

## DEGREE THEORY AND GAUSSIAN MEASURES IN INFINITE DIMENSIONS

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In this talk, I will explain a new approach to the definition of the degree of a map between infinite dimensional spaces. First, recall the definition of the Leroy-Schauder degree, as given by Elworthy and Tromba [2]:

Let  $F : B \rightarrow B$  be a nonlinear map on the Banach space  $B$  such that

- i)  $F$  is compact, in the sense that  $d_x F$  is a compact linear map on  $B$  for all  $x \in B$ ;
- ii)  $I + F$  is a proper map.

Then the degree of  $F$  is defined to be

$$\deg(F) = \sum_{x+F(x)=y} \operatorname{sgn}(I + d_x F),$$

where  $y$  is any regular value of the map  $I + F$ ; that is  $I + d_x F$  is invertible for all  $x$  such that  $x + F(x) = y$ . (The number  $\operatorname{sgn}(I + d_x F) = \pm 1$ , according to whether the linear map  $I + d_x F$  preserves or reverses the orientation of  $B$ .) The basic result of the theory is that this definition of degree is independent of the regular value  $y \in B$  that is used, and that the set of regular values of  $I + F$  is generic, and hence, non-empty.

Consider the following map, from the space of  $C^\infty$  maps from the circle  $S^1$  to  $\mathbb{R}^N$  into the space of  $C^{\infty-1}$  maps from the circle to  $\mathbb{R}^N$ :

$$f(t) \mapsto \frac{df(t)}{dt} + (VV)(f(t)),$$

where  $V$  is a  $C^\infty$  real function on  $\mathbb{R}^N$ , and hence  $VV$  is a  $C^\infty$

nonlinear map from  $\mathbb{R}^N$  to itself. In the end, we would like to have a definition of the degree of this map. There are two problems with applying the Leroy-Schauder theory, though: the map is not proper, and does not map a Banach space into itself, but rather maps one Banach space into another. The solution of the second objection is very simple. Let  $\Lambda$  be the operator  $\frac{d}{dt} + 1$ , which is an isomorphism between  $C^\alpha(S, \mathbb{R}^N)$  and  $C^{\alpha-1}(S, \mathbb{R}^N)$ . Then by composing the map we are considering with the operator  $\Lambda^{-1}$ , we obtain the map

$$f(t) \mapsto f(t) + \Lambda^{-1}(\nabla V - 1)(f(t)) ,$$

which maps  $C^\alpha(S, \mathbb{R}^N)$  into itself smoothly for  $\alpha \geq 0$ . Since  $\Lambda$  is a linear isomorphism, we feel that whatever the degree of  $\frac{d}{dt} + \nabla V$  is, it must be the same as the degree of  $I + \Lambda^{-1}(\nabla V - 1)$ , which is of the form  $I + \text{compact}$ .

The other difficulty, that  $\frac{d}{dt} + \nabla V$  is not proper, cannot be overcome, and this prevents us from applying the Leroy-Schauder theory. What we will do is imitate another approach to the definition of degree in finite dimensions, which makes use of differential forms:

if  $\phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ , then the degree of  $\phi$  is defined by

$$\int_{\mathbb{R}^N} \phi^* \omega = \text{deg}(\phi) \cdot \int_{\mathbb{R}^N} \omega ,$$

where  $\omega$  is an  $N$ -form on  $\mathbb{R}^N$ . The fact that this definition of  $\text{deg}(\phi)$  agrees with our earlier one follows from Sard's theorem, which tells us that the set of regular values of  $\phi$  has full measure in  $\mathbb{R}^N$ , that is, its complement has Lebesgue measure zero.

To imitate this definition in infinite dimensions, we need a class of objects to stand in for volume forms. We will use measures of the form  $f \cdot d\mu$ , where  $d\mu$  is a fixed Gaussian measure on  $B$ , and  $f$  is an  $L^\infty$  function. The pullback is defined by using the change of

variables formula for Gaussian measures which, formulated correctly, holds in infinite dimensions as well as in finite.

In order to state the change of variables formula, we need some background in Wiener spaces, that is, Banach spaces carrying a Gaussian measure. The theory of calculus on Wiener spaces is generally called Malliavin calculus after the mathematician who first used it to study hypoelliptic differential operators. For more information on the subject, see the excellent review by Ikeda and Watanabe [4].

A Gaussian measure on a Banach space  $B$  is characterized by its Fourier transform:

$$\int_B e^{i\langle \alpha, x \rangle} d\mu(x) = e^{-\langle \alpha, \alpha \rangle / 2}$$

for every  $\alpha \in B^*$ , the dual of  $B$ , where  $(\cdot, \cdot)$  is an inner product on  $B^*$  which makes  $B^*$  into a pre-Hilbert space. The Hilbert completion of  $B^*$  under this inner product is called  $H$ , and we obtain a triplet

$$B^* \hookrightarrow H \hookrightarrow B.$$

This triplet (or equivalently, the specification of the inclusion  $H \hookrightarrow B$ ) specifies the measure  $d\mu$  completely. It can be proved that both inclusions are compact.

The prototypical Wiener space is the Banach space  $C^0(S, \mathbb{R}^n)$  of loops in  $\mathbb{R}^n$ , with  $H$  given by  $H^1(S, \mathbb{R}^n)$ , the Sobolev space of loops for which

$$\|f\|_{H^1}^2 = \int_S \left| \frac{df}{dt} \right|^2 + |f|^2 = \int_S |\Lambda f|^2$$

is finite. This measure was first constructed by Wiener.

In doing calculus on a Wiener space, it proves to be very useful to have a fixed domain of "smooth" functions on which to operate. A

convenient choice here is the set of cylinder functions, which have the form

$$F(x) = F(\langle \alpha_1, x \rangle, \dots, \langle \alpha_n, x \rangle),$$

where  $\alpha_i \in B^*$  and  $F \in C_0^\infty(\mathbb{R}^n)$ . It is a simple exercise in Fourier transforms to show that

$$\int_B F d\mu(x) = \det(2\pi A)^{-1/2} \int_{\mathbb{R}^n} F(x_1, \dots, x_n) e^{-A^{ij} x_i x_j / 2} dx_1 \dots dx_n,$$

where  $A$  is the  $n \times n$  matrix  $A^{ij} = \langle \alpha_i, \alpha_j \rangle$ . This formula shows that  $d\mu$  is a probability measure, and that the cylinder functions are dense in  $L^p(B, d\mu)$  for  $p < \infty$ .

The main difference between calculus in finite dimensions and calculus on a Wiener space is that derivatives are only taken in directions corresponding to vectors in  $H \subset B$ . The gradient operator from the dense domain of cylinder functions to  $L^p(B; H)$ ,  $p < \infty$ , is defined by

$$(\nabla F)(x) = \sum_{j=1}^n \left( \frac{\partial F}{\partial x_j} \right) (\langle \alpha_1, x \rangle, \dots, \langle \alpha_n, x \rangle) \otimes \alpha_j.$$

This operator has as its adjoint  $\nabla^*$  the operator on  $H$ -valued cylinder functions given by

$$(\nabla^*(F \otimes \alpha))(x) = \left( - \sum_{j=1}^n (\alpha, \alpha_j) \left( \frac{\partial F}{\partial x_j} \right) + \langle \alpha, x \rangle F \right) (\langle \alpha_1, x \rangle, \dots, \langle \alpha_n, x \rangle).$$

Since both  $\nabla$  and  $\nabla^*$  have dense domains, it follows that they are closeable - we shall denote their closures by the same symbols.

The operator  $\Delta = \nabla^* \nabla$  is known as the Ornstein-Uhlenbeck operator - it is an infinite dimensional version of the harmonic oscillator, and has eigenvalues  $\{0, 1, 2, \dots\}$ . Using it as an analogue of the Laplacian, we can define Sobolev spaces on  $B$ :

$$L^{p,s}(B) = \{F \mid \Delta^{s/2} F \in L^p(B)\} .$$

Similarly, if  $G$  is a Hilbert space, there is a Sobolev space  $L^{p,s}(B;G)$  of measurable maps from  $B$  to  $G$ . It has been proved, by Meyer and Krée, using Littlewood-Paley methods, that  $\nabla$  is bounded from  $L^{p,s}(B)$  to  $L^{p,s-1}(B,H)$ , and  $\nabla^*$  is bounded from  $L^{p,s}(B,H)$  to  $L^{p,s-1}(B)$ , for any  $s$ .

As an analogue of the space of test functions in finite dimensions, we have the Frechet space

$$W^\infty(B) = \bigcap_{\substack{p < \infty \\ s < \infty}} L^{p,s}(B) .$$

But elements of  $W^\infty(B)$  need not be continuous functions of  $B$  - for example, Ito integrals with smooth data.

A Wiener map or a Wiener space  $B$  is a map of the form  $x \mapsto x + F(x)$ , where  $F \in W^\infty(B,H)$ . For example, a linear Wiener map takes the form

$$x \mapsto x + Ax + h ,$$

where  $A$  is a bounded linear operator from  $B$  to  $H$ , and  $h \in H$ .

As another example of a Wiener map, we have the map defined above on  $C^\alpha(S, \mathbb{R}^n)$ , where  $\alpha < \frac{1}{2}$  in order that  $C^\alpha(S, \mathbb{R}^n)$  should be a Wiener space:

$$f \mapsto \Lambda^{-1} \left( \frac{d}{dt} + \nabla V \right) f ,$$

so that  $F(f) = \Lambda^{-1}(\nabla V - I)f$ . It is an agreeable exercise in Gaussian integration to show that  $F \in W^\infty(B,H)$  if  $V$  satisfies

$$|\nabla^k V| \leq e^{O(|x|)} \quad \text{for } k \geq 1 .$$

We have the following analogue of Sard's theorem for Wiener maps.

A proof may be found in Getzler [3].

Theorem 1 If  $I + F$  is a Wiener map such that in addition  $F \in C^1(B, H)$ , then its critical set  $(I + F)$  [the set of  $x \in B$  such that  $I + \nabla_x F$  is not invertible on  $H$ ] has  $\mu$ -measure zero.  $\square$

Using this theorem, we can define the pullback of the Gaussian measure  $d\mu$  by a  $C^1$  Wiener map. Since the derivative of  $F$  is a Hilbert-Schmidt map on  $H$ , it has a well defined orientation if it is invertible - that is, the group

$$GL_2(H) = \{I + A \in I + HS(H) \mid I + A \text{ is invertible}\}$$

has two components. Thus, we can define  $\text{sgn}(I + A) = \pm 1$  according to which component  $I + A$  lies in. The pullback is now defined by the following formula:

$$\int_B f(x) d(I + F)^* \mu(x) = \int_B \sum_{x + F(x) = y} \text{sgn}(I + \nabla_x F) f(x) d\mu(y).$$

We would like a change of variables formula that would enable us to calculate  $d(I + F)^* \mu(x)$ . Suppose for the moment that  $B$  is a finite dimensional Wiener space. Then the ordinary change of variables formula gives

$$\frac{d(I + F)^* \mu}{d\mu} = \det(I + \nabla F) e^{-\langle x, F(x) \rangle - |F(x)|^2/2}.$$

This makes no sense in infinite dimensions, but Ramer [5] showed how to rearrange it in such a way that it does, using the renormalized determinant  $\det_2$ . This is a holomorphic function on  $GL_2(H)$ , defined by

$$\begin{aligned} \det_2(I - A) &= \exp\left(\sum_{n=2}^{\infty} \frac{1}{n} \text{Tr } A^n\right) \quad \text{if } \|A\|_{\infty} < 1, \\ &= \det(I - A) e^{-\text{Tr } A} \quad \text{if } \text{Tr } |A| < \infty. \end{aligned}$$

Notice that  $\det_2(I + A) = 0$  if and only if  $I + A$  is singular, and

otherwise, the sign of  $\det_2(I+A)$  is just  $\text{sgn}(I+A)$ .

We can now renormalize the change of variables formula:

$$\begin{aligned} \det(I+\nabla F) e^{-(x, F(x)) - |F(x)|^2/2} \\ = \det_2(I+\nabla F) e^{[\text{Tr}\nabla F - (x, F(x))] - |F(x)|^2/2} \\ = \det_2(I+\nabla F) e^{-\nabla^* F - |F(x)|^2/2}. \end{aligned}$$

We shall call this quantity  $\delta(F)$ . It is hardly surprising that the following change of variables formula holds.

**Theorem 2** *If  $I+F$  is a  $C^1$  Wiener map, then*

$$\frac{d(I+A)^* \mu}{d\mu} = \delta(F). \quad \square$$

Actually, since  $\delta(F)$  makes sense for any  $F \in W^1(B, H)$ , it seems reasonable to define the pullback of  $d\mu$  by such a Wiener map  $I+F$  to be  $\delta(F) d\mu$ . In this way, we will even be able to define the degree of some  $W^1$  but non-continuous Wiener maps!

We can now define the degree of a Wiener map:

$$\begin{aligned} \text{deg}(I+F) &= \int_B d(I+F)^* \mu \\ &= \int_B \delta(F) d\mu. \end{aligned}$$

The following theorem gives a sufficient condition for this to be a reasonable definition.

**Theorem 3** *If  $I+F$  is a  $W^1$  Wiener map such that*

$$\| \nabla F \|_{\text{HS}}^2 - \text{div } F - |F(x)|^2/2 \in L^p(d\mu)$$

for some  $p > 1$ , then for all  $f \in L^\infty(B)$ ,

$$\int_B (I+F)^* f \delta(F) d\mu = \text{deg}(I+F) \int_B f d\mu. \quad \square$$

Of course, it is sufficient to prove this theorem for  $f$  of the form  $e^{i\langle \alpha, x \rangle}$ ,  $\alpha \in B^*$ . This is done by an argument which makes use of integration by parts.

If  $I + F$  is actually  $C^1$ , then by the usual reasoning, making use of the implicit function theorem, we see that a geometric form of degree theory holds too: if  $y \in B$  is a regular value of  $I + F$ , then

$$\sum_{x+F(x)=y} \operatorname{sgn}(I + \nabla_x F) = \operatorname{deg}(I + F).$$

It follows that  $\operatorname{deg}(I + F)$  is actually an integer; whether this is true for more general Wiener maps is less clear.

Let us now see how the above theory applies to our map  $A = \frac{d}{dt} + \nabla V$ . Ceccotti and Ghirardello [1] gave the following heuristic calculation of  $\operatorname{deg}(A)$ .

Assume first that  $V$  has only non-degenerate critical points.

Then if  $f$  is a solution of  $A(f) = 0$ , we have

$$\int |A(f)|^2 = \int \left| \frac{df}{dt} \right|^2 + 2 \frac{d}{dt} V(f(t)) + |\nabla V(f)|^2 = 0,$$

so that  $f$  is a constant and equals one of the critical points of  $V$ . Furthermore, if  $f$  is a constant, we have  $\operatorname{sgn}(\nabla_f A) = \operatorname{sgn}(\nabla_f V)$ , so that

$$\sum_{A(f)=0} \operatorname{sgn}(\nabla_f A) = \operatorname{deg}(\nabla V : \mathbb{R}^N \rightarrow \mathbb{R}^N).$$

Of course, this calculation is merely suggestive since, as explained above, Leroy-Schauder theory cannot be applied to the map  $A$ . However, our theory of degree does apply:

Theorem 4 a) If  $V \in C^\infty(\mathbb{R}^N)$  satisfies

- i)  $|\nabla^k V|$  is exponentially bounded for  $k \geq 1$ ,
- ii)  $|\nabla V| \rightarrow \infty$  as  $|x| \rightarrow \infty$ ,



iii) for a sufficiently large  $c \gg 0$ ,

$$|\nabla^2 V| \leq c(1 + |\nabla V|),$$

then the hypothesis of Theorem 3 is satisfied for the Wiener map

$$\Lambda^{-1} \circ A.$$

b) The degree of  $A$  (that is, of  $\Lambda^{-1} \circ A$ ) equals  $\deg(\nabla V)$ .  $\square$

The proof of part a) is a straightforward calculation with mollifiers, in which  $\Lambda^{-1} \circ A = I + F$  is replaced by  $I + P_\varepsilon F P_\varepsilon$ , and  $\varepsilon$  is taken to zero. To calculate the degree of  $A$ , we use the fact that the family of measures  $d\lambda_\varepsilon(f) = d\lambda(\varepsilon^{-1}f)$  converge weakly to a Dirac measure at zero as  $\varepsilon \rightarrow 0$ . Instead of pulling back the measure  $d\lambda$ , we pull back  $d\lambda_\varepsilon$ , and find that

$$\begin{aligned} \deg(A) &= \int \lim_{\varepsilon \rightarrow 0} d(I + F) * \lambda_\varepsilon \\ &= \int \sum_{\nabla V(x)=0} \operatorname{sgn}(\nabla_x^2 V) \delta(f(t) - x) \\ &= \deg(\nabla V). \end{aligned}$$

This semiclassical calculation is reminiscent of the heat kernel proof of the Atiyah-Singer index theorem.

## REFERENCES

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