REGULARITY OF THE SINGULAR SETS IN IMMISCIBLE FLUID INTERFACES AND SOLUTIONS TO OTHER PLATEAU-TYPE PROBLEMS

Brian White

Existence and almost everywhere regularity of solutions to a wide variety of Plateau-type problems follows from several geometric measure theory theorems (due primarily to DeGiorgi, Federer, Fleming, Reifenberg, Almgren, and Allard.) But singularities do occur, and the size and nature of the singular set depends strongly on the particular problem. In this paper we describe our recent discovery that, for many problems, the singular sets are also fairly regular. The pioneering work on regularity of singular sets was done by Jean Taylor [T1,T2] for certain two-dimensional surfaces in $\mathbb{R}^3$; our work is a simplification and generalization of hers.

An important ingredient in the proofs is a slight extension of a stratification of singularities theorem due to Almgren [A2,2.27]. The theorem says that an m-dimensional surface S that is stationary for any of the Plateau-type problems we consider stratifies naturally as

$$S = \cup_{i=0}^{m} \Sigma_i$$

where Σ_i has Hausdorff dimension $\leq i$. The stratification is by tangent cone type. In particular, Σ_m consists of all points at which each tangent cone is an m-plane, and Σ_{m-1} consists of all points at which each tangent cone is a union of half-planes meeting along an (m-2) dimensional subspace. (The lower levels of the stratification need not concern us here.) In the particular problems we consider, it is also possible to show that Σ_m is an open subset of S and consists of smooth minimal or constant mean curvature manifolds. (This is not always true. For instance it fails for two dimensional mass minimizing integral currents in $\mathbb{R}^4 \approx \mathbb{C}^2$: the variety $z^2 = w^3$ is an example.)

We will first describe the results for several problems and then sketch some ideas involved in the proofs.

1. IMMISCIBLE FLUIDS

Consider an energy-minimizing configuration of three immisicible fluids (such as mercury, water, and sesame oil.) Mathematically we model the situation as follows. To any partition of a given region in $\mathbb{R}^{m+1}$ into regions V_1, V_2, and V_3, we assign an energy:

$$\Sigma_{i<j}\alpha_{ij}\times(\text{m-dimensional area of the } V_i\text{-}V_j \text{ interface})$$

where the α_{ij}'s are surface energy densities which depend on the physical properties of the fluids. We could also take into account the gravitational energy of the configuration. Now we ask, what is the structure of a partition that minimizes energy subject to prescribed volumes of the three regions? We prove that for the interface $S = \partial V_1 \cup \partial V_1 \cup \partial V_3$, Σ_m consists of smooth constant mean curvature hypersurfaces, and Σ_{m-1} consists of (m-1)-dimensional $C^{1,\alpha}$ manifolds along which three sheets of Σ_m meet at fixed angles (the angles depending on the α_{ij}.)

In the case of four or more immiscible fluids, we prove the slightly weaker result that a dense open subset of $\overline{\Sigma_{m-1}}$ consists of (m-1)-dimenensional manifolds along which three or more sheets of Σ_m meet.

2. AREA MINIMIZING HYPERSURFACES MOD p

Two rectifiable integer-multiplicity currents S and S' are said to be *congruent mod p* provided $S - S' = pT$ for some integer multiplicity current T; we then write $S \equiv S'$ (mod p). (For integral flat chains, the definition of congruence mod p is slightly more complicated [W2,1.2].) We say that S is *mass-minimizing mod p* provided

$$\mathbb{M}(S) \le \mathbb{M}(S') \quad \text{whenever} \quad \partial S \equiv \partial S' (\text{mod } p)$$

Given any (m-1) dimensional T in $\mathbb{R}^{m+1}$ with $\partial T \equiv 0$ (mod p), there exists an S with $\partial S \equiv T$ (mod p) that minimizes mass mod p. Our methods show that if S is a mass minimizing hypersurface mod p and if p is odd, then Σ_{m-1} consists of $C^{1,\alpha}$ manifolds along which p sheets of Σ_m meet. This was established by Jean Taylor [T1] when $m = 2$ and $p = 3$. There are known examples that illustrate this behaviour.

The case of even p is considerably more subtle. Even almost everywhere regularity is not known except for $p = 2$ or $p = 4$ [W1]. Indeed, examples (such as [W1, fig.2]) show that Σ_m need not be open in S and may contain singular points.

3. SOAP FILMS

Let S be an m-dimensional subset of an open subset $U \subset \mathbb{R}^{m+1}$ such that

$$\mathcal{H}^m(S) \le \mathcal{H}^m(\varphi(S))$$

for any Lipschitz map $\varphi : U \to U$ such that $x \mapsto \varphi(x) - x$ has compact support in U. We show that Σ_{m-1} is a $C^{1,\alpha}$ manifold along which three sheets of Σ_m meet at equal angles.

We think it likely that the method will also show that Σ_{m-2} is a $C^{1,\alpha}$ manifold, but we have not yet checked this.

In case $m = 2$, the result is due to Jean Taylor [T2].

4. IDEAS IN THE PROOFS

Consider the case of mass minimizing hypersurfaces mod p. A minimizing mod p cone C (with C contained in the unit ball $\mathbb{B} \subset \mathbb{R}^{m+1}$ and $(\partial C) \cap \operatorname{int} \mathbb{B} \equiv 0 \pmod p$) is said to have the epiperimetric property provided there exists positive ϵ and δ such that

$$(*) \qquad \mathbb{M}(S) - \mathbb{M}(C) \leq (1-\epsilon) \left[\mathbb{M}(0 ⨳ (S \cap \partial\mathbb{B})) - \mathbb{M}(C)\right]$$

whenever S is mass-minimizing mod p and $\mathcal{F}^p(\partial S - \partial C) < \delta$. Note that for $\epsilon = 0$ (and $\delta = \infty$) this inequality holds trivially, and it leads immediately to the monotonicity of $r^{-m}\,\mathbb{M}(S \cap \mathbb{B}_r)$ and the existence of tangent cones. But for all we know, there may in general be more than one tangent cone at a given point. Furthermore, even if the tangent cone is unique, $S \cap \mathbb{B}_r$ may resemble its tangent cone in only a weak, measure-theoretic way.

Reifenberg [R] discovered that if S has a tangent cone C with the epiperimetric property (i.e. for which * holds with $\epsilon > 0$), then $r^{-m}\,\mathbb{M}(S \cap \mathbb{B}_r)$ actually grows like a power of r, and $S \cap \mathbb{B}_r$ (after scaling by r^{-1}) converges rapidly to its (unique) tangent cone C. In many cases one can further show that this rapid convergence forces $S \cap \mathbb{B}_r$ to be a $C^{1,\alpha}$ perturbation of C itself.

In practise, one proves the epiperimetric property for a cone C by constructing comparison surfaces. That is, given a cycle (mod p)

$T \subset \partial\mathbb{B}$ weakly near ∂C, one has to construct a surface S' with $\partial S' \equiv T \pmod p$ and such that:

$$\mathbb{M}(S') - \mathbb{M}(C) \leq (1-\epsilon)\left[\mathbb{M}(0 \times (S' \cap \partial\mathbb{B})) - \mathbb{M}(C)\right]$$

This immediately implies the epiperimetric property. In doing this, one is faced with the difficulty that, even if ∂C is rather simple, T may be extremely complicated. Typically this sort of difficulty is handled by proving Lipschitz approximation theorems. Our contribution to the theory is showing that it suffices to consider only those T which themselves have a certain minimization property (essentially Almgren's $\mathbb{M},\epsilon,\delta)$ minimization property [A1]). Since T is $(m-1)$ dimensional, we may then by induction assume that T has the regularity properties we are trying to prove. Finally, the comparison surface S' is constructed by linearizing the minimal surface equation of C.

REFERENCES

[A1] F.J.Almgren, Jr., "Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints", Mem. A.M.S. 4, No. 165.

[A2] ——————, "Q-valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension two", preprint.

[R] R.E. Reifenberg, "An epiperimetric inequality related to the analyticity of minimal surfaces", Ann. of Math. 80 (1964), 1-21.

[T1] J.E. Taylor, "Regularity of the singular sets of two dimensional area minimizing flat chains modulo 3 in $\mathbb{R}^3$", Inv. Math. 22 (1973), 119-139.

[T2] ——————, "The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces," Ann. of Math. 103 (1976), 489-539.

[W1] B. White, "The structure of minimizing hypersurfaces mod 4", Inv. Math. 53 (1979), 45-58.

W2] ————, "A regularity theorem for minimizing hypersurfaces mod p", A.M.S. Proc. Symp. Pure Math., Arcata 1984.

Department of Mathematics
Stanford University
STANFORD CA 94305 U.S.A.