

EXISTENCE OF MINIMAL SURFACES OF BOUNDED TOPOLOGICAL
TYPE IN THREE-MANIFOLDS

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In this paper we describe various recent results on the existence of minimal surfaces in three-manifolds. In a variety of contexts, we are able to establish the existence of smooth, embedded, two-dimensional, minimal submanifolds of three-manifolds, where the genus and index of instability of the minimal surface are bounded independently of the metric on the three-manifold. In one version of our theorem, we obtain closed minimal surfaces in compact three-manifolds; in a second version, we obtain minimal surfaces with boundary lying in the boundary of a uniformly convex subset of \mathbb{R}^3 . As a consequence of our theorems, we are able to obtain a number of new examples in which minimal surfaces are realized in three-manifolds in topologically interesting ways. The details will appear in [PR]. Our methods are those of geometric measure theory.

We now describe our results and methods in more detail. We begin with several definitions. The *genus* of a compact, two dimensional, topological manifold (with or without boundary) is defined to be the number of handles in the manifold if it is orientable, and the number of cross caps if it is not orientable. Let Σ be a smooth, compact, connected, oriented, three dimensional, Riemannian manifold with Heegard genus H . The *Heegard genus* of Σ is the least genus for which there is a smooth, compact, connected, embedded, two dimensional submanifold M of Σ of that genus such that $\Sigma \sim M$ has exactly two connected components, each of which is a handlebody. Such an M is called a *Heegard surface* in Σ . M is necessarily orientable.

As is well known [SJ], whenever S is a smooth, compact, embedded, two dimensional, minimal submanifold (with or without boundary) of Σ , there exists a

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second order, symmetric, elliptic differential operator (*the second variation operator*) $f \mapsto -\text{Lap } f - f|A|^2 - f \text{ Ric}(\nu, \nu)$, where ν is any (not necessarily continuous) unit normal vectorfield to S in Σ and f is any smooth, real valued function on S which vanishes on $\text{Bdry } S$. S need not be orientable. The spectrum of this operator is discrete, and its eigenvalues $\{\lambda_i(S)\}$ satisfy

$$\lambda_1(S) < \lambda_2(S) \leq \lambda_3(S) \leq \dots \rightarrow \infty.$$

The *index* and *nullity* of S are defined by

$$\text{index}(S) = \text{card}\{j : \lambda_j(S) < 0\}, \quad \text{and} \quad \text{nullity}(S) = \text{card}\{j : \lambda_j(S) = 0\}.$$

One of our main results is the following.

THEOREM 1. Σ supports a nonempty, smooth, compact, embedded, two dimensional, minimal submanifold M such that $\text{genus}(M) \leq H$ and

$$\text{index}(M) \leq 1 \leq \text{index}(M) + \text{nullity}(M).$$

Among other things, Theorem 1 generalizes the theorem of L. Simon and F. Smith [SSm] that a smooth three-sphere with any metric supports an embedded minimal two-sphere. We state a more general version of this result in Theorem 2 below.

The minimal surface M in Theorem 1 is obtained by a minimum/maximum construction in the calculus of variations in the large. In order to describe the construction, we must make several definitions. We assume without loss of generality that Σ is a properly embedded, three dimensional submanifold of some fixed Euclidean space \mathbf{R}^ν . We fix an (oriented) Heegard surface Λ in Σ of class two, and denote by K that unique closed subset of Σ characterized by the conditions that $K \sim \Lambda$ and $\Sigma \sim K$ are the distinct connected components of $\Sigma \sim \Lambda$, and that K is an oriented manifold with boundary Λ under the orientation induced from Σ .

We denote by

$$Z_2(\Sigma) = \mathbf{I}_2(\mathbf{R}^\nu) \cap \{T : \text{spt } T \subset \Sigma \text{ and } \partial T = 0\}$$

the space of two dimensional integral cycles on Σ [FH, 4.1.24]. To each compact, oriented, two dimensional submanifold M with boundary of class one in Σ , we associate an integral current

$$t(M) = \mathcal{H}^2 \llcorner M \wedge \eta \in \mathbf{I}_2(\mathbf{R}^\nu)$$

naturally associated with M and its given orientation (here η is the orienting unit two-vectorfield of M). We denote by

$$\mathcal{V}_2(\Sigma)$$

the space of two dimensional varifolds on Σ ; i.e., the space of Radon measures on $\mathbf{G}_2(\Sigma)$. Here $\mathbf{G}_2(\Sigma)$ denotes the Grassmann bundle over Σ whose fibre at any $x \in \Sigma$ is the space of (unoriented) two dimensional linear subspaces of the tangent space $T_x(\Sigma)$ to Σ at x . Associated with each $V \in \mathcal{V}_2(\Sigma)$ is the total variation measure $\|V\|$ on Σ by the formula $\|V\|(A) = V(\mathbf{G}_2(\Sigma) \cap \{(x, S) : x \in A\})$, for $A \subset \Sigma$. To each two dimensional, embedded submanifold M of class one of Σ with $\mathcal{H}^2(M) < \infty$, we associate a varifold

$$v(M) \in \mathcal{V}_2(\Sigma)$$

defined by $v(M)(A) = \mathcal{H}^2[M \cap \{x : (x, T_x M) \in A\}]$ for $A \subset \mathbf{G}_2(\Sigma)$. Here \mathcal{H}^2 denotes Hausdorff two dimensional measure in \mathbf{R}^ν . If $\phi: \Sigma \rightarrow \Sigma$ is of class one and $V \in \mathcal{V}_2(\Sigma)$, then we define $\phi_{\#}V \in \mathcal{V}_2(\Sigma)$,

$$\phi_{\#}V(\alpha) = \int \alpha(\phi(x), D\phi(x)[S]) |\wedge_2 D\phi(x) \circ S| dV(x, S), \quad \text{for } \alpha \in \mathcal{K}(\mathbf{G}_2(\Sigma)).$$

Here, as usual, we identify a two dimensional linear subspace S in $T_x(\Sigma)$ with the element of $\text{Hom}(T_x(\Sigma), T_x(\Sigma))$ which is orthogonal projection onto S , and $\mathcal{K}(\mathbf{G}_2(\Sigma))$ denotes the continuous real-valued functions on $\mathbf{G}_2(\Sigma)$.

We define

$$\mathbf{M}: \mathcal{V}_2(\Sigma) \rightarrow \mathbf{R}, \quad \mathbf{M}(V) = \|V\|(\Sigma), \quad V \in \mathcal{V}_2(\Sigma).$$

We write $\mathbf{M}(M) = \mathbf{M}(v(M)) = \mathcal{H}^2(M)$ when M is an embedded, two dimensional submanifold of class one of Σ and $\mathcal{H}^2(M) < \infty$. We define

$$\mathbf{F}: \mathcal{V}_2(\Sigma) \times \mathcal{V}_2(\Sigma) \rightarrow \mathbf{R},$$

$$F(V, W) = \sup\{V(f) - W(f) : f \in \mathcal{K}(\mathbf{G}_2(\Sigma)), |f| \leq 1, \text{Lip } f \leq 1\}, \quad \{V, W\} \subset \mathcal{V}_2(\Sigma).$$

The F metric topology is equivalent to the weak* topology on \mathbf{M} bounded subsets of $\mathcal{V}_2(\Sigma)$.

Suppose g is a smooth tangent vectorfield to Σ and $\phi_t: \Sigma \rightarrow \Sigma$ is the one-parameter group of diffeomorphisms generated by g . The map

$$\mathbf{R} \times \mathcal{V}_2(\Sigma) \rightarrow \mathcal{V}_2(\Sigma), \quad (t, V) \mapsto \phi_{t\#}V, \quad (t, V) \in \mathbf{R} \times \mathcal{V}_2(\Sigma),$$

is jointly continuous, and the map

$$\mathbf{R} \times \mathcal{V}_2(\Sigma) \rightarrow \mathbf{R}, \quad (t, V) \mapsto \mathbf{M} \circ \phi_{t\#}V, \quad (t, V) \in \mathbf{R} \times \mathcal{V}_2(\Sigma),$$

is smooth in t for each fixed $V \in \mathcal{V}_2(\Sigma)$. We define the first variation

$$\delta^1 V(g) = \left. \frac{d}{dt} \right|_{t=0} \mathbf{M} \circ \phi_{t\#}V,$$

for all $V \in \mathcal{V}_2(\Sigma)$. We say that V is stationary in Σ provided $V \in \mathcal{V}_2(\Sigma)$ and $\delta^1 V(g) = 0$ for all smooth tangent vectorfields g to Σ .

To each map $\phi: [0, 1] \times \Sigma \rightarrow \Sigma$ of class one, there corresponds a continuous function

$$\mathbf{v}(\phi): [0, 1] \rightarrow \mathcal{V}_2(\Sigma), \quad \mathbf{v}(\phi)(t) = \phi_{t\#}\mathbf{v}(\Lambda), \quad t \in [0, 1].$$

We denote by

$$\Phi_0 \quad (\text{resp. } \Phi)$$

the set of all functions $\phi: [0, 1] \times \Sigma \rightarrow \Sigma$ of class two such that for each $t \in (0, 1)$ (resp. $t \in [0, 1]$), $\phi_t: \Sigma \rightarrow \Sigma$ is an orientation preserving diffeomorphism. More generally, we denote by

$$\Phi_0^S \quad (\text{resp. } \Phi^S)$$

the set of all functions $\phi: [0, 1] \times \Sigma \rightarrow \Sigma$ of class two for which there are numbers $0 = a_0 < a_1 < \dots < a_n = 1$ such that for each $k \in \{1, \dots, n\}$, the map

$$[0, 1] \times \Sigma \rightarrow \Sigma, \quad (t, x) \mapsto \phi((t - a_{k-1})/(a_k - a_{k-1})), \quad (t, x) \in [0, 1] \times \Sigma,$$

belongs to Φ_0 ; for each $k \in \{1, \dots, n-1\}$ (resp. $k \in \{0, \dots, n\}$), there is a finite set $S \subset \Sigma$ such that

$$\phi_{a_k}[\Lambda \sim \phi_{a_k}^{-1}[S]: \Lambda \sim \phi_{a_k}^{-1}[S] \rightarrow \phi_{a_k}[\Lambda] \sim S$$

is an orientation preserving diffeomorphism; and $\mathbf{v}(\phi)$ is continuous in the \mathbb{F} metric topology. One notes that $\phi_{a_k}[\Lambda]$ is not necessarily smooth only if $k \in \{1, \dots, n-1\}$ (resp. $k \in \{0, \dots, n\}$). We denote by E the set of all smooth submanifolds of Σ of the form $\phi_1[\Lambda]$ for some $\phi \in \Phi$; similarly, E^S denotes the set of all surfaces in Σ of the form $\phi_1[\Lambda]$ for some $\phi \in \Phi^S$.

We define

$$\Pi^S = \{\mathbf{v}(\phi) : \phi \in \Phi_0^S, \mathcal{K}^3(\phi_0[K]) = 0, \phi_1[K] = \Sigma,$$

$$\mathbf{M}(\mathbf{v}(\phi))(i) = 0, i \in \{0, 1\}\};$$

$$\Pi = \Pi^S \cap \{\mathbf{v}(\phi) : \phi \in \Phi_0\};$$

$$\mathbf{L} = \inf\{\sup \text{image}(\mathbf{M} \circ g) : g \in \Pi\}.$$

For any sequence $S = \{g_n\}_{n=1}^\infty$ in Π , we define

$$\mathbf{K}(S) = \mathcal{V}_2(\Sigma) \cap \{V : V = \lim_{j \rightarrow \infty} g_{n_j}(t_j) \text{ for some sequences}$$

$$n_1 < n_2 < \dots \text{ in } \mathbb{Z} \text{ and } t_1, t_2, \dots \text{ in } [0, 1]\}.$$

We say that a sequence S in Π is a *critical sequence* if

$$\mathbf{L} = \sup\{\mathbf{M}(V) : V \in \mathbf{K}(S)\},$$

in which case we define

$$\mathbf{C}(S) = \mathbf{K}(S) \cap \{V : \mathbf{M}(V) = \mathbf{L}\}.$$

We also define \mathbf{L}^S , $\mathbf{K}^S(S)$, critical sequences in Π^S , and $\mathbf{C}^S(S)$ for sequences $S \in \Pi^S$ in the obvious way. Elements of $\mathbf{C}(S)$ or $\mathbf{C}^S(S)$ are called *critical surfaces*. The most important properties of critical sequences and critical surfaces are contained in this theorem.

THEOREM 2. *The following two statements are true.*

(1) *There exists a critical sequence S in Π^S .*

(2) *For any critical sequence S in Π^S , $C^S(S)$ is compact and nonempty.*

Furthermore, there is a set $A^S \subset \mathcal{V}_2(\Sigma)$ such that the following two statements are true.

(3) *For any critical sequence S in Π^S , there exists a critical sequence \tilde{S} in Π^S such that $\emptyset \neq C^S(\tilde{S}) \subset C^S(S) \cap A^S$.*

(4) *If $V \in A^S$, then V is stationary in Σ , and there exist positive integers n_1, \dots, n_k and pairwise disjoint, smooth, two dimensional, compact, connected, embedded, minimal submanifolds M_1, \dots, M_k of Σ such that*

$$V = \sum_{j=1}^k n_j \mathbf{v}(M_j).$$

If M_j is one-sided in Σ , then n_j is even. Furthermore, if we denote by \mathcal{O} (resp. \mathcal{U}) the set of all $j \in \{1, \dots, k\}$ such that M_j is two-sided (resp. one-sided) in Σ , then

$$\sum_{j \in \mathcal{O}} n_j \text{genus}(M_j) + \sum_{j \in \mathcal{U}} (n_j/2) \text{genus}(M_j) \leq H,$$

and

$$\begin{aligned} \sum_{j \in \mathcal{O}} n_j \text{index}(M_j) + \sum_{j \in \mathcal{U}} (n_j/2) \text{index}(M_j) &\leq 1 \\ &\leq \sum_{j \in \mathcal{O}} n_j [\text{index}(M_j) + \text{nullity}(M_j)] + \sum_{j \in \mathcal{U}} (n_j/2) [\text{index}(M_j) + \text{nullity}(M_j)]. \end{aligned}$$

REMARK. Statements (1) and (2) remain true if Π^S and $C^S(S)$ are replaced by Π and $C(S)$. Furthermore, there is a set $A \subset \mathcal{V}_2(\Sigma)$ having most of the properties in (3) and (4) with Π^S , $C^S(S)$, A^S replaced by Π , $C(S)$, A . The only exception is that the lower bound on index + nullity in (4) is not yet known to be true. Paths in Π are induced by ambient isotopies, while paths in Π^S are induced by slightly more general maps in which some degeneracy is permitted. One result of using paths in Π^S is a considerable technical simplification of some arguments. A second result is

that sometimes one obtains better estimates, as we have just seen. A more important example of an improved estimate is discussed in Theorem 5.

The set A (*resp.* A^S) is a subset of the set of all $V \in \mathcal{V}_2(\Sigma)$ for which there exists $0 < d < F(V, 0)$ such that for every neighborhood \mathcal{W} of V with $\text{diam}_{\mathbb{F}}(\mathcal{W}) \leq d$, there exists $0 < \rho < \infty$ such that for every $0 < \epsilon < \rho$, there exist $0 < \delta < \infty$ and $g \in \Pi$ (*resp.* $g \in \Pi^S$) with

$$M \circ g \leq L + \epsilon \quad \text{and} \quad \text{image}(g) \cap \{V : F(V, \mathcal{V}_2(\Sigma) \sim \mathcal{W}) > 2\rho\} \neq \emptyset,$$

having the property that if $\tilde{g} \in \Pi$ (*resp.* $\tilde{g} \in \Pi^S$), $M \circ \tilde{g} \leq M \circ g + \delta$, $F(g, \tilde{g}) \leq \rho$, and $F(g(t), \mathcal{V}_2(\Sigma) \sim \mathcal{W}) \geq \epsilon$ whenever $t \in [0, 1]$ and $\tilde{g}(t) \neq g(t)$, then $M \circ \tilde{g}(t) \geq L - \epsilon$ for some $t \in [0, 1]$ with $F(\tilde{g}(t), \mathcal{V}_2(\Sigma) \sim \mathcal{W}) \geq \rho$.

Both the genus estimate and the index estimates in Theorem 2 are sharp, as examples show. Before discussing highlights of the proof, we give several applications.

APPLICATION 1. If $\Sigma \simeq \mathbf{RP}^3$, then the Heegard genus of Σ is one, and Σ supports an embedded minimal surface which is either a two-sphere, a torus, or a projective plane. If Σ is endowed with a metric of positive Ricci curvature, then Σ supports an embedded minimal surface which is either a torus or a projective plane.

APPLICATION 2. Let Σ be any three-dimensional spherical space form ($[\text{OP}]$); i.e., Σ is a three-manifold of the form \mathbf{S}^3/Γ , where Γ is a finite subgroup of $\text{SO}(4)$ which acts freely on \mathbf{S}^3 . Assume that Σ is endowed with a metric of positive Ricci curvature, which is always possible. We consider several possibilities.

(a) Σ is any lens space other than \mathbf{RP}^3 ; i.e., Γ is cyclic of order greater than two.

The Heegard genus of Σ is one, and there is surface in Σ which is a minimal Heegard torus.

(b) Σ is prism manifold; i.e., Γ is product of a dihedral or binary dihedral group with a (possibly trivial) cyclic group. The Heegard genus of Σ is two, and Σ supports a minimal Heegard surface of genus two or a minimal Klein bottle.

(c) Σ is a binary polyhedral three-manifold; i.e., Γ is a product of a generalized binary tetrahedral group, a binary octahedral group, or a binary icosahedral group with a (possibly trivial) cyclic group. Note that there is an infinite family

of generalized binary tetrahedral groups. In all cases, Σ supports a minimal Heegard surface of genus two. In particular, by applying Theorem 2 in the special case where Σ is the Poincaré homology sphere ($\Sigma = \mathbb{S}^3/\Gamma$ and Γ is the binary icosahedral group) endowed with a metric of constant curvature $+1$, we are able to settle affirmatively a conjecture of Frankel ([FT]).

APPLICATION 3. Assume that Σ is any three-manifold of Heegard genus two endowed with a metric of non-positive sectional curvature. Such manifolds are known to occur abundantly (see [TW], for example). One concludes that Σ supports a minimal Heegard surface of genus two, a totally geodesic flat torus, or a totally geodesic flat Klein bottle.

- (a) If Σ has strictly negative curvature, then the totally geodesic flat torus or Klein bottle cannot occur. For example, in the important special case that Σ is a hyperbolic manifold (constant curvature -1), one concludes that Σ supports a minimal Heegard surface of genus two.
- (b) By the results of [GW] and [LY], a totally geodesic flat torus or Klein bottle T in Σ must be π_1 -injective; i.e., the homomorphism $\pi_1(T) \rightarrow \pi_1(\Sigma)$ is injective. Thus, if Σ has no such π_1 -injective tori or Klein bottles, then Σ supports a minimal Heegard surface of genus two. This case arises for certain of the Seifert fiber spaces Σ with three exceptional fibers and the two-sphere as orbit surface where the first homology group $H_1(\Sigma, \mathbb{Z})$ is finite. (See [OP] for Seifert fiber space terminology.) In the terminology of [TW] or [SP], such spaces have geometric structures of type $\widetilde{\text{SL}}_2\mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$. Seifert fiber spaces of the latter type possess a metric of nonpositive curvature, and thus they support minimal Heegard surfaces of genus two. Furthermore, these spaces occur as Brieskorn homology three-spheres (see [MJ], for example); i.e., as the links of singularities of complex varieties of dimension two. Some of these examples Σ have been shown to have inequivalent Heegard decompositions of genus two ([BGM]). In such cases, Σ possesses minimal Heegard surfaces M and M' of genus two, for which there is no diffeomorphism $\phi: \Sigma \rightarrow \Sigma$ with $\phi(M) = \phi(M')$. Examples of this type are also known for hyperbolic three-manifolds of Heegard genus two as in (a).

REMARK. Although Seifert fiber spaces Σ with geometric structures of type $\widetilde{\mathrm{SL}}_2\mathbb{R}$ do not have metrics of nonpositive sectional curvature, we are able nevertheless to obtain minimal Heegard surfaces of genus two in Σ , provided Σ has no π_1 -injective tori or Klein bottles. This follows from the observation that *if Σ is a closed three-manifold with a geometric structure of type $\widetilde{\mathrm{SL}}_2\mathbb{R}$, then any minimal torus or Klein bottle in Σ must be π_1 -injective*. Such π_1 -injective minimal surfaces have been classified by J. Hass [HJ], and our observation follows by methods similar to those of [HJ, Lemma 1.9].

These applications illustrate how our methods may be used to realize minimal surfaces in three-manifolds in topologically interesting ways. These results require the full generality of Theorem 2, including the sharp genus and index estimates. One notes especially that many of the minimal surfaces in applications 2 and 3 attain the upper bound on the genus.

The techniques involved in proving Theorem 2 are both combinatorial and geometric. One of the technically important accomplishments in Theorem 2 is the proof that there is a critical sequence in Π or Π^S whose critical set is contained *entirely* in \mathbf{A} or \mathbf{A}^S . The proof is a nontrivial combinatorial argument. This has turned out to be extremely useful, since regularity and essentially all interesting topological and analytic estimates studied so far are true of all surfaces in \mathbf{A} or \mathbf{A}^S . We shall see an example of this in the discussion of almost minimizing properties below.

In order to prove the regularity and genus estimates for varifolds V in \mathbf{A} or \mathbf{A}^S , one first shows that each such V satisfies a so-called almost minimizing condition. The first almost minimizing condition was introduced in [PJ1], and its use was essential in the proof of existence and optimum regularity for minimal hypersurfaces in arbitrary Riemannian manifolds ([PJ1], [SS]). In order to study minimal two-spheres on three-spheres, Simon and Smith [SSm] have studied a second, uniform version of the almost minimizing condition which has the advantage of yielding not only regularity but also genus control in their context. Essentially the same condition has also been employed in the study of minimal surfaces with free boundary ([GJ], [JJ]). We, too, utilize an almost minimizing condition which is necessarily weaker than that in [PJ1] or [SSm],

but for which it is possible nevertheless to establish regularity and the genus bound. To be precise, we say that V satisfies the *uniform local Λ -almost minimizing condition with respect to \mathcal{U}* provided:

- (1) $V \in \mathcal{V}_2(\Sigma)$ is stationary in Σ .
- (2) \mathcal{U} is a collection of pairs of disjoint open subsets of Σ .
- (3) there exists $0 < d < \infty$ such that for every neighborhood \mathcal{W} of V in $\mathcal{V}_2(\Sigma)$ with $\text{diam}_{\mathbb{F}}(\mathcal{W}) \leq d$, there exists $0 < \rho < \infty$ such that for every $0 < \epsilon < \infty$, there exist $0 < \delta < \infty$ and $M \in E^S$ with $\mathbf{v}(M) \in \mathcal{W}$ such that for any pair $\{U_0, U_1\} \in \mathcal{U}$, there exists $i \in \{0, 1\}$ such that $\mathbf{M}(\psi_1[M]) > \mathbf{M}(M) - \epsilon$ whenever $\psi \in \Phi$ satisfies $\psi_0 = \mathbf{1}_\Sigma$,

$$\text{Clos } \Sigma \cap \{x : \psi_t(x) \neq x \text{ for some } 0 \leq t \leq 1\} \subset U_i,$$

$$\mathbf{F}(\mathbf{v}(M), \mathbf{v}(\psi_t[M])) \leq \rho \text{ for all } 0 \leq t \leq 1,$$

and

$$\mathbf{M}(\psi_t[M]) \leq \mathbf{M}(M) + \delta \text{ for all } 0 \leq t \leq 1.$$

We say that \mathcal{U} is an *admissible collection* if \mathcal{U} is collection of pairs of open subsets of Σ such that $U_i^0 \cap U_j^1 = \emptyset$ for some $\{i, j\} \subset \{0, 1\}$ whenever (U_0^0, U_1^0) and (U_0^1, U_1^1) belong to \mathcal{U} . The main conclusion is the following.

THEOREM 3.

- (1) If $V \in \mathbf{A}$ or $V \in \mathbf{A}^S$, then V satisfies the uniform local Λ -almost minimizing condition with respect to \mathcal{U} for any admissible collection \mathcal{U} .
- (2) If V satisfies the uniform local Λ -almost minimizing condition with respect to \mathcal{U} for any admissible collection \mathcal{U} , then there exist positive integers n_1, \dots, n_k and pairwise disjoint, smooth, two dimensional, compact, connected, embedded, minimal submanifolds M_1, \dots, M_k of Σ such that

$$V = \sum_{j=1}^k n_j \mathbf{v}(M_j).$$

Furthermore, if M_j is one-sided in Σ , then n_j is even, and if we denote by \mathcal{O} (resp. \mathcal{U}) the set of all $j \in \{1, \dots, k\}$ such that M_j is two-sided (resp. one-sided) in Σ , then

$$\sum_{j \in \mathcal{O}} n_j \text{genus}(M_j) + \sum_{j \in \mathcal{U}} (n_j/2) \text{genus}(M_j) \leq H.$$

One notes that this genus bound is slightly better than one would expect (cf. [MSY]).

The almost minimizing conditions are local, hence one has no hope of proving global results (such as the index estimates) for surfaces known only to satisfy such a condition. An important reason why the sets A and A^S were introduced is to be able to study simultaneously both the almost minimizing property and the index estimates. An added benefit has been that the study of the sets A and A^S has somewhat simplified the analysis of the second variation of area of critical surfaces. For example, the basic estimate in Theorem 1 bounding the index from above by one is essentially an automatic consequence of belonging to the set A^S . This type of estimate may be substantially generalized to surfaces both of higher dimension and of higher codimension (cf. [PJ2]).

Our methods also apply in other settings. One result is the following.

THEOREM 4. *If*

- (1) $\{T_0, T_1\} \subset Z_2(\Sigma)$ and there is a positive number δ such that $M(T_i + S) > M(T_i)$ whenever $S \in Z_2(\Sigma)$, $0 < M(S) < \delta$, and $i \in \{0, 1\}$;
- (2) M is a smooth, compact, oriented, two dimensional submanifold of Σ ;
- (3) there exists a map $\phi \in \Phi_0$ such that

$$\lim_{t \rightarrow 0^+} t(\phi_t[M]) = T_0 \quad \text{and} \quad \lim_{t \rightarrow 1^-} t(\phi_t[M]) = T_1,$$

$$\lim_{t \rightarrow 0^+} M(t(\phi_t[M])) = M(T_0) \quad \text{and} \quad \lim_{t \rightarrow 1^-} M(t(\phi_t[M])) = M(T_1);$$

then there exist a nonzero varifold $V \in \mathcal{V}_2(\Sigma)$, positive integers n_1, \dots, n_k , and pairwise disjoint, smooth, two dimensional, compact, connected, embedded, minimal submanifolds M_1, \dots, M_k of Σ such that V is stationary in Σ ,

$$M(V) > \max\{M(T_0), M(T_1)\}, \quad \text{and} \quad V = \sum_{j=1}^k n_j \mathbf{v}(M_j).$$

Furthermore, if M_j is one-sided in Σ , then n_j is even, and if we denote by O (resp. U) the set of all $j \in \{1, \dots, k\}$ such that M_j is two-sided (resp. one-sided) in Σ , then

$$\sum_{j \in O} n_j \text{genus}(M_j) + \sum_{j \in U} (n_j/2)(\text{genus}(M_j) - 1) \leq \text{genus}(M),$$

and

$$\begin{aligned} \sum_{j \in \mathcal{O}} n_j \operatorname{index}(M_j) + \sum_{j \in \mathcal{U}} (n_j/2) \operatorname{index}(M_j) &\leq 1 \\ &\leq \sum_{j \in \mathcal{O}} n_j [\operatorname{index}(M_j) + \operatorname{nullity}(M_j)] + \sum_{j \in \mathcal{U}} (n_j/2) [\operatorname{index}(M_j) + \operatorname{nullity}(M_j)]. \end{aligned}$$

The following is a similar result for manifolds with boundary.

THEOREM 5. *Let $B \subset \mathbb{R}^3$ be a uniformly convex body such that $\operatorname{Bdry} B$ is smooth and let Γ be a smooth, compact, oriented, not necessarily connected, one dimensional submanifold of $\operatorname{Bdry} B$ spanned by two locally area minimizing, oriented minimal surfaces S_0, S_1 in B . Then there exists a third smooth, embedded, minimal surface M spanning Γ of greater area than that of S_0 and S_1 , such that*

$$\operatorname{genus}(M) \leq \max\{\operatorname{genus}(S_0), \operatorname{genus}(S_1)\},$$

and

$$\operatorname{index}(M) \leq 1 \leq \operatorname{index}(M) + \operatorname{nullity}(M).$$

In proving this theorem, one defines spaces of maps Φ_0 and Φ_0^S in \mathbb{R}^3 similarly to the spaces of the same names defined in Σ above, with the additional condition that ϕ_t fixes Γ for all $\phi \in \Phi_0$ or $\phi \in \Phi_0^S$ and $t \in (0, 1)$. Under the conditions given, there are a smooth, oriented, embedded submanifold S in B with boundary Γ and a map $\phi \in \Phi_0^S$ such that

$$\operatorname{genus}(S) \leq \max\{\operatorname{genus}(S_0), \operatorname{genus}(S_1)\};$$

$$\lim_{t \rightarrow 0^+} t(\phi_t[S]) = S_0 \quad \text{and} \quad \lim_{t \rightarrow 1^-} t(\phi_t[S]) = S_1;$$

$$\lim_{t \rightarrow 0^+} M(t(\phi_t[S])) = M(S_0) \quad \text{and} \quad \lim_{t \rightarrow 1^-} M(t(\phi_t[S])) = M(S_1).$$

One proves the theorem by applying the minimum/maximum construction among all such maps ϕ . One defines a set A^S as we did in the construction on Σ , and the critical surfaces V which one initially obtains are elements of A^S as before. The proof then proceeds as in Theorem 2. It is a consequence of the proof that $V = v(M)$;

i.e., each component of M occurs with multiplicity one in V . One also notes that the genus bound on M depends in an essential way on the corresponding bound for S . Although one may not necessarily assume that S_0 and S_1 are isotopic (cf. [HP]), manifolds S and isotopies $\phi \in \Phi_0$ always exist having the properties above, except that the genus bound on S (and therefore on M) is not so good. (The genus bound in isotopy is

$$\text{genus}(S) \leq \text{genus}(S_0) + \text{genus}(S_1) + N - 1,$$

where N is the number of connected components in Γ .) In particular, use of the more general maps in Φ_0^S gives a sharper theorem.

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