THE TOPOLOGY OF ASYMPTOTICALLY EUCLIDEAN STATIC PERFECT FLUID SPACE-TIME

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1. INTRODUCTION

In this paper we prove that a geodesically complete, asymptotically Euclidean, static perfect fluid space-time having a connected fluid region and satisfying the time-like convergence condition is diffeomorphic to $\mathbb{R}^3 \times \mathbb{R}$. It is believed that such a space-time would be spherically symmetric at least for physically reasonable conditions on the density function ρ and the pressure function p. The above assertion (that the space-time is diffeomorphic to $\mathbb{R}^3 \times \mathbb{R}$) has been claimed in [1] provided the Poincaré conjecture is valid. In fact a theorem due to Gannon [2] says that such a space-time is diffeomorphic to $N \times \mathbb{R}$ where N is a simply connected complete 3-manifold. The asymptotic conditions then imply that N has the same homotopy as $\mathbb{R}^3$ ([1]). Thus Gannon's result reduced the question to proving the non-existence of fake 3-cells in N. In particular it would give the full result if the 3 dimensional Poincaré conjecture were known to be true.

2. STATIC PERFECT FLUID SPACE-TIME

By a static perfect fluid spacetime we mean a geodesically complete space-time $(M, {}^4g)$ such that:

(i) M is a C^∞ manifold diffeomorphic to $N \times \mathbb{R}$ where for each $t \in \mathbb{R}$, $N_t = N \times \{t\}$ is a spacelike three-manifold.

(ii) The Lorentz metric ${}^4g$ can be written as

$$ {}^4g = -V^2(dt \otimes dt) + g \tag{2.1} $$

where V is a positive $C^{1,1}$ function and g is a tensor such that g restricted to N is a Riemannian metric on N, and V and g are independent of t. We assume that ${}^4g$ is at least $C^{1,1}$.

(iii) $(M, {}^4g)$ satisfies Einstein's equation

$$ \mathrm{Ric}({}^4g)_{AB} - \frac{1}{2}\,\mathrm{Scalar}({}^4g)\,{}^4g_{AB} = 8\pi\big((\rho+p)u_Au_B + p\,{}^4g_{AB}\big) \tag{2.2} $$

where ρ and p are bounded measurable functions and u_A is a unit timelike vector field on M.

By virtue of the Gauss-Codazzi embedding equations for the Lorentzian metric ${}^4g$, (2.2) decomposes into

$$ \mathrm{Ric}(g)_{\alpha\beta} = V^{-1}V_{;\alpha\beta} + 4\pi(\rho-p)g_{\alpha\beta}\ , \tag{2.3} $$

and

$$ \Delta V = 4\pi V(\rho+3p) \quad \text{on} \quad N\ , \tag{2.4} $$

where $;$ denotes the covariant derivative with respect to g and Δ denotes the Laplacian with respect to g ([3]). It is clear that ρ and p are independent of t. It follows from (2.2) that if ${}^4g$ satisfies the timelike convergence condition, namely,

$$ \mathrm{Ric}({}^4g)(W, W) \geq 0 \tag{2.5} $$

for all timelike vectors W, then $\rho + 3p \geq 0$. By continuity (2.5) implies the null convergence condition, namely, $\mathrm{Ric}({}^4g)(K, K) \geq 0$ for all null vectors K. By virtue of (2.2) the latter condition is satisfied if and only if $\rho + p \geq 0$.

We also assume that there exists an open connected region $Q \subset N$ such that $\operatorname{ess}\inf_K (\rho + p) > 0$ for all compact $K \subset Q$ and $\rho = p = 0$ in $N \sim \overline{Q}$. The functions ρ and p are respectively called the density and the pressure of the fluid. We assume that ${}^4g$ satisfies the timelike convergence condition so that by (2.4), ΔV is non-negative. However when Q is unbounded, the null convergence condition will be sufficient for our purpose. We say that $(M, {}^4g)$ is "asymptotically Euclidean" if (N, g) satisfies the following condition: There exists an open connected set $N_0 \subset N$ such that $\overline{N}_0$ is compact and $N \sim \overline{N}_0$ is diffeomorphic to $\mathbb{R}^3 \sim \overline{B}_1$ where $\overline{B}_1$ is the closed unit ball centered at the origin and, with respect to the standard co-ordinate system in $\mathbb{R}^3$, we have, on $N \sim \overline{N}_0$,

$$(2.6) \qquad g_{\alpha\beta} = \delta_{\alpha\beta} + O\left(|x|^{-\lambda}\right) \quad \text{and} \quad \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} = O\left(|x|^{-1-\lambda}\right)$$

for some $\lambda \in (0, 1)$, where $|x| \equiv \left(\sum_{\alpha=1}^{3} (x^\alpha)^2\right)^{\frac{1}{2}} \to \infty$.

3. THE MAIN RESULT

From the above condition it follows that there exists a smooth sphere S_r in $N \sim \overline{Q}$ given by $|x| = r$ in the asymptotic coordinate system such that the mean curvature of S_r in (N, g) with respect to the outward normal is strictly positive. Now by Gannon's Theorem (Proposition 1.2 in [2]) N is simply connected. Let $\overline{N}_1$ be the simply connected compact submanifold of N with boundary $\partial\overline{N}_1 = S_r$ (that is, in asymptotic co-ordinate system $N \sim \overline{N}_1 = \mathbb{R}^3 \sim \overline{B}_r(0)$). Then a fundamental existence theorem due to Meeks, Simon and Yau implies,

THEOREM 1. [4]. *Either $\overline{N}_1$ is diffeomorphic to a closed unit ball in $\mathbb{R}^3$ or there exists a $C^{2,\alpha}$, $\alpha \in (0, 1)$, embedded area minimizing minimal sphere S in the interior N_1 of $\overline{N}_1$.*

We apply the above theorem to obtain,

THEOREM 2. *Either* N *is diffeomorphic to* $\mathbb{R}^3$ *or there exists a* $C^{2,\alpha}$ $\alpha \in (0, 1)$, *embedded totally geodesic sphere* S *in* $N \sim Q$.

Proof. The proof is essentially a straightforward modification of a result due to Frankel and Galloway (corollary to Theorem 1 in [5]). The area minimizing minimal sphere S of Theorem 1 satisfies the stability inequality

$$\int_S \left(|A|^2 + \mathrm{Ric}(n,n)\right)\xi^2 \le \int |\nabla_S \xi|^2 \ , \ \xi \in C^1(S)$$

where n is the unit normal vector field on S . (Here we have assumed that the metric is C^2 in a neighbourhood of S . In general, since the Ricci curvature is only defined almost everywhere we have to use an approximation argument. For details see [6].) We put $\xi = V$ and use (2.3), (2.4) and $\Delta V = \Delta_S V + V_{;\alpha\beta} n^\alpha n^\beta$ to get $\int_S \left(|A|^2 + 8\pi(\rho+p)\right)\xi^2 \le 0$. Hence the theorem follows. □

Now S separates N, and $N \sim S$ has exactly two closed components, say N_1 and N_2 having boundary S (see Lemma 4.4 and Theorem 4.6 on page 107 in [7]). It follows from the asymptotic condition that exactly one of the components, say N_1 , is bounded. Since Q is connected we may have either

Case I: $Q \subset \overline{N}_2$ or Case II: $Q \subset \overline{N}_1$.

To rule out these cases we first deduce some formulae.

LEMMA 3. *Let* S *be a* (C^2) *totally geodesic embedded sphere in* (N, g) *such that* $S \subset N \sim Q$. *We suppose* n *is a continuous unit normal form* *on* S . *Then*

(3.1) (*i*) $g(n, \nabla V) = m'$, *a constant on* S ;

(2.3) (*ii*) $\int_S \frac{|\nabla_S V|^2}{V^2} = 4\pi$, *where* ∇_S *is the gradient operator on* S *with respect to the metric induced from* g ;

and provided $V < 1$,

(*iii*) *for a sequence* T_l *of smooth spheres in* $N \sim \overline{Q}$ *converging to* S *in the* C^2 *sense*

(3.3) $$\lim_{l\to\infty} \int_{T_l} \frac{cV^2 + a}{V(1-V^2)^3} g(\tilde{n}, \nabla w) = \int_S 2m' \left\langle \nabla_S \left(\frac{cV^2+a}{V(1-V^2)^3} \right) , \nabla_S V \right\rangle$$

where $w = |\nabla V|^2$, $\tilde{n} = \tilde{n}(l)$ *is the smooth unit normal form on* T_l *consistent in direction with* n , $\langle \ , \ \rangle$ *denotes induced inner product on* T_l *and* c, a *are arbitrary constants to be specified later.*

Proof. (Outline only. Details can be found in [6]). (3.1) follows by virtue of (2.3) and Codazzi's equation. (3.2) follows from (2.3), (2.4), contracted Gauss' equation and Gauss-Bonnet Theorem. For C^2 metric (3.3) follows from (3.2) and $g(n, \nabla w) = -m'\overline{R}V$ on S where $w = |\nabla V|^2$ and $\overline{R}$ is the scalar curvature of S . For $C^{1,1}$ metric we use approximation. □

Lemma 3 immediately rules out Case I: because on N_1 , $\Delta V = 0$ giving $g(n, \nabla V) = 0$ on S . Thus $\int_{N_1} |\nabla V|^2 = 0$ which contradicts (3.2). If Case II occurs, then Q is compact. Hence using elliptic theory we may take

(3.4) $$V = 1 - \frac{m}{|x|} + \eta \quad \text{as} \quad |x| \to \infty$$

where $m > 0$, $\eta = O\left(|x|^{-1-\beta}\right)$, $\dfrac{\partial\eta}{\partial x^\tau} = O\left(|x|^{-2-\beta}\right)$ and the L^2 average of $\dfrac{\partial^2\eta}{\partial x^\sigma \partial x^\tau}$ over $B_{2|x|}(0) \sim B_{|x|}(0)$ is $O\left(|x|^{-3-\beta}\right)$ for some $\beta \in (0, 1)$.

Now we shall use Robinson's divergence form inequality ([8]) on $N \sim \overline{Q}$, viz.,

$$\left(FV^{-1}w^{;\alpha} + GwV^{;\alpha}\right)_{;\alpha} \geq 0 \tag{3.5}$$

where

$$F = \left(cV^2 + a\right)/\left(1 - V^2\right)^3 \tag{3.6}$$

and

$$G = -2c\left(1 - V^2\right)^3 + 6\left(cV^2 + a\right)/\left(1 - V^2\right)^4 , \tag{3.7}$$

c and a being constants such that $F > 0$ on $N \sim \overline{Q}$.

Integrating (3.5) over N_2 , and using (3.1), (3.3) and $m' = -4\pi m/|S|$ we get

$$\int_S \left[\left\langle -2\nabla_S\left(\frac{cV^2 + a}{V\ 1-V^2\ 3}\right) , \nabla_S V\right\rangle + \frac{2cw}{\left(1-V^2\right)^3} - \frac{6\left(cV^2+a\right)\overline{w}}{\left(1-V^2\right)^4}\right] \geq (c+a)\,|S|/8m^2$$

Now using $w = |\nabla_S V|^2 + m'^2$ and choosing $(c,a) = (-1, 1)$ and $(c,a) = (1, 0)$ we get respectively,

$$\int_S \frac{2\left(1-9V^2\right)}{V^2\left(1-V^2\right)^3} |\nabla_S V|^2 \geq 8m'^2 \int_S \frac{1}{\left(1-V^2\right)^3} > 8m'^2 |S| \tag{3.8}$$

and

$$\int_S \left[\frac{-18V^2}{\left(1-V^2\right)^4} |\nabla_S V|^2 + \frac{2m'^2\left(1-4V^2\right)}{\left(1-V^2\right)^4}\right] \geq |S|/8m^2 \tag{3.9}$$

Finally, using $\dfrac{1-9V^2}{\left(1-V^2\right)^3} < 1$ and (3.2) in (3.8) we get $|S| > 16\pi m^2$

whereas using $\frac{1-4V^2}{(1-V^2)^4} < 1$ in (3.9) we get $|S| < 16\pi m^2$. Thus Case II also does not occur. Hence we have proved,

THEOREM 4. *A geodesically complete asymptotically Euclidean static perfect fluid space-time having connected fluid region and satisfying the timelike convergence condition is diffeomorphic to* $\mathbb{R}^3 \times \mathbb{R}$.

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