{D}(A)) \subset \mathcal{D}(A)$, that $B = -i(\overline{GA-AG}) \in L(H)$, and that $\{B\}$ and $\{B^*\}$ are q-compatible with A, with $q_1 < \infty$. Then $\overline{G|A|-|A|G} \in L(H)$, and

$$\|G|A| - |A|G\| \leq \frac{2}{\pi}(1+4\sqrt{2}q_1)\,\|B\| \ .$$

Proof Let $\delta = \delta_G$ be defined as above, and use the properties already developed. In particular,

$$\delta(A(I+itA)^{-1}) = (I+itA)^{-1}\delta(A)(I+itA)^{-1}$$
$$= (P_t-iQ_t)B(P_t-iQ_t)$$
$$(\text{as } \delta(A) \subset B) \ .$$

$$\therefore \qquad \delta(|A|) \subset \frac{1}{\pi}\ \text{p.v.}\int_{-\infty}^{\infty} i\delta(A(I+itA)^{-1})\ \frac{dt}{t}$$

$$= \frac{1}{\pi}\ \text{p.v.}\int_{-\infty}^{\infty} i(P_t - iQ_t)B(P_t - iQ_t)\ \frac{dt}{t}$$

$$= \frac{2}{\pi}\int_0^{\infty} (P_tBQ_t + Q_tBP_t)\ \frac{dt}{t}\ .$$

As $\{B\}$ and $\{B^*\}$ are q-compatible with A, it follows from (d) of theorem 2.1 that the right hand side is in $L(H)$, and that it has norm $\leq 2 \cdot \frac{2}{\pi} \cdot \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}} + 4q_1)\|B\| = \frac{2}{\pi}(1 + 4\sqrt{2}q_1)\|B\|$.

Now $\mathcal{D}(\delta(|A|)) = \mathcal{D}(|A|) = \mathcal{D}(A)$, which is dense in H, so $\overline{\delta(|A|)} \in L(H)$, and

$$\|\delta(|A|)\| \leq \frac{2}{\pi}(1 + 4\sqrt{2}q_1)\|B\|\ . \qquad \|$$

We remark that, in the case when $H = L_2(\mathbb{R})$, $A = D = \frac{1}{i}\frac{d}{dt}$, and G is multiplication by an L_∞-function g with L_∞-derivative $g' = b$, then B is multiplication by b, $\|B\| = \|b\|_\infty$ and $q_1 \leq \frac{1}{\sqrt{2}} + 6(1 + \frac{1}{e\sqrt{2}})$, as is shown in §3. Therefore

$$\|\delta(|D|)\| \leq \frac{2}{\pi}(5 + 24\sqrt{2}(1 + \frac{1}{e\sqrt{2}}))\|B\|$$

$$< 31\|b\|_\infty\ .$$

So we have a new proof of the L_2-estimate proved by Calderon [Cl], together with an explicit bound on the norm. We leave comment on what happens when g is not bounded to the next section, where we generalize this result to the higher commutators.

7. Higher Commutators. We now prove estimates for certain operators which are related to the commutator integrals of Calderon. The relationship of these operators with commutators is given in part (c) of the next theorem, and their relationship with Calderon's integrals later on. Note however, that it is the fact that they satisfy the identity (b) of theorem 7.1 which is important, rather than the fact that, under suitable conditions, they are commutators. In fact, part (c) is not used again.

Throughout this section A denotes a self-adjoint operation in $\mathcal{H}$, and P_t and Q_t are defined as before. We also let

$$R_t = (I+itA)^{-1} = P_t - iQ_t , \qquad t \in \mathbb{R}, \quad t \neq 0 ,$$

and note that $\|R_t\| \leq 1$.

DEFINITION Given $B, B_1, B_2, \ldots, B_n \in L(\mathcal{H})$, define the operators $\underset{\sim}{C}(B_n,\ldots,B_1)$, $C(B_n,\ldots,B_1)$ and $C_n(B)$ by

$$\underset{\sim}{C}(B_n,\ldots,B_1) = \frac{i}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty} R_tB_nR_tB_{n-1} \cdots R_tB_1R_t \frac{dt}{t} ,$$

$$C(B_n,\ldots,B_1) = \text{symmetric part of } \underset{\sim}{C}(B_n,\ldots,B_1) ,$$

and

$$C_n(B) = C(B,\ldots,B) = \underset{\sim}{C}(B,\ldots B) .$$

When $n = 0$, we take $\underset{\sim}{C}(B_n,\ldots,B_1)$ to be

$$C_0(B) = \frac{i}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty} R_t \frac{dt}{t} = \frac{2}{\pi}\,\text{p.v.}\int_0^{\infty} Q_t \frac{dt}{t}$$

$$= H = \text{sgn}(A) .$$

THEOREM 7.1 Suppose $\mathcal{B}$ is q-compatible with A with $q_1,\ldots,q_n < \infty$. Suppose $B_j, B_j^* \in \mathcal{B}_1$, $j = 1,\ldots n$. Let

$$p_n = \frac{2}{\pi} \sum_{\ell=0}^{n} q_\ell \left\{ 3 \sum_{j=0}^{n-\ell-1} q_j + 4q_{n-\ell} \right\} .$$

Then the following are true

(a) $\underset{\sim}{C}(B_n,\ldots,B_1), C(B_n,\ldots,B_1)$ and $C_n(B_1) \in L(H)$, and have norms $\leq p_n$.

(b) If there exists $G_n \in L(H)$ such that $B_n = \overline{\delta_n(A)}$, and $\delta_n(B_j) = 0$, $j = 1,2,\ldots n-1$, where $\delta_n = \delta_{G_n}$, then $\overline{\delta_n(C(B_{n-1},\ldots,B_1))A} \in L(H)$, and

$$nC(B_n,\ldots,B_1) = \overline{\delta_n(C(B_{n-1},\ldots,B_1))A} + \sum_{k=1}^{n} C(B_n,\ldots,B_{k+1},B_{k-1},\ldots,B_1)B_k .$$

(For $n = 1$, this becomes $C(B_1) = \overline{\delta_1(H)A} + HB_1$.)

(c) If there exists $G_1,\ldots,G_n \in L(H)$ such that $B_j = \overline{\delta_j(A)}$, $j = 1,\ldots n$, and $\delta_k(B_j) = 0$, $1 \leq j < k \leq n$, where $\delta_j = \delta_{G_j}$, and if $B_i(\mathcal{D}(A^k)) \subset \mathcal{D}A^k)$, $1 \leq i \leq n$, $1 \leq k \leq n$, then

$$C(B_n,\ldots,B_1) = \frac{1}{n!} \overline{\delta_n \delta_{n-1} \ldots \delta_1(HA^n)} .$$

Proof

(a) $$C(B_1,\ldots,B_n) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} R_t B_1 R_t B_2 \ldots R_t B_n R_t \frac{dt}{t}$$

$$= \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \prod_{k=1}^{n} (P_t B_k) P_t \frac{dt}{t} + \frac{1}{\pi} \sum_{\ell=0}^{n} \text{p.v.} \int_{-\infty}^{\infty} \prod_{k=1}^{\ell} (P_t B_k)(-iQ_t) \prod_{k=1}^{n-\ell} (B_{\ell+k} P_t) \frac{dt}{t}$$

$$+ \frac{1}{\pi} \sum_{\ell=0}^{n} \sum_{j=0}^{n-\ell-1} \text{p.v.} \int_{-\infty}^{\infty} \prod_{k=1}^{\ell} (P_t B_k)(-iQ_t) B_{\ell+1} \prod_{k=\ell+2}^{n-j} (R_t B_k)(-iQ_t) \prod_{k=1}^{j} (B_{n-j+k} P_t) \frac{dt}{t} .$$

Now the first integral on the right hand side is zero, as P_t is even in t. The others can be estimated using theorem 2.1. So

$$\|\underset{\sim}{C}(B_1,\ldots,B_n)\| \le \frac{2}{\pi}\sum_{\ell=0}^{n}\left[q_\ell \sum_{j=0}^{n-\ell-1} q_j + q_{n-\ell}\sum_{j=0}^{\ell-1} q_j + 4q_\ell q_{n-\ell}\right] + \frac{2}{\pi}\sum_{\ell=0}^{n}\sum_{j=0}^{n-\ell-1} q_\ell q_j$$

$$= \frac{2}{\pi}\sum_{\ell=0}^{n} q_\ell\left\{3\sum_{j=0}^{n-\ell-1} q_j + 4q_{n-\ell}\right\}.$$

The estimates for $C(B_1,\ldots,B_n)$ and $C_n(B_1)$ follow.

(b) It is straightforward to show that

$$\delta_n C(B_{n-1},\ldots,B_1) = -\,itnC(B_n,\ldots,B_1)\ .$$

Moreover, on noting that $(-it)R_tA \subset R_t - I$, it follows that

$$\delta_n(C(B_{n-1},\ldots,B_1))A \subset nC(B_n,\ldots,B_1) - \sum_{k=1}^{n} C(B_n,\ldots,B_{k+1},B_{k-1},\ldots,B_1)B_k\ .$$

Since $\mathcal{D}(A)$ is dense in $\mathcal{H}$, the result follows.

(c) We first prove that, for $1 \le m \le n$,

$$\delta_m\delta_{m-1}\ \ldots\ \delta_1(HA^n) - (\delta_m\delta_{m-1}\ldots\delta_1(HA^{n-1}))A$$

$$\subset \sum_{j=1}^{m} (\delta_m\ldots\delta_{j+1}\delta_{j-1}\ldots\delta_1(HA^{n-1}))B_j\ . \qquad (*)$$

When $m = 1$, this becomes

$$\delta_1(HA^n) - \delta_1(HA^{n-1})A \subset HA^{n-1}B_1\ ,$$

which is valid. So assume (*) holds for $m = k - 1$, and prove for $m = k$. Set $S = HA^{n-1}$, and note that, by the assumption on domains,

$$\mathcal{D}(\delta_k\ldots\delta_1(S)A) = \mathcal{D}(\delta_k(\delta_{k-1}\ldots\delta_1(S)A))$$

$$= \mathcal{D}(\delta_k\ldots\delta_1(SA)) = \mathcal{D}(A^n) \ . \tag{1}$$

Therefore

$$\delta_k \ \ldots \ \delta_1(S)A = \delta_k(\delta_{k-1}\ldots\delta_1(S)A) + \delta_{k-1} \ \ldots \ \delta_1(S)B_k \ . \tag{2}$$

Also, note that, if $k > j$

$$\delta_k(XB_j) = \delta_k(X)B_j \tag{3}$$

for all operators X . (Apply (ii) of §5) .

We now prove (*) :

$$\delta_k \ \ldots \ \delta_1(SA) - (\delta_k\ldots\delta_1(S))A$$

$$= \delta_k \ \ldots \ \delta_1(SA) - \delta_k((\delta_{k-1}\ldots\delta_1(S))A) + (\delta_{k-1}\ldots\delta_1(S))B_k \qquad \text{(by (2))}$$

$$\subset \delta_k(\delta_{k-1}\ldots\delta_1(SA) - (\delta_{k-1}\ldots\delta_1(S))A) + (\delta_{k-1}\ldots\delta_1(S))B_k$$

$$\subset \delta_k \sum_{j=1}^{k-1} (\delta_{k-1}\ldots\delta_{j+1}\delta_{j-1}\ldots\delta_1(S))B_j) + (\delta_{k-1}\ldots\delta_1(S))B_k \qquad \text{(by induction)}$$

$$= \sum_{j=1}^{k-1} (\delta_k\delta_{k-1}\ldots\delta_{j+1}\delta_{j-1}\ldots\delta_1(S))B_j + (\delta_{k-1}\ldots\delta_1(S))B_k \qquad \text{(by (3))} \ ,$$

and this equals the right hand side of (*).

As $C(B_n,\ldots,B_1)$ and $\overline{\frac{1}{n!}\delta_n \ \ldots \ \delta_1(HA^n)}$ satisfy similar identities, it is now easy to see that they are equal. First note that, when $n = 0$, they both equal H . Now suppose they are equal for $n = k - 1$, and prove for $n = k$.

$$\frac{1}{n!}\,\delta_n\delta_{n-1}\,\cdots\,\delta_1(HA^n)$$

$$\subset \frac{1}{n!}\,(\delta_n\delta_{n-1}\cdots\delta_1(HA^{n-1}))A + \sum_{j=1}^{m}\,(\delta_m\cdots\delta_{j+1}\delta_{j-1}\cdots\delta_1(HA^{n-1}))B_j$$

(by (*) and (1))

$$\subset \frac{1}{n}\,\delta_n C(B_{n-1},\ldots,B_1)A + \frac{1}{n}\sum_{n=1}^{m} C(B_m,\ldots,B_{j+1},B_{j-1},\ldots,B_1)B_j$$

(by induction)

$$\subset C(B_n,\ldots,B_1)\ .$$

The result now follows from the fact that $\mathcal{D}(\delta_n\cdots\delta_1(HA^n)) = \mathcal{D}(A^n)$, which is dense in H . ||

8. <u>Calderon's Commutator Integrals.</u> In this section $H = L_2(\mathbb{R})$, $A = D = \frac{1}{i}\frac{d}{dx}$, and $M = \{B \in L(H) \mid Bu = bu\ ,\ b \in L_\infty(\mathbb{R})\}$. Recall that M is q-compatible with D with $q_k = \frac{1}{\sqrt{2}} + 6k(1 + \frac{1}{e\sqrt{2}})$. For $B_j \in M$, the operators $C(B_1,\ldots,B_n)$ and $C_n(B)$, which are defined as in §7 with respect to $A = D$, belong to $L(L_2(\mathbb{R}))$, and

$$C(B_1,\ldots,B_n) \le p_n\|B_1\|\ \cdots\ \|B_n\|,$$

where p_n is defined in theorem 7.1. Note that $p_n = 0(n^4)$.

If $b_1,\ldots,b_n \in L_\infty(\mathbb{R})$, and $u \in L_2(\mathbb{R})$, define

$$C(b_1,\ldots,b_n)u(x) = \frac{i}{\pi}\ \text{p.v.}\int_{-\infty}^{\infty}\frac{1}{(x-y)^{n+1}}\prod_{j=1}^{n}(g_j(x)-g_j(y))u(y)dy\ ,$$

where g_j is absolutely continuous and satisfies $g_j' = b_j$, $j = 1,\ldots,n$. When $n = 0$, this becomes

$$Hu(x) = \frac{\text{p.v.}}{\pi i}\int_{-\infty}^{\infty}\frac{1}{x-y}\,u(y)dy\ ,$$

where $H = \mathrm{sgn}(D)$ is the Hilbert transform, appropriately normalized. Our first theorem is the following. In it, B_j and G_j denote the operators of multiplication by b_j and g_j, and δ_j denotes commutation by $-ig_j$. That is,

$$\delta_j(S)u = -ig_j(Su) + iS(g_ju) ,$$

for operators S in $L_2(\mathbb{R})$, and $u \in \mathcal{D}(G_jS) \cap \mathcal{D}(SG_j)$.

<u>THEOREM 8.1</u> Let $b_1,\ldots,b_n \in L_\infty(\mathbb{R})$. Then

(i) if $u \in L_2(\mathbb{R})$, then $C(b_1,\ldots,b_n)u$ is defined almost everywhere and belongs to $L_2(\mathbb{R})$,

(ii) $C(b_1,\ldots,b_n) = C(B_1,\ldots,B_n)$,

(iii) $\|C(b_1,\ldots,b_n)u\|_2 \le p_n\|b_1\|_\infty \ldots \|b_n\|_\infty\|u\|_2$ for all $u \in L_2(\mathbb{R})$,

(iv) if $\phi \in C_0^\infty(\mathbb{R})$, then, for $n \ge 1$,

$$nC(b_n,\ldots,b_1)\phi = \delta_n(C(b_{n-1},\ldots,b_1))D\phi + \sum_{j=1}^{n} C(b_n,\ldots,b_{j+1},b_{j-1},\ldots,b_1)(b_j\phi) .$$

<u>Proof</u> When $n = 0$, the equality (ii) becomes $H = H$ (where $H = \mathrm{sgn}(D)$). Let us assume (i), (ii) and (iii) are proved for $n = k - 1$, and prove the theorem for $n = k$. We first prove (iv). Let $\phi \in C_0^\infty(\mathbb{R})$. Then, for almost all x,

$\delta_k(C(b_{k-1},\dots,b_1))D\phi(x)$

$$= -\frac{i}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty}\frac{1}{(x-y)^k}\prod_{j=1}^{k}(g_j(x)-g_j(y))\phi'(y)\,dy$$

$$= \frac{-i}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty}\left[\frac{d}{dy}\left\{\frac{1}{(x-y)^k}\prod_{j=1}^{k}(g_j(x)-g_j(y))\phi(y)\right\}\right.$$

$$-\frac{k}{(x-y)^{k+1}}\prod_{j=1}^{k}(g_j(x)-g_j(y))\phi(y)$$

$$\left.+\frac{1}{(x-y)^k}\sum_{\ell=1}^{k}\left\{\prod_{j=1}^{\ell-1}(g_j(x)-g_j(y))b_\ell(y)\prod_{j=\ell+1}^{k}(g_j(x)-g_j(y))\right\}\phi(y)\right]dy$$

$$= kC(b_k,\dots,b_1)\phi(x) - \sum_{j=1}^{k}C(b_k,\dots,b_{j+1},b_{j-1},\dots,b_1)(b_j\phi)(x)$$

(by the induction assumption).

This proves (iv) when $n = k$.

We now continue under the additional assumption that g_k is bounded. Then $G_k \in L(H)$, and the preceding theorem can be invoked. Note that $\delta_k = \delta_{G_k}$, $B_k = \overline{\delta_k(D)}$, and $\delta_k(B_j) = 0$, $j = 1,\dots,k-1$. So, if $\phi \in C_0^\infty(\mathbb{R})$,

$$kC(b_k,\dots,b_1)\phi = \delta_k(C(B_{k-1},\dots,B_1))D\phi$$

$$+\sum_{j=1}^{k}C(B_k,\dots,B_{j+1},B_{j-1},\dots,B_1)(B_j\phi)$$

(by (iv) and the induction assumption)

$$= kC(B_k,\dots,B_1)\phi$$

(by theorem 7.1)

As $C(B_k,\dots,B_1) \in L(L_2(\mathbb{R}))$, parts (i) and (ii) follow, for $n = k$, using, for example, theorem 21 of [CMO]. Part (iii) is now immediate.

It remains for us to remove the assumption that g_k is bounded. Define $b_{k,m} = \chi_{(-m,m)} b_k$. The theorem has been proved with b_k replaced by $b_{k,m}$. It is not hard to show now that it is valid for b_k itself. $\|$

The results in chapter IV of [CMO] can now be used to obtain results concerning L_p-estimates, maximal functions, etc. The same applies to the following theorem, which is the main theorem of the paper. In it, $C_n(b) = C(b,\ldots,b)$ and $C_0(b) = H$.

<u>THEOREM 8.2</u> Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ denote an analytic function on the disc $\{z \in \mathbb{C} \mid |z| < r\}$. If F_b is defined by

$$F_b u(x) = \frac{i}{\pi} \text{ p.v.} \int_{-\infty}^{\infty} \frac{1}{x-y} F\left(\frac{g(x)-g(y)}{x-y}\right) u(y)\,dy \ ,$$

where g is an absolutely continuous function satisfying $g' = b$, and if $\|b\|_\infty < r$, and if $u \in L_2(\mathbb{R})$, then $F_b u(x)$ is defined for almost all x , $F_b u = \sum_{n=0}^{\infty} a_n C_n(b) u \in L_2(\mathbb{R})$, and

$$\|F_b u\|_2 \le \sum_{n=0}^{\infty} p_n |a_n| \|b\|_\infty^n \|u\|_2 < \infty \ .$$

<u>Proof</u> Apply the previous theorem. The series $\Sigma p_n |a_n| \|b\|_\infty^n$ is convergent because $p_n = O(n^4)$. To verify that the series $\Sigma a_n C_n(b) u(x)$ converges for almost all x , use the fact that there is a sequence $N_k \to \infty$ such that

$$\sum_{n=0}^{N_k} a_n C_n(b) u \to \sum_{n=0}^{\infty} a_n C_n(b) u \qquad \text{(a.e.)}$$

$\|$

9. The Hilbert Transform on Lipschitz Curves. Suppose γ is a curve in the complex plane which is parametrized by $z = \rho(x+g(x))$, $x \in \mathbb{R}$, with $\rho > 0$ and $\|b\|_\infty < 1$, where $b = g' \in L_\infty(\mathbb{R})$. (A curve has such a parametrization if and only if it has a parametrization $z = x + ih(x)$ with h *real-valued* and $\|h'\|_\infty < \infty$.) The Hilbert transform H_γ on γ is defined by

$$(H_\gamma U)(z) = \frac{i}{\pi} \text{ p.v.} \int_\gamma \frac{1}{z-\zeta} U(\zeta)d\zeta , \quad U \in L_2(\gamma) , \quad z \in \gamma .$$

Define an isomorphism between $L_2(\gamma)$ and $L_2(\mathbb{R})$ as follows :

$$L_2(\gamma) \cong L_2(\mathbb{R})$$

$$U(z) \leftrightarrow \rho\{1+b(x)\}^{-\frac{1}{2}}u(x)$$

Therefore

$$(H_\gamma U)(x) = \frac{i}{\pi} \text{ p.v.} \int_{-\infty}^{\infty} \frac{\sqrt{1+b(x)}\sqrt{1+b(y)}}{(x+g(x))-(y+g(y))} u(y)dy .$$

(The existence of the p.v. is independent, a.e., of the parametrization.)

That is,

$$H_\gamma = (I+B)^{\frac{1}{2}}F_b(I+B)^{\frac{1}{2}} ,$$

where F_b is defined by $F(z) = \frac{1}{1+z}$.

Hence, by the preceding theorem we have

THEOREM 9.1 If $\|b\|_\infty < 1$ and $u \in L_2(\mathbb{R})$, then $H_\gamma u(x)$ is defined for almost all x , and $H_\gamma u \in L_2(\mathbb{R})$. Indeed

$$\|H_\gamma\| \le 24(\frac{9}{\pi})(1 + \frac{1}{e\sqrt{2}})^2(1 - \|b\|_\infty)^{-5}\|1+b\|_\infty$$

$$< \frac{110\|1+b\|_\infty}{(1-\|b\|_\infty)^5} .$$

Proof Apply the preceding theorem with

$$F(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n .$$

Then

$$\|F_b\| \le \sum_{n=0}^{\infty} p_n \|b\|_\infty^n \le c(1-\|b\|_\infty)^{-5} ,$$

since $p_n = O(n^4)$. Checking that c can be chosen as above is hardly a pursuit that needs be followed here. ||

It follows also that the Hilbert transform on a simple closed Lipschitz curve is L_2-bounded, as one can localize and apply the above. (This does not apply to Lipschitz curves with arbitrary behaviour at ∞ .)

One can also obtain results for L_p-norms, as noted in the last section.

As a corollary of the above result we obtain the L_2-boundedness of the following operators used in potential theory :

$$(G_1 u)(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{x-y}{(x-y)^2+(h(x)-h(y))^2}\, u(y)\,dy ,$$

$$(G_2 u)(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{h(x)-h(y)}{(x-y)^2+(h(x)-h(y))^2}\, u(y)\,dy .$$

Here h is a real-valued Lipschitz function. Corresponding estimates in higher dimensions can be obtained using Calderon's method of rotation.

10. Square Roots of Accretive Operators.

The background to this section is contained in [K0] and [K1].

An operator S in a Hilbert space H is called *maximal accretive* if

(i) $\mathcal{D}(S)$ is dense in $\mathcal{H}$,

(ii) $\mathrm{Re}(Su,u) \geq 0$ for all $u \in \mathcal{D}(S)$, and

(iii) S cannot be extended to an operator with larger domain also satisfying (ii).

Every maximal accretive operator is closed.

If S is maximal accretive then there is a unique maximal accretive operator $S^{\frac{1}{2}}$ satisfying $(S^{\frac{1}{2}})^2 = S$. Note that $\mathcal{D}(S^{\frac{1}{2}}) \supset \mathcal{D}(S)$, and, by lemma V. 3.43 of [KO],

$$S^{\frac{1}{2}}u = \frac{2}{\pi}\int_0^\infty (I+t^2S)^{-1}Su\,dt\ , \qquad u \in \mathcal{D}(S)\ .$$

Note also that $\mathcal{D}(S)$ is dense in $\mathcal{D}(S^{\frac{1}{2}})$ under $\{\|u\|^2+\|S^{\frac{1}{2}}u\|^2\}^{\frac{1}{2}}$. Also note that S^* is maximal accretive, and $S^{*\frac{1}{2}} = S^{\frac{1}{2}*}$.

A *closed regularly accretive sesquilinear form* J in $\mathcal{H}$ is a map $J : \mathcal{V}_J \times \mathcal{V}_J \to \mathbb{C}$, where $\mathcal{V}_J$ is a dense linear subspace of $\mathcal{H}$, such that

(i) $J[u,v]$ is linear in u and conjugate linear in v ,

(ii) $\mathrm{Re}\, J[u,u] \geq 0 \ \ \forall u \in \mathcal{V}_J$.

(iii) $\exists \kappa > 0 \ni |\mathrm{Im}\, J[u,u]| \leq \kappa\, \mathrm{Re}\, J[u,u] \ \ \forall u \in \mathcal{V}_J$,

(iv) $\mathcal{V}_J$ is complete under $\|u\|_J^2 = \mathrm{Re}\, J[u,u] + \|u\|^2$.

<u>THEOREM 10.1</u> If J is closed regularly accretive then the associated operator A_J is maximal accretive and $\mathcal{D}(A_J)$ is dense in $\mathcal{V}_J$. Here A_J is defined to be the operator with largest domain satisfying

$$J[u,v] = (A_Ju,v)\ , \qquad u \in \mathcal{D}(A_J)\ ,\ v \in \mathcal{V}_J\ . \qquad \|$$

<u>THEOREM 10.2</u> If $\mathcal{D}(A_J^{\frac{1}{2}}) = V_J$, then $\mathcal{D}(A_J^{*\frac{1}{2}}) = V_J$, and

$$J[u,v] = (A_J^{\frac{1}{2}}u, A_J^{*\frac{1}{2}}v) , \qquad u,v \in V_J . \qquad \|$$

It is not always true that $\mathcal{D}(A_J^{\frac{1}{2}}) = V_J$. [Mc2]. Our aim is to give conditions under which $\mathcal{D}(A_J^{\frac{1}{2}}) = V_J$. Indeed we shall prove the following theorem.

<u>THEOREM 10.3</u> Suppose $A : H \to K$ is a closed densely defined operator and $F \in L(K)$ with $\operatorname{Re} F \geq \rho > 0$. Let $V_J = \mathcal{D}(A)$ and define $J : V_J \times V_J \to \mathbb{C}$ by $J[u,v] = (FAu,Av)$. Then

(i) J is a closed regularly accretive form;

(ii) the maximal accretive operator A_J associated with J is given by $A_J = A^*FA$; and also $A_J^* = A^*F^*A$;

(iii) $\|I-\rho F^{-1}\| \leq (1-\rho^2\|F\|^{-2})^{\frac{1}{2}} < 1$;

(iv) if $\{I-\rho F^{-1}\}$ and $\{I-\rho F^{*-1}\}$ are q-compatible with A , and if

$$c = \frac{2}{\pi} \sum_{k=0}^{\infty} p_k \|I-\rho F^{-1}\|^k < \infty ,$$

where

$$p_k = \frac{1}{\sqrt{2}} \left(\sum_{j=0}^{k-1} q_j + 4q_k \right) ,$$

then $\mathcal{D}((A^*FA)^{\frac{1}{2}}) = \mathcal{D}(A)$, and

$$\frac{\sqrt{\rho}}{c} \|Au\| \leq \|(A^*FA)^{\frac{1}{2}}u\| \leq c\sqrt{\rho}\|Au\| , \qquad u \in \mathcal{D}(A) .$$

<u>Proof</u> Parts (i) and (ii) are straightforward.

(iii) Let $\kappa^2 = (1-\rho^2\|F\|^{-2})$. If $w \in K$, then

$$\|Fw-\rho w\|^2 - \kappa^2\|Fw\|^2 = (1-\kappa^2)\|Fw\|^2 + \rho^2\|w\|^2 - 2\rho \ \mathrm{Re}(Fw,w)$$

$$\leq \rho^2\|F\|^{-2}\|Fw\|^2 + \rho^2\|w\|^2 - 2\rho^2\|w\|^2$$

$$\leq 0 \ .$$

$$\therefore \ \|(I-\rho F^{-1})u\| \leq \kappa\|u\| \ , \quad u \in K \ .$$

$$\therefore \ \|I-\rho F^{-1}\| \leq \kappa \ .$$

(iv) By scaling, we can reduce to the case $\rho = 1$. We assume this.

We first note the following identity :

$$(I+t^2A^*FA)^{-1}tA^*Fw = \sum_{k=0}^{\infty} Q_t^* \{(I-F^{-1})P_t\}^k w \ , \quad w \in \mathcal{D}(A^*F) \ .$$

To prove this recall that $tA^*P_t = Q_t^*$ and $I - P_t = tAQ_t^*$. Therefore (using the conventions of §4) ,

$$tA^*F(I-(I-F^{-1})P_t) = tA^*F(I-P_t) + tA^*P_t$$

$$= t^2A^*FAQ_t^* + Q_t^* \ .$$

$$= (t^2A^*FA+I)Q_t^* \ .$$

$$\therefore \ (I+t^2A^*FA)^{-1}tA^*F \subset Q_t^*(I-(I-F^{-1})P_t)^{-1}$$

$$= \sum_{k=0}^{\infty} Q_t^* \{(I-F^{-1})P_t\}^k \ .$$

Therefore, if $u \in \mathcal{D}(A^*FA)$,

$$(A^*FA)^{1/2}u = \frac{2}{\pi}\int_0^\infty (I+t^2A^*FA)^{-1}tA^*FAu\,\frac{dt}{t}$$

$$= \frac{2}{\pi}\int_0^\infty \sum_{k=0}^\infty Q_t^*\{(I-F^{-1})P_t\}^k Au\,\frac{dt}{t}$$

$$= \frac{2}{\pi}\sum_{k=0}^\infty \int_0^\infty Q_t^*\{(I-F^{-1})P_t\}^k Au\,\frac{dt}{t}\ .$$

This last step follows from part (c) of theorem 2.1, and lemma 1.4, with $u_k = Q_t^*\{(I-F^{-1})P_t\}^k Au$, and $c_k = (1+p_k)\|I-F^{-1}\|^k\|Au\|$. It follows also from that lemma that

$$\|(A^*FA)^{1/2}u\| \le \frac{2}{\pi}\sum_{k=0}^\infty p_k\|I-F^{-1}\|^k\|Au\| = c\|Au\|\ .$$

Since $\mathcal{D}(A^*FA)$ is dense in V_J , and $(A^*FA)^{1/2}$ is closed, it follows that $V_J \subset \mathcal{D}((A^*FA)^{1/2})$ and

$$\|(A^*FA)^{1/2}u\| \le c\|Au\|$$

for all $u \in V_J$. Similarly, $V_J \subset \mathcal{D}((A^*F^*A)^{1/2})$, and

$$\|(A^*F^*A)^{1/2}u\| \le c\|Au\|$$

for all $u \in V_J$.

Choose $u \in \mathcal{D}(A^*FA) \subset V_J \subset \mathcal{D}((A^*F^*A)^{1/2})$. Then

$$\|Au\|^2 \le |(FAu,Au)| = |(A^*FAu,u)|$$

$$= |((A^*FA)^{1/2}u,(A^*F^*A)^{1/2}u)|$$

$$\le c\|(A^*FA)^{1/2}u\|\|Au\|\ .$$

$$\therefore\quad \|Au\| \le c\|(A^*FA)^{1/2}u\|\ .$$

Now A is closed, and $\mathcal{D}(A^*FA)$ is dense in $\mathcal{D}((A^*FA)^{\frac{1}{2}})$ under $\{\|u\|^2 + \|(A^*FA)^{\frac{1}{2}}u\|^2\}^{\frac{1}{2}}$, so $\mathcal{D}((A^*FA)^{\frac{1}{2}}) \subset V_J$, and

$$\|Au\| \le c\|(A^*FA)^{\frac{1}{2}}u\|$$

for all $u \in \mathcal{D}((A^*FA)^{\frac{1}{2}})$. $\|$

We apply this theorem to obtain the following result about differential operators. In it, $H^1(\mathbb{R})$ denotes the Sobolev space, $H^1(\mathbb{R}) = \{u \in L_2(\mathbb{R}) \mid Du \in L_2(\mathbb{R})\}$.

<u>THEOREM 10.4</u> Let $f \in L_\infty(\mathbb{R})$, with $\operatorname{Re} f \ge \rho > 0$, and define $J : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}$ by

$$J[u,v] = \int_{-\infty}^{\infty} f(x)Du(x)\overline{Dv(x)}dx .$$

Then A_J is maximal accretive, where

$$A_J u = DfDu ,$$

with

$$\mathcal{D}(A_J) = \{u \in H^1(\mathbb{R}) \mid fDu \in H^1(\mathbb{R})\} .$$

Also

$$\mathcal{D}(A_J^{\frac{1}{2}}) = H^1(\mathbb{R})$$

and

$$\frac{\sqrt{\rho}}{c}\|Du\| \le \|A_J^{\frac{1}{2}}u\| \le c\sqrt{\rho}\|Du\| , \quad u \in H^1(\mathbb{R}) ,$$

where

$$c = 6 \sup_{x \in \mathbb{R}} \left(1 - \left|1 - \frac{\rho}{f(x)}\right|\right)^{-3} \le 6 \frac{\|f\|_\infty^6}{\rho^6} .$$

<u>Proof</u> Apply the preceding theorem with $\mathcal{H} = \mathcal{K} = L_2(\mathbb{R})$, $A = D$, $\mathcal{V}_J = \mathcal{D}(A) = H^1$, let F denote multiplication by f , and given q_j the values specified in theorem 3.1. All that remains to be done is to find the value of c . Let

$$\kappa = \|I - \rho F^{-1}\| = \sup_{x \in \mathbb{R}} \left|1 - \frac{\rho}{f(x)}\right| \le 1 - \frac{\rho^2}{\|f\|_\infty^2} .$$

Then

$$c = \frac{2}{\pi} \sum_{k=0}^{\infty} p_k \kappa^k \le \frac{1}{2} \alpha \sum_{k=0}^{\infty} (k+1)(k+2)\kappa^k$$

for some $\alpha > 0$, as $q_j = 0(j)$ and $p_k = 0(k^2)$.

$$\begin{aligned} \therefore \quad c &\le \alpha(1-\kappa)^{-3} \\ &= \alpha \sup\left(1 - \left|1 - \frac{\rho}{f(x)}\right|\right)^{-3} \\ &\le \alpha \rho^{-6} \|f\|_\infty^6 . \end{aligned}$$

One can show, if one wishes, that α can be taken as 6 . ||

One can use this result to show that, if $f_z \in L_\infty(\mathbb{R})$ depends analytically on $z \in \Omega$, where Ω is an open subset of $\mathbb{C}$, then so does the corresponding operator $A_{J_z}^{1/2} \in \mathcal{L}(H^1(\mathbb{R}), L_2(\mathbb{R}))$. Details will be published elsewhere. It can also be shown that lower order terms can be added to J_z as in the following theorem. Details of this will also be given elsewhere.

<u>THEOREM 10.5</u> Suppose f_z, g_z, h_z, k_z are L_∞-functions depending analytically on $z \in \Omega$, where Ω is an open subset of $\mathbb{C}$, and $\mathrm{Re}\, f_z \geq \rho_z > 0$. Let J_z be the form with domain $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ defined by

$$J_z[u,v] = \int_{-\infty}^{\infty} \left\{ f_z(Du)(\overline{Dv}) + g_z(Du)\bar{v} + h_z u(\overline{Dv}) + k_z u\bar{v} \right\} .$$

There exists κ_z° such that

$$\mathrm{Re}\, J_z[u,u] + \kappa_z^\circ \|u\|^2 \geq 0 ,$$

for all $u \in H^1(\mathbb{R})$. Let $A_z = A_{J_z}$.

Then, for all $\kappa_z > \kappa_z^\circ$, $A_z + \kappa_z I$ is maximal accretive, $\mathcal{D}((A_z+\kappa_z I)^{1/2}) = H^1(\mathbb{R})$ and, for some $\lambda_z > 0$,

$$\frac{1}{\lambda_z}\|u\|_{H^1} \leq \|(A_z+\kappa_z I)^{1/2}u\| \leq \lambda_z \|u\|_{H^1} .$$

Suppose κ_z is chosen to depend analytically on $z \in \Omega$. Then, for each $u \in H^1(\mathbb{R})$, $(A_z+\kappa_z I)^{1/2}u$ depends analytically on $z \in \Omega$. ||

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