VECTOR VALUED POLYMEASURES AND PERTURBATIONS
OF SEMIGROUPS OF OPERATORS

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Let X be a Banach space and $S : [0,\infty) \to L(X)$ a continuous semigroup of operators acting on X with infinitesimal generator A.

Let Λ be a locally compact Hausdorff space, $\mathcal{B}$ the σ-algebra of Baire sets in Λ and $P : \mathcal{B} \to L(X)$ a spectral measure. For a scalar valued P-integrable function W on Λ, we write

$$P(W) = \int_{\Lambda} W(\lambda) P(d\lambda) .$$

Assuming that such a scalar valued function W is given on Λ, let

$$T(t) = \exp(tP(W)) ,$$

for $t \geq 0$. So, $T : [0,\infty) \to L(X)$ is the semigroup with infinitesimal generator $P(W)$.

The aim is to describe the effects of the two semigroups S and T acting simultaneously.

For example, let us consider the diffusion in $\mathbb{R}^3$ of a substance reacting with the environment. In this case, $\Lambda = \mathbb{R}^3$ and X is the space of all finite real-valued measures on $\mathcal{B}$. The semigroup S describes the spontaneous diffusion of the substance disregarding the creation/destruction process due to the reaction with the environment. So, if the spacial distribution of the substance at time $t = 0$ is given by a

measure $\varphi \in X$ and, if the reaction of the substance with the environment is neglected, then the distribution at a time $t \geq 0$ is represented by the measure $S(t)\varphi$. The semigroup T describes the creation/destruction process due to the reaction disregarding the diffusion. For any $B \in \mathcal{B}$ and $\varphi \in X$, let $P(B)\varphi$ be the indefinite integral of the characteristic function of the set B with respect to the measure φ . Then, given a function W on Λ , integrable with respect to P , and a measure $\varphi \in X$, the measure $P(W)\varphi$ is the indefinite integral of W with respect to φ . So, if W is the reaction rate, φ is the distribution of the substance at time $t = 0$ and if the diffusion is neglected, the distribution at a time $t \geq 0$ is represented by the measure $T(t)\varphi = \exp(tP(W)\varphi$. We wish to find out how the distribution of the substance will change in time if both processes, diffusion and creation/destruction, go on simultaneously.

As another example, take the Schrodinger semigroup S (which is, in fact, a group) describing the possible evolutions of states of a free nonrelativistic quantum mechanical particle with three degrees of freedom. In this case $\Lambda = \mathbb{R}^3$, $X = L^2(\mathbb{R}^3)$ and, for any $B \in \mathcal{B}$, the value $P(B)$ of the spectral measure P is the operator of multiplication by the characteristic function of the set B . We wish to obtain the motions of the particle in a force field with potential V . The evolution of the states of the particle is due to simultaneous action of the semigroups S and T , where $W = iV$.

Given a $t > 0$, let Ω_t be the set of all continuous paths $\omega : [0,t] \rightarrow \Lambda$. Let $\mathcal{Q}_t$ be the family of all sets

$$(1) \qquad E = \{\omega \in \Omega_t : \omega(t_j) \in B_j , j = 1,2,\ldots,k\}$$

corresponding to any $k = 1, 2, \ldots,$ any instants $0 \leq t_1 < t_2 < t_{k-1} < t_k \leq t$ and any sets $B_1, B_2, \ldots, B_k$ from $\mathcal{B}$. Let S_t be the σ-algebra of subsets of Ω_t generated by $\mathcal{Q}_t$. Let

$$M_t(E) = S(t-t_k)P(B_k)S(t_k-t_{k-1})P(B_{k-1}) \ldots P(B_2)S(t_2-t_1)P(B_1)S(t_1) ,$$

and, given $\varphi \in X$, let $\mu_t(E) = M_t(E)\varphi$, for every set E in $\mathcal{Q}_t$ given by (1).

The set functions M_t and μ_t are coordinate-wise σ-additive. That is, they are σ-additive in each B_j separately without necessarily being σ-additive as functions of E. They are both bounded on $\mathcal{Q}_t$.

Set functions in products of two spaces, σ-additive in each coordinate separately, have been given some attention in the literature and are called bimeasures. By analogy, M_t and μ_t are called polymeasures.

In some cases, for example when S is the diffusion semigroup, both M_t and μ_t are σ-additive and can even be extended to $L(X)$-valued or X-valued, respectively, measures on the whole of S_t. However, it is intuitively clear that, for the calculation of integrals of functions like

$$(2) \qquad \omega \to \exp \int_0^\infty W(\omega(q)) dq , \qquad \omega \in \Omega_t ,$$

the separate σ-additivity in each coordinate should suffice. Indeed, (2) is, in a sense, a product of functions, each depending on just one coordinate of ω.

Let $\text{sim}(\mathcal{Q}_t)$ be the family of all $\mathcal{Q}_t$-simple functions on Ω_t. The integral of any function $f \in \text{sim}(\mathcal{Q}_t)$ with respect to the polymeasure

μ_t is uniquely determined by the requirements of linearity and that the integral of the characteristic function of any set $E \in \mathcal{Q}_t$ should be equal to $\mu_t(E)$. If $f \in \text{sim}(\mathcal{Q}_t)$, then the function on $\mathcal{Q}_t$, whose value at any set $E \in \mathcal{Q}_t$ is equal to the integral of the function $f\chi_E$, is an X-valued polymeasure and is called the indefinite integral of the function f with respect to μ_t.

The set $L(\mu_t)$ of all μ_t-integrable functions and the integral with respect to the polymeasure μ_t can be defined in such a way that these objects have properties analogous to those of integrable functions and the integral with respect to a σ-additive measure. The main point of difference is that the indefinite integral of a function $f \in L(\mu_t)$ with respect to μ_t is a polymeasure on $\mathcal{Q}_t$ rather than a measure. The natural topology on $L(\mu_t)$ is the topology of convergence in mean. The convergence in mean induces and is induced by the uniform convergence on $\mathcal{Q}_t$ of indefinite integrals. The space $L(\mu_t)$ is complete with respect to the topology of convergence in mean and $\text{sim}(\mathcal{Q}_t)$ is a dense subspace in it.

The effects of semigroups S and T, acting simultaneously can be expressed in terms of integration with respect to polymeasures μ_t, $t \geq 0$.

Under the simultaneous action of semigroups S and T, an element φ of the space X evolves, in time $t > 0$, into the element

$$(3) \qquad u(t) = \int_{\Omega_t} \left[\exp \int_0^t W(\omega(q))\,dq \right] \mu_t(d\omega) .$$

This statement has an intuitively obvious meaning which can be made precise by an argument amounting to a proof of a version of the Trotter-Kato formula. So, formula (3) solves the set problem and the matter can rest here.

Nevertheless, it is desirable to tie (3) with the traditional
formulations of the problem. Traditionally, the problem of the evolution
of an element $\varphi \in X$ under the simultaneous action of semigroups S and
T, is formulated as a one-parameter family of integral equations or as a
differential equation with given initial data. Such formulations are
usually more restrictive, imposing various extraneous conditions on the
data or the entries of these equations, like the function W, or both.
On the other hand, there are various methods available for solving
traditional problems. Hence, the traditional forms of the problem can
be viewed as methods for evaluating the integral in (3). Conversely,
formula (3) can be used for the analysis of solutions, even generalized
solutions, of some differential or integral equations.

Interestingly enough, it can be shown, by a single interchange of
the order of integration, that the application $t \to u(t)$, $t \geq 0$,
defined by (3), satisfies the integral equation

(4)
$$u(t) = S(t)\varphi + \int_0^t S(t-s)P(W)u(s)\,ds , \quad t \geq 0 .$$

Indeed,

$$\exp \int_0^t W(\omega(q))\,dq = 1 + \int_0^t \left[W(\omega(s)) \exp \int_0^s W(\omega(q))\,dq \right] ds ,$$

$$\int_{\Omega_t} \mu_t(d\omega) = S(t)\varphi , \quad \int_{\Omega_t} \left[(W(\omega(s)) \exp \int_0^s W(\omega(q))\,dq \right] \mu_t(d\omega) = S(t-s)P(W)u(s) .$$

Hence, to obtain (4), it suffices to interchange in (3) the integration with
respect to μ_t and the integration with respect to s.

The integral equation (4) represents the initial-value problem

(5)
$$\dot{u}(t) = Au(t) + P(W)u(t) , \quad t > 0 ; \quad u(0) = \varphi .$$

For that matter, the formula

$$U(t) = \int_{\Omega_t} \left[\exp \int_0^t W(\omega(q))dq \right] M_t(d\omega) \ , \quad t \geq 0 \ ,$$

gives the fundamental solution $t \to U(t)$, $t \geq 0$, of the differential equation in (5).

The function $t \to u(t)$ can also be defined by (3) in some cases when the function W is unbounded, for example, when W is real and non-positive (diffusion) or purely imaginary (quantum mechanics). Then the integrand in (4) might not be defined. However, (4) can still be rescued by the following device.

The space X is continuously included in a locally convex space Y as a dense subset. Assume that the operators $P(B)$, $B \in \mathcal{B}$, and $S(t)$, $t \geq 0$, have continuous extensions on the whole of Y and that W is integrable with respect to the resulting $L(Y)$-valued spectral measure so that $P(W)$ is a continuous linear operator on Y . Then (4) has a perfectly good meaning and is related to (3) as indicated.

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