HIGHER ORDER PROPERLY ELLIPTIC BOUNDARY VALUE PROBLEMS WITH WEAKLY SMOOTH COEFFICIENTS

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1. INTRODUCTION

Consider the boundary value problem

(1.1) $$\text{find } u \text{ such that } Au = f \text{ on } \Omega\,,\quad B_P u = g_P \text{ on } \partial\Omega$$

where Ω is a bounded domain in $\mathbb{R}^n$, $A = \sum_{|\alpha|\le 2m} a_\alpha D^\alpha$ is an elliptic operator and B_P is a system of boundary operators $B_j = \sum_{|\alpha|\le j} b_{j\alpha} D^\alpha$, $j \in P$, $P = \{j_1,\ldots,j_m\}$. For the moment assume the coefficients of A and B_P and the boundary $\partial\Omega$ are smooth, and for convenience assume the orders $j \in P$ satisfy $j \le 2m-1$.

Let $H^s(\Omega)$ denote the Sobolev space of order s on Ω, and set $N^s(A) = \{u \in H^s(\Omega) : Au = 0\}$ and $\Pi_P = \prod_{j\in P} H^{s-j-\frac{1}{2}}(\partial\Omega)$. So there is determined a bounded operator

(1.2) $$B_P : N^s(A) \to \Pi_P\,.$$

By the work of Agmon, Douglis, Nirenberg, Browder, Schechter, Lions, Magenes and others, it is known that if A is properly elliptic and B_P covers A (that is, the supplementary and complementing conditions are satisfied) then (1.2) defines a Fredholm operator for $s \ge 2m$.

Moreover, if B_P is a normal boundary system, the restriction $s \geq 2m$ can be removed using duality and interpolation. Indeed, suppose B_P is normal, so that in particular the orders of the B_j are distinct. Let $Q = M \sim P$ where $M = \{0,1,\ldots,2m-1\}$. Extending B_P arbitrarily to a normal system B_M of boundary operators B_j of order $j \in M$ with smooth coefficients, it follows that there is a unique adjoint system C_M of boundary operators C_j of order $2m - j - 1$, $j \in M$, such that for smooth u, v the following Green's formula holds :

$$(1.3) \qquad (Au,v) - (u,A'v) = \sum_{j\in M} \langle B_j u, C_j v \rangle ,$$

where A' is the formal adjoint of A, and $(\ ,\ )$ and $\langle\ ,\ \rangle$ denote respectively the $L^2(\Omega)$ and $L^2(\partial\Omega)$ inner products. For each real s then there is determined a bounded operator

$$(1.4) \qquad C_Q : N^{2m-s}(A') \to \Pi_Q^*$$

where $\Pi_Q^* = \prod_{j\in Q} H^{-s+j+\frac{1}{2}}(\partial\Omega)$ is the dual of Π_Q.

In fact, for elliptic A and normal B_P, (1.2) is Fredholm if and only if (1.4) is. If also A is properly elliptic and covered by B_P then A' is properly elliptic and covered by C_Q. By the previously mentioned results, (1.2) is Fredholm for $s \leq 0$. By an interpolation argument it follows that (1.2) is Fredholm for all real s.

These results, and the definitions and proofs can be found in the book of Lions and Magenes [3].

Also in [3] (problem 2.11.2) the question is raised of avoiding the use of interpolation when $0 < s < 2m$. In this paper we outline a procedure for doing that.

The advantage to be gained by avoiding interpolation is that weaker smoothness assumptions are required of the coefficients. Roughly speaking, (1.2) is Fredholm if there is smoothness of order $|s-m| + m$, whereas the interpolation argument for $0 < s < 2m$ requires smoothness of order $2m$.

Of course, if A is strongly elliptic then, particularly in the second order case, it is well known that smoothness assumptions on coefficients and boundary can be greatly relaxed. See for example the books of Necas [5] and Gilbarg and Trudinger [1]. On the other hand, McIntosh [4] has obtained results for second order operators which are just properly elliptic with discontinuous coefficients on a plane domain Ω satisfying only the condition that $H^1(\Omega)$ be compactly embedded in $L^2(\Omega)$.

2. SESQUILINEAR FORMS

As is customary for problems with normal boundary conditions, we find an equivalent variational formulation. This requires the construction of an associated sesquilinear form.

So consider the operator (1.2) when $0 < s < 2m$, and for convenience take s to be an integer. Set $\theta = \max(s, 2m-s)$.

We make the assumption

(2.1) the boundary of Ω is smooth.

This assumption can be relaxed, but not to a significant extent. In particular it allows us to make smooth changes of coordinates under which conditions (2.2) and (2.3) below are invariant.

Next assume A can be written in the generalized divergence form

$$(2.2) \qquad A = \sum_{\substack{|p|\leq 2m-s \\ |q|\leq s}} D^p a_{pq} D^q \quad \text{where} \quad a_{pq} \in C^{\max(|p|-1,|q|-1),1}(\bar{\Omega}) \ .$$

This degree of Holder continuity of the coefficients ensures that both A and A' map smooth functions into $L^2(\Omega)$.

To state our assumptions on B_P we make the following definition. We say a differential operator E in the boundary is of class (a,b) where a , b are integers, if E can be expressed locally with respect to boundary coordinates in the form

$$(2.3) \qquad E = \sum_{|p+q|\leq a-b} D^p E^{pq} D^q \quad \text{where} \quad E^{pq} \in C^{f(a,b,p,q),1} \ ,$$

$$(2.4) \qquad f(a,b,p,q) = |p| - a - 1 + \max(b+\theta+|q|, 2m-1) \ .$$

We follow the conventions that the only differential operator of negative order is the zero operator, and that $C^{-1,1} = L^\infty$.

An operator E of class (a,b) maps smooth functions into $L^2(\partial\Omega)$ and determines a bounded operator $E : H^{s-b-\frac{1}{2}}(\partial\Omega) \to H^{s-a-\frac{1}{2}}(\partial\Omega)$. If also F is of class (b,c) then EF is of class (a,c) .

We make the assumption that the B_j for $j \in P$ are of the form

$$(2.5) \qquad B_j = \sum_{\ell=0}^{j} B_{j\ell}\gamma_\ell \quad \text{where} \quad B_{j\ell} \ \text{is of class} \ (j,\ell) \ .$$

As usual, γ_ℓ is the trace operator of order ℓ .

In this section we also assume that A is elliptic and that B_P is a normal system. This latter condition means that the orders $j \in P$ are distinct, satisfy $0 \le j \le 2m-1$, and the functions B_{jj} are non-vanishing.

Extend B_P to a normal system B_M with B_j for $j \in M$ satisfying (2.5). Then $(B_{j\ell})$, $0 \le j, \ell \le 2m-1$, is an invertible lower triangular matrix of differential operators of class (j,ℓ). Its inverse $(B^{j\ell})$ has the same properties.

We can now construct the associated sesquilinear form. Firstly define the bounded form J_1 on $H^s(\Omega) \times H^{2m-s}(\Omega)$ by

$$J_1[u,v] = \sum_{\substack{|p| \le 2m-s \\ |q| \le s}} \left(a_{pq} D^q u, D^p v\right) . \tag{2.6}$$

Restricting firstly to functions with support in small neighbourhoods of $\partial\Omega$, changing coordinates so that one is the distance function from the boundary, integrating by parts and then using a partition of unity, we obtain for all smooth u, v,

$$J_1[u,v] = (Au,v) - \sum_{j=s}^{2m-1} \langle F_j u, \gamma_{2m-j-1} v \rangle \tag{2.7}$$

where $F_j = \sum_{\ell=0}^{j} F_{j\ell}\gamma_\ell$ and $F_{j\ell}$ is of class (j,ℓ).

Integrating by parts in the reverse order gives

$$J_1[u,v] = (u,A'v) + \sum_{j=0}^{s-1} \langle \gamma_j u, G_j v \rangle \tag{2.8}$$

where $G_j = \sum_{\ell=j}^{2m-1} G_{j\ell}\gamma_{2m-\ell-1}$ and $G_{j\ell}$ is of class $(2m-j-1, 2m-\ell-1)$.

Defining $F_j = \gamma_j$ for $0 \le j \le s - 1$ and $G_j = \gamma_{2m-j-1}$ for $s \le j \le 2m - 1$ we obtain Green's formula

(2.9) $$(Au,v) - (u,A'v) = \sum_{j\in M} \langle F_j u, G_j v \rangle .$$

By the ellipticity of A, F_M and G_M are normal systems.

Denote by $\langle\ ,\ \rangle_a$ for a real, the natural extension of the $L^2(\partial\Omega)$ inner product $\langle\ ,\ \rangle$ on smooth functions to $H^a(\partial\Omega) \times H^{-a}(\partial\Omega)$.

Then for smooth u, v we have $\langle F_j u, G_j v \rangle = \langle F_j u, G_j v \rangle_{s-j-\frac{1}{2}}$. Further $F_{jk}B^{k\ell}$ is of class (j,ℓ) and so its formal adjoint maps $H^{-s+j+\frac{1}{2}}(\partial\Omega)$ into $H^{-s+\ell+\frac{1}{2}}(\partial\Omega)$. Hence

$$\sum \langle F_j u, G_j v \rangle = \sum \left\langle F_{jk}B^{k\ell}B_{\ell t}\gamma_t u, G_{jr}\gamma_{2m-r-1} v \right\rangle_{s-j-\frac{1}{2}}$$

$$= \sum \left\langle B_{\ell t}\gamma_t u, \left(F_{jk}B^{k\ell}\right)' G_{jr}\gamma_{2m-r-1} v \right\rangle_{s-\ell-\frac{1}{2}}$$

$$= \sum \langle B_\ell u, C_\ell v \rangle_{s-\ell-\frac{1}{2}}$$

where $C_\ell = \sum \left(F_{jk}B^{k\ell}\right)' G_{jr}\gamma_{2m-r-1}$ is a normal boundary operator of order $2m - \ell - 1$. It can be expressed in the form

$C_\ell = \sum_{r=\ell}^{2m-1} C_{\ell r}\gamma_{2m-r-1}$ where locally

(2.10) $$C_{\ell r} = \sum_{|p+q|\le r-\ell} D^p C_{\ell r}^{pq} D^q \quad \text{where} \quad C_{\ell r}^{pq} \in C^{g(\ell,r,p,q),1} ,$$

(2.11) $$g(\ell,r,p,q) = \max\{|p+q|-r+\ell+\theta-1, \min(|p|+\ell, |q|-r+2m-2)\} .$$

In particular, for smooth u , v we have Green's formula

(2.12) $$(Au,v) - (u,A'v) = \sum_{j\in M} \langle B_j u, C_j v \rangle_{s-j-\frac{1}{2}} .$$

We seek now to replace F_j and γ_{2m-j-1} in (2.7) by B_j and C_j respectively. For smooth u , v define

(2.13) $$J_2[u,v] = \sum_{j=s}^{2m-1} \left(\langle F_j u, \gamma_{2m-j-1} v \rangle - \langle B_j u, C_j v \rangle_{s-j-\frac{1}{2}} \right) .$$

Then it follows that

(2.14) $$J_2[u,v] = \sum_{j=0}^{s-1} \sum_{\ell=s}^{2m-1} \langle \left(C'_{j\ell} B_j - G'_{j\ell} F_j \right) u, \gamma_{2m-\ell-1} v \rangle_{s-\ell-\frac{1}{2}}$$

and, using a computation of Grubb [2], that there exist $c_{pq} \in C^{|q|-1,1}(\bar{\Omega})$ such that for smooth u , v ,

(2.15) $$J_2[u,v] = \sum_{\substack{|p|\leq 2m-s \\ |q|\leq s}} \left(c_{pq} D^q u, D^p v \right) .$$

Extend the definition of J_2 to all $H^s(\Omega) \times H^{2m-s}(\Omega)$ by (2.15), and set $J = J_1 + J_2$. Then for smooth u , v ,

(2.16) $$J[u,v] = (Au,v) - \sum_{j=s}^{2m-1} \langle B_j u, C_j v \rangle_{s-j-\frac{1}{2}} .$$

3. VARIATIONAL FORMULATION

We are now in a position to give a variational formulation of the boundary value problem. Firstly, if X is the index set P or Q , decompose it in the form $X = X' \cup X''$ where $X' = \{j \in X : j \geq s\}$.

THEOREM 3.1 *Assume* Ω *is a bounded domain in* $\mathbb{R}^n$ *satisfying* (2.1), A *is an elliptic operator of the form* (2.2) *and* B_P *is a normal boundary system satisfying* (2.5). *The following are equivalent :*

(a) $B_P : N^s(A) \to \Pi_P$ *is Fredholm;*

(b) the form J *is Fredholm on* $N^s(B_{P''}) \times N^{2m-s}(C_{Q'})$;

(c) the form $\mathcal{J}$ *defined by*

$$\mathcal{J}[(u,\phi),(v,\psi)] = J[u,v] + \langle B_{P''}u,\psi\rangle + \langle \phi,C_{Q'}v\rangle$$

is Fredholm on $(H^s(\Omega) \times \Pi_{Q'}) \times (H^{2m-s}(\Omega) \times \Pi^*_{P''})$.

That *(a)* is equivalent to *(b)* expresses the fact that the boundary value problem

(3.2) given $g_P \in \Pi_P$, $g_{P''} = 0$, find $u \in H^s(\Omega)$ so that $Au = 0$ and $B_P u = g_P$,

has the variational formulation

(3.3) given $g_{P'} \in \Pi_{P'}$, find $u \in N^s(B_{P''})$ so that $J[u,v] = -\langle g_{P'},C_P,v\rangle$ for all $v \in N^{2m-s}(C_{Q'})$.

The form $\mathcal{J}$ is introduced because its domain is independent of its coefficients and so corresponding *a priori* estimates are localizable. Indeed, write U for a typical element of $H^s(\Omega) \times \Pi_{Q'}$ with norm $\|\mathsf{U}\|_s$ and V for an element of $H^{2m-s}(\Omega) \times \Pi_{P''}$ with norm $\|\mathsf{V}\|_{2m-s}$. Let $\|\mathsf{U}\|_{s-1}$ and $\|\mathsf{V}\|_{2m-s-1}$ denote the corresponding norms in the larger spaces obtained by replacing each Sobolev space H^t by the larger space H^{t-1} , in which it is compactly embedded.

By Peetre's lemma, $\mathcal{J}$ is Fredholm if and only if

$$\|u\|_s \le c\left(\sup \frac{|\mathcal{J}[u,v]|}{\|v\|_{2m-s}} + \|u\|_{s-1}\right) \quad \text{for all } u , \tag{3.4}$$

and

$$\|v\|_{2m-s} \le c\left(\sup \frac{|\mathcal{J}[u,v]|}{\|u\|_s} + \|v\|_{2m-s-1}\right) \quad \text{for all } v . \tag{3.5}$$

These estimates can now be localized in the usual way. Indeed, using a partition of unity, it suffices to prove (3.4) and (3.5) for u and v with small support. We can take these supports in a smooth coordinate neighbourhood at the boundary and pass to the case of half balls in $\mathbb{R}^n$. By the continuity of the coefficients of all highest order terms and the compactness of lower order terms we can freeze coefficients and pass to the case of homogeneous operators with constant coefficients in $\mathbb{R}^n_+$. If A is properly elliptic and covered by B_P, the localized versions of estimates (3.4) and (3.5) follow by standard techniques. Hence

THEOREM 3.6 *Assume in addition to the conditions of theorem* 3.1 *that* A *is properly elliptic and covered by* B_P. *Then* $B_P : N^s(A) \to \Pi_P$ *is Fredholm.*

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