BEST CONTINUITY AT THE BOUNDARY FOR SOLUTIONS OF THE MINIMAL SURFACE EQUATION

Graham H. Williams

We consider the Dirichlet problem for the minimal surface equation. We assume that Ω is a bounded open set in $\mathfrak{R}^n$ and ϕ is a continuous function defined on $\partial\Omega$. Then we consider the problem:

Find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that

(i) $u = \phi$ on $\partial\Omega$,

(ii) u satisfies the minimal surface equation in Ω, that is,

$$\sum_{i=1}^{n} D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right) = 0 \quad \text{in} \quad \Omega.$$

Additionally we assume that $\partial\Omega$ is C^2 and has nonnegative mean curvature, H, everywhere. This assumption means that there is a unique solution for the problem ([JS]). We examine the way the regularity of u depends on the regularity of ϕ.

If $k \geq 2$, $0 < \alpha < 1$ and $\phi, \partial\Omega \in C^{k,\alpha}$ then the estimates of Jenkins and Serrin [JS] plus standard theory ([GT]) show that $u \in C^{k,\alpha}(\overline{\Omega})$. The case $k = 1$ has been considered by Lieberman [L] and also Giaquinta and Giusti [GG], and they showed that if $\phi \in C^{1,\alpha}(\partial\Omega)$ then $u \in C^{1,\alpha}(\overline{\Omega})$. The case $k = 0$ was studied by Giusti [G] and also by the author [W1], [W2], [W3]. The results of Giusti, from 1972, may be summarized as follows:

THEOREM 1 [G]

(A) *If* $\partial\Omega$ *has strictly positive mean curvature and* $\phi \in C^{0,\alpha}(\partial\Omega)$, $0 < \alpha \le 1$ *then* $u \in C^{0,\frac{\alpha}{2}}(\overline{\Omega})$.

(B) *If* $\partial\Omega$ *has nonnegative mean curvature and* $\phi \in C^{0,1}(\partial\Omega)$ *then there exists* $\alpha > 0$ *such that* $u \in C^{0,\alpha}(\overline{\Omega})$.

Giusti also gave an example where $\partial\Omega$ has strictly positive mean curvature, $\phi \in C^{0,1}(\partial\Omega)$ and $u \in C^{0,\frac{1}{2}}(\overline{\Omega})$ but $u \notin C^{0,\alpha}(\overline{\Omega})$ for any $\alpha > \frac{1}{2}$. The results of [W1] and [W2] generalized part (A) of this theorem by allowing an arbitrary modulus of continuity γ for ϕ and also by allowing a growth condition on H instead of strict positivity. They also made the result of part (B) of the theorem more precise by giving an exact value for α.

THEOREM 2 [W1], [W2]

(A1) *If* $x_0 \in \partial\Omega$, $H(x) \ge c_1|x - x_0|^k$ *where* $k \ge 0$, $c_1 > 0$ *and* $|\phi(x) - \phi(x_0)| \le \gamma(|x - x_0|)$ *then there is a constant* c *such that*

$$|u(x) - u(x_0)| \le c\gamma(c|x - x_0|^{\frac{1}{k+2}}), \qquad x \in \overline{\Omega}.$$

(B1) *If* $H(x) \ge 0$ *on* $\partial\Omega$ *and* $0 < \alpha < 1$ *then there is a constant* $K = K(n, \alpha)$, *such that if* $\mathrm{Lip}(\phi) < K$ *then* $u \in C^{0,\alpha}(\overline{\Omega})$. *Furthermore if* $\varepsilon > 0$, *there exists* ϕ *with* $\mathrm{Lip}(\phi) < K + \varepsilon$ *but such that* $u \notin C^{0,\alpha}(\overline{\Omega})$.

REMARKS

(i) The constant $K(n, \alpha)$ may be given explicity. For example $K(2, \alpha) = \text{cotangent}\left(\frac{\pi\alpha}{2}\right)$.

(ii) (A1) and (B1) may be combined to give the best value for α in Giusti's Theorem part (B).

(iii) In the last part of (B1) it is assumed that $\alpha > \frac{1}{2}$, if $H > 0$, or $\alpha > \frac{1}{k+2}$ if the situation is as in (A1).

The paper [W3] considers the problem of generalizing results (B) and (B1) above to the case where ϕ is not Lipschitz. For example one may conjecture that, by analogy with (B), if $H \geq 0$ and $\phi \in C^{0,\alpha}(\partial\Omega)$ then $u \in C^{0,\beta}(\overline{\Omega})$ for some β. However the counterexamples of (B1) show that this is not true in general. Thus we are led to ask whether there is some other measure of continuity which always applies. A somewhat precise answer was obtained to this question in [W3]. We shall assume a modulus of continuity γ (which can be assumed concave) for the boundary values ψ and since the case of Lipschitz data, that is $\gamma(t) = Kt$, has been considered in (B1) above we also assume $\frac{\gamma(t)}{t} \to \infty$ as $t \to 0$. We make corresponding assumptions on the higher derivatives or, more precisely, we assume

(1) $$\lim_{r \to 0} (\gamma^{-1}(r))' + r|(\gamma^{-1}(r))''| + r^{-2}|(\gamma^{-1}(r))'''| = 0.$$

THEOREM 3

(I) *Suppose $H \geq 0$ and ψ has modulus of continuity γ which satisfies* (1). *For $\lambda \in R$ let*

(2) $$F_\lambda(r) = (\gamma^{-1}(r))^\lambda \exp\left\{-a \int_r^1 \frac{1}{\gamma^{-1}(t)} dt\right\}$$

where a is the first positive zero of the Bessel function $J_{\frac{n-3}{2}}(x)$. Then, for any $\lambda > -\frac{(n-2)}{2}$, u has modulus of continuity $\beta(t)$ where

(3) $$\beta(t) = F_\lambda^{-1}(ct)$$

for some constant c.

(II) *Suppose $H \leq 0$ near $x_0 \in \partial\Omega$ and γ is a modulus of continuity satisfying* (1). *Then for any $\lambda < -\frac{(n-2)}{2}$ there are boundary values ϕ having modulus of continuity λ such that u does not have modulus of continuity β given by* (3) *and* (2).

REMARKS

(i) The function β given by (3) is, in general, not going to be a simple one. However if one notices that the dominant factor in $F_\lambda(r)$ is the exponential (irrespective of the value of λ) it is possible to obtain a somewhat simpler modulus of continuity, namely,

(4) $$\beta_1(t) = F^{-1}\left(-\frac{\ln t}{a} + c\right)$$

$$\text{with} \qquad F(r) = \int_r^1 \frac{1}{\gamma^{-1}(t)}\,dt.$$

Part (II) of Theorem 3 shows that a cannot be replaced by any smaller constant.

(ii) In the special case of Hölder continuous boundary data ψ we have $\gamma(t) = Kt^\alpha$ and, while β given by (3) is not especially simple, it is easy to find β_1 given by (4). In fact

$$\beta_1(t) = \left[\frac{aK^{\frac{1}{\alpha}}\alpha}{(1-\alpha)(-\ln t + c)}\right]^{\frac{\alpha}{1-\alpha}}.$$

The constant a is as in the Theorem and cannot be improved.

(iii) There are other methods for deriving a modulus of continuity for u. In particular we mention the results of Simon [S1], which give a modulus without any conditions on the mean curvature H, and also section 13.5 of [GT].

While the answer obtained for the best modulus of continuity may not be particularly interesting or useful we believe that the way in which this answer arises in the proof is of interest.

PROOF OUTLINE: The proofs involve the construction of appropriate barriers and the main ingredient in this construction is to use an idea of Simon's [S2]. The idea is to construct the barrier as a function over the tangent plane to the boundary cylinder at $(x_0, \psi(x_0))$ instead of directly as a function over the domain Ω. We must of course ensure that the function so constructed is strictly increasing in the x_{n+1}- direction so that it is indeed a function over a portion of Ω. The advantage of this approach is that in the new coordinates the gradient of the function approaches 0 at x_0 whereas previously it became unbounded. When the gradient approaches zero the minimal surface operator is well approximated by the Laplacian. Consequently we need to consider functions v, on the cusped domain $D = \{y \in R^n : y_n > \gamma(|y'|),\ y' = (y_1, \ldots, y_{n-1})\}$, which are positive inside D but zero on ∂D and whose Laplacian is near zero. Writing D in polars as $D = \{(r, \theta) : |\theta| < \tilde{\gamma}(r)\}$ the required functions v have the form

$$v(r,\theta) = F_\lambda(r) G(\tilde{\gamma}^{-1}\theta) H(r,\theta)$$

where $G(t) = t^{\frac{3-n}{2}} J_{\frac{n-3}{2}}(at)$ and

$$H(r,\theta) = \exp\left\{\frac{b}{2} r \tilde{\gamma}' \tilde{\gamma}^{-2}\theta^2 + \frac{d}{2}\theta^2 \tilde{\gamma}^{-1}\right\}$$

for suitably chosen constants b and d.

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Department of Mathematics,
University of Wollongong,
Wollongong, N.S.W. 2500