THE NEUMANN PROBLEM FOR EQUATIONS OF MONGE-AMPÈRE TYPE

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In the paper [10] we are concerned with the existence of classical solutions to the semilinear Neumann problem for equations of Monge-Ampère type

$$\det D^2u = f(x,u,Du) \tag{1}$$

in convex domains Ω in Euclidean n-space, $\mathbb{R}^n$, where f is a prescribed positive function on $\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n$. In conjunction with (1), we treat Neumann boundary conditions of the form

$$D_\nu u = \varphi(x,u) \tag{2}$$

on the boundary $\partial\Omega$, where ν denotes the unit inner normal on $\partial\Omega$ and φ is a given function on $\partial\Omega\times\mathbb{R}$. For the main existence theorem, whose statement follows, we assume that Ω is uniformly convex with boundary $\partial\Omega \in C^{3,1}$, $f \in C^{1,1}(\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n)$ is positive and non-decreasing in z, for all $(x,z,p) \in \bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n$, and $\varphi \in C^{2,1}(\partial\Omega\times\mathbb{R})$ is non-decreasing in z with

$$\varphi_z(x,z) \geq \gamma_0 \tag{3}$$

for all $(x,z) \in \partial\Omega\times\mathbb{R}$, for some positive constant γ_0. Furthermore we assume the structural inequality

$$(4) \qquad f(x,N,p) \le g(x)/h(p)$$

for all $(x,p) \in \Omega\times\mathbb{R}^n$, where N is a constant and $g \in L^1(\Omega)$, $h \in L^1_{loc}(\mathbb{R}^n)$ are positive functions satisfying

$$(5) \qquad \int_\Omega g < \int_{\mathbb{R}^n} h \ .$$

THEOREM 1 *Under the above hypotheses on the domain* Ω *and functions* f,φ, *the boundary value problem* (1),(2) *has a unique convex solution* $u \in C^{3,\alpha}(\bar{\Omega})$ *for all* $\alpha < 1$.

When the domain Ω and functions f and φ are C^∞, then the solution $u \in C^\infty(\bar{\Omega})$. Two special cases embraced by Theorem 1 are the standard Monge-Ampère equation,

$$(6) \qquad \det D^2u = f(x) \ ,$$

and the equation of prescribed Gauss curvature

$$(7) \qquad \det D^2u = K(x)\ (1+|Du|^2)^{(n+2)/2} .$$

Theorem 1 yields a unique convex solution of the boundary value problem (6),(2) for arbitrary positive $f \in C^{1,1}(\bar{\Omega})$ while a unique convex solution of the boundary value problem (7),(2) is obtained provided

(8) $$\int_\Omega K < \omega_n .$$

Condition (8) is also necessary for the existence of a classical solution; (see [2],[15]). It is interesting to compare Theorem 1 with the known results for the Dirichlet problem,

(9) $$u = \varphi(x) \quad \text{on} \quad \partial\Omega ;$$

(see for example [1],[3],[4],[6],[7],[9],[12],[14]). Here the problem (7),(9) is solvable classically for arbitrary $\varphi \in C^{1,1}(\partial\Omega)$ if and only if the function K also vanishes on $\partial\Omega$ [14] and we cannot then necessarily infer further global regularity of the solution u beyond $u \in C^{0,1}(\bar{\Omega})$. Corresponding necessary conditions also hold for the general Dirichlet problem (1),(9); (see [14]). Note that in the extremal case, $\int_\Omega K = \omega_n$, the equation (7) has a bounded classical solution in Ω which is unique up to additive constants but satisfies $D_\nu u = -\infty$ on $\partial\Omega$ [15].

The proof of Theorem 1 depends on the method of continuity which requires the *a priori* estimation of solutions in the Schauder space $C^{2,\alpha}(\bar{\Omega})$ for some $\alpha > 0$. The main concern here is the estimation of the second derivatives for which we have had to introduce techniques somewhat different from those associated with the Dirichlet problem [3],[5]. Second derivative Hölder estimates are already provided in [14]; (see also [8],[13]). Our maximum modulus estimates are obtained through an interesting extension of the Aleksandrov Bakel'man maximum principle to oblique boundary conditions, while gradient estimates result from convexity.

As an application of Theorem 1 we derive the following existence result pertaining to the case when f and φ are independent of z.

THEOREM 2 *Suppose that the hypotheses of* Theorem 1 *hold except that* f *and* φ *are independent of* z. *Then there exists a unique number* λ *and convex function* $u \in C^{3,\alpha}(\bar{\Omega})$ *for all* $\alpha < 1$, *unique up to additive constants, solving the boundary value problem*

$$\det D^2 u = f(x, Du) \quad \text{in} \quad \Omega , \tag{10}$$

$$D_\nu u = \lambda + \varphi(x) \quad \text{on} \quad \partial\Omega . \tag{11}$$

The vanishing of the functional λ thus provides the necessary compatibility condition for the given Neumann problem to be solvable. However such a condition cannot be made explicit as is the case with linear operators. In this connection, for functions f of the form

$$f(x,p) = g(x)/h(p) \tag{12}$$

it would be more interesting to consider, instead of (2), the prescription of $Du(\Omega)$, because then the compatability condition becomes

$$\int_{Du(\Omega)} h = \int_{\Omega} f . \tag{13}$$

In particular for the boundary condition,

$$|Du|^2 = 1 \quad \text{on} \quad \partial\Omega , \tag{14}$$

$Du(\Omega)$ is the unit ball. We remark that (14) becomes an oblique condition, that is $\nu \cdot Du < 0$ on $\partial\Omega$, through imposition of the convexity of the solution u.

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