APPLICATIONS OF MINIMAX TO MINIMAL SURFACES
AND THE TOPOLOGY OF 3-MANIFOLDS

*Jon T. Pitts and J.H. Rubinstein**

In this paper, we describe some recent constructions of and applications of minimal surfaces in 3-manifolds. These minimal surfaces arise from a minimum/maximum construction developed in [PR2] (and summarized in detail in [PR1]). (See also [PJ1] and [SS] for earlier versions of the method.) An outline of the basic procedure is as follows:

Let Σ be a closed connected oriented Riemannian 3-manifold and suppose Λ is an oriented Heegard surface in Σ; i.e., the closures of the two components of $\Sigma \sim \Lambda$ are handlebodies K and K'. We consider one-parameter smooth families $\Lambda_t, t \in [0,1]$, sweeping out Σ and having these properties: Λ_0 and Λ_1 are graphs; Λ_t is isotopic to Λ for all $0 < t < 1$; the handlebody K_t for $0 < t < 1$ is chosen so that the orientation on K_t coming from Σ induces the given orientation on Λ_t; and K_t converges to Λ_0 as $t \to 0+$ and the limit of K_t as $t \to 1-$ is Σ. The fundamental theorem is the following.

THEOREM 1. *There are a sequence of families* Λ_t^i *and choices of parameter* t_i *so that as* $i \to \infty$, $\Lambda_{t_i}^i$ *converges (in the* $\mathbf{F}$ *metric for varifolds) to a smooth closed embedded minimal surface* M *such that* genus$(M) \le$ genus(Λ) *and* index$(M) \le 1 \le$ index$(M) +$ nullity(M).

REMARKS. (1) Theorem 1 has turned out to be a versatile and powerful tool. In this paper we describe six interesting applications. In practice, applying Theorem 1 is comparatively straightforward. In a given situation, where one seeks minimal surfaces with specific properties, it typically suffices to specify a single (non-minimaxing)

"sweep-out" Λ_t having the sought properties. Theorem 1 or its variants then produce the surface. It is for this reason that in this paper we have emphasized constructing the sweep-outs.

(2) See [PR2] for the definition of the $\mathbf{F}$ metric. Also as in [PR2], index(M) is the number of negative eigenvalues of the second variation operator on normal vector-fields to M (counted with multiplicity), and nullity(M) is the multiplicity of zero eigenvalues of this operator.

(3) M is not necessarily connected, and some components of M may be nonorientable or may occur with multiplicity greater than one (as varifolds). Then

$$\operatorname{genus}(M) = \sum_{j} n_j \operatorname{genus}(M_j) + \sum_{k} (n_k/2) \operatorname{genus}(M_k),$$

where M_j (respectively M_k) is an orientable (respectively nonorientable) component of M with multiplicity n_j (respectively n_k). As usual, genus(M_j) is the number of handles of M_j and genus(M_k) is the number of crosscaps of M_k.

(4) M is obtained from Λ by deformations, possibly including a sequence of *compressions* of Λ, and followed perhaps by *projections* of some components of the resulting surface onto double covers of nonorientable components M_k of M. More precisely, a compression of Λ comes from an embedded disk D in Σ with $D \cap \Lambda = \partial D$. Let $N(D)$ be a small product neighborhood of D, chosen so that $N(D) \cap \Lambda$ is an annular neighborhood $N(\partial D)$ of ∂D. Then a compression Λ' of Λ is obtained by taking $\Lambda' = (\Lambda \cup \partial N(D)) \sim \operatorname{Int} N(\partial D)$. All the components of the surface Λ_0 obtained by a sequence of compressions of Λ are orientable. A nonorientable component M_k of M arises from a component Λ_k of Λ_0 which bounds a twisted line bundle over a surface which is isotopic to M_k. In the minimax limit, Λ_k projects onto M_k as a double covering.

This method is very general; in this paper we sketch six recent applications. Note that different sweep-out procedures are needed, including multi-parameter families of surfaces, equivariant sweep-outs, and sweep-outs using hypersurfaces in dimensions greater than three. These six applications are as follows.

(1) In joint work with J. Hass, we show the existence of many triply periodic minimal surfaces in $\mathbf{R}^3$ and generalizations.

(2) We construct complete minimal surfaces of finite area in complete noncompact hyperbolic 3-manifolds of finite volume.

(3) We develop a very general theory for equivariant minimum/maximum constructions in 3-manifolds. (See also [PR3] for many more results than are included here.)

(4) We sketch a scheme for classifying 3-manifolds of positive Ricci curvature. (See also [HR]).

(5) We describe an equivariant minimum/maximum construction in manifolds of dimension greater than three.

(6) We give a simple proof of the existence of complete minimal surfaces in $\mathbf{R}^3$ having genus ≥ 1 and three ends.

§1 Triply periodic minimal surfaces in $\mathbf{R}^3$.

Let $L = \{m\mathbf{u} + n\mathbf{v} + p\mathbf{w} : m, n, p \in \mathbf{Z}\}$, where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent vectors in $\mathbf{R}^3$. Thus L is a discrete rank 3 lattice in $\mathbf{R}^3$ and $T^3 = \mathbf{R}^3/L$ is a flat 3-torus. A triply periodic (connected) surface Λ^* in $\mathbf{R}^3$ is by definition invariant under translations by vectors in L and so descends to a closed (connected) surface Λ in $\mathbf{R}^3/L$. For certain lattices L, W. Meeks [MW1] constructed examples of triply periodic minimal surfaces, and the question was raised whether every lattice L supports such surfaces (see also [MW2]). Meeks [MW3] has also shown that a closed orientable (connected) minimal surface Λ in T^3 of genus ≥ 1 is always a Heegard surface (and hence has genus at least 3). Our main result is the following.

THEOREM 2. *Every lattice L admits infinitely many distinct (nonisometric) triply periodic minimal surfaces Λ^* which project to surfaces of genus 3 in $T^3 = \mathbf{R}^3/L$.*

Our method is quite general; there are substantial generalizations below among Other Results. We remark that Meeks [MW4] has recently announced a similar result to Theorem 2 using an entirely different proof.
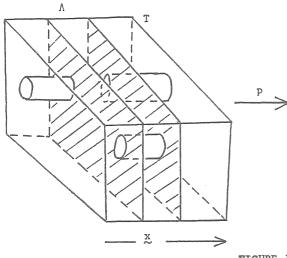
PROOF: (Sketch) Let us choose any generating vector $\mathbf{x}$ for L; i.e., $\mathbf{x}$ is in some basis

for L as an abelian group. If x/2 is adjoined to L, the result is a lattice L' which contains L as a subgroup of index 2. The flat 3-torus $T' = \mathbf{R}^3/L'$ is double covered by $T = \mathbf{R}^3/L$. Let $p: T \to T'$ be the projection. Then the covering translation can be viewed as translation by x/2.

Suppose an embedded minimal surface Λ' in T' can be found, which is nonorientable of genus 4. Then Λ' is double covered by an orientable surface Λ of genus 3 in T (see Figure 1) which lifts to a triply periodic minimal surface Λ^* in $\mathbf{R}^3$, and which is invariant under translation by x/2. Suppose y is a different generator for L and Λ_0 is a y/2-invariant orientable minimal surface of genus 3 in T. Then Λ and Λ_0 cannot be ambiently isometric by a translation of T. In fact, if such an isometric translation existed, then as translations commute we would have that Λ is invariant under translation by both x/2 and y/2. This free action of $\mathbf{Z}_2 + \mathbf{Z}_2$ on Λ would induce a projection of Λ onto a nonorientable surface of genus 3 in a 3-torus. But it is well known that such surfaces cannot embed in any 3-torus, which gives a contradiction. (For example, the Browder-Livesay invariant shows that only embedded nonorientable surfaces of even genus can occur.)

On the other hand, T has only finitely many isometries which are not translations (e.g., rotations). Consequently we conclude that if, for any choice of generator x for L, embedded orientable minimal surfaces of genus 3 in T can be constructed which are invariant by translation by x/2, then there are infinitely many such distinct minimal surfaces. So it suffices to find nonorientable minimal surfaces Λ' in T'.

We now describe a sweep-out in T' by nonorientable surfaces of genus 4, denoted Λ'_t, $t \in (0, 1)$. Λ'_t converges as $t \to 0+$ to a flat 2-torus A and converges as $t \to 1-$ to another flat 2-torus B. A and B are chosen to be $\mathbf{Z}_2$-homologous, but *not* $\mathbf{Z}$-homologous. We can view the sweep-out as occurring in the space of cycles with coefficients in $\mathbf{Z}_2$. In Figure 2, T' is drawn as a fundamental domain (parallelipiped) in $\mathbf{R}^3$, so that opposite faces are to be identified. A nonorientable handle is attached to A which forms the prototype for Λ'_t. For t near 0, the handle is almost pinched out. As t increases from 0 to 1, the handle expands and then compresses again so that Λ'_t collapses onto B.
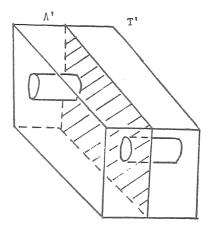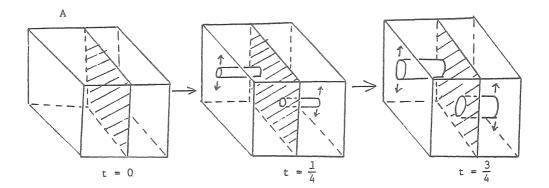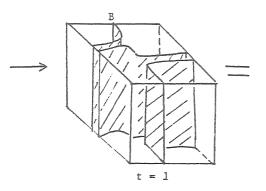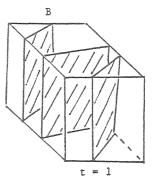
FIGURE 1

FIGURE 2

To finish the argument, we apply the minimax procedure to the collection of such sweep-outs Λ'_t. A slightly strengthened version of Theorem 4 of [PR1] is needed here (where $M(T_i + S) \geq M(T_i)$ is allowed). The conclusion is that there is an embedded minimal surface in the isotopy class of Λ'. Note that the only possible compressions of Λ' give some 2-torus C of multiplicity one as the result. It remains to show that this cannot happen for the minimax process.

Notice first that by Gauss-Bonnet, any minimal 2-torus is flat. So if a 2-torus C was the minimax limit of a sequence $\Lambda''^{i}_{t_i}$, then there would be nonorientable surfaces which are arbitrarily close to C (in the $\mathbf{F}$ metric for varifolds) but of smaller area than C. We can use the coarea formula plus a monotonicity argument to compress the nonorientable handle, yielding a 2-torus isotopic to C but having less area than C. This is impossible, as C minimizes area in its isotopy class.

OTHER RESULTS. (a) The method described above does not depend in an essential way on having a flat metric on $\mathbf{R}^3$ and T^3. So whenever a 3-torus has a Riemannian metric which is invariant under the translation by $\mathbf{x}/2$, then we obtain existence of an orientable minimal surface of genus 3 which is mapped to itself by $\mathbf{x}/2$, so long as minimal 2-tori have local stability properties similar to the flat case.

(b) The same procedure applies in a closed orientable 3-manifold Σ which is a surface bundle over a circle so that the monodromy map has the following three properties. First, the monodromy can be written as $\phi^2 \colon S \to S$, where S is the fiber of the bundle. Second, there are disjoint embedded noncontractible loops C and $\phi(C)$ on S. Third, if Σ' is the 3-manifold which is an S-bundle over a circle with monodromy ϕ, then there is an obvious double-covering map $\Sigma \to \Sigma'$. We assume Σ is given a Riemannian metric so that the covering transformation for this projection is an isometry.

The conclusion is that Σ supports a minimal surface which is a Heegard surface of genus $2g + 1$, where $g = \mathrm{genus}(S)$. In the special case that ϕ (and ϕ^2) is the identity, then for example, Σ can be given a product metric, $\Sigma = S \times S^1$. It can be shown easily that $2g + 1$ is the smallest Heegard genus possible for Σ.

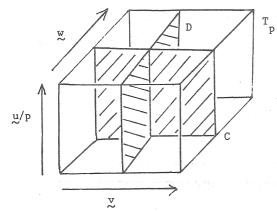(c) Finally, the most general case is that of a closed connected orientable 3-manifold

Σ with the property that its second Betti number is at least 2. Again, if we can find a suitable sweep-out by $\mathbf{Z}_2$-cycles between classes of the form α and $\alpha + 2\beta$ in $H_2(\Sigma, \mathbf{Z})$, then a nonorientable minimal surface can be found in Σ. (See [HPR] for details.)
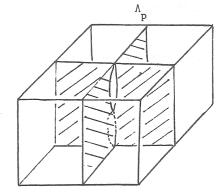
CONVERGENCE OF TRIPLY PERIODIC MINIMAL SURFACES. A further observation is that a sequence of triply periodic minimal surfaces can be found as above in $\mathbf{R}^3$, which converges to a lattice of (skew) planes. Let us define the *orthogonal lattice of planes* as the collection of planes which pass through a point of the form $(m, 0, 0)$ or $(0, m, 0)$ where $m \in \mathbf{Z}$, and which have normals parallel to the x_1-axis or x_2-axis in $\mathbf{R}^3$. It is well known that the periodic Scherk's surface can be scaled to converge to the orthogonal plane lattice. In our examples, however, the limiting lattice consists of planes which are not orthogonal in general.
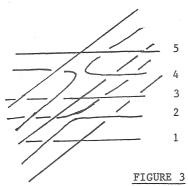
Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ span a lattice L in $\mathbf{R}^3$ as before, and let L_p denote the lattice generated by $\mathbf{u}/p$, $\mathbf{v}$, $\mathbf{w}$, where p is any odd positive integer. Assume also that $\mathbf{v}$, $\mathbf{w}$ have been chosen so that the lengths of the vector products $|\mathbf{u} \times \mathbf{v}|$ and $|\mathbf{u} \times \mathbf{w}|$ are as small as possible amongst all generating sets for L. As previously, we can find an embedded nonorientable minimal surface Λ_p of genus 4 in $T_p = \mathbf{R}^3/L_p$, by taking the minimax of sweepouts between the flat 2-tori A spanned by $\{\mathbf{u}/p, \mathbf{v} + \mathbf{w}\}$ and B spanned by $\{\mathbf{u}/p, \mathbf{v} - \mathbf{w}\}$.

We claim that the Area$(\Lambda_p) \leq l/p$, where $l = |\mathbf{u} \times \mathbf{v}| + |\mathbf{u} \times \mathbf{w}|$. Let C and D be the flat 2-tori in T_p spanned by $(\mathbf{u}/p, \mathbf{v})$ and $(\mathbf{u}/p, \mathbf{w})$, respectively. In Figure 3, start with $C \cup D$ with two tubes patched in along the singular circle $C \cap D$ to form a non-orientable surface in the isotopy class of Λ_p. If one of the tubes is compressed, then we get a torus which can be isotoped to A or B (depending on the choice of tube) in an *area-decreasing* fashion. This gives an explicit sweep-out from A to B with maximum area $\leq l/p$. Hence the minimax limit surface Λ_p has area bounded above by l/p.

Now let us lift each Λ_p to a surface Γ_p in $T = \mathbf{R}^3/L$. Since Area$(\Gamma_p) \leq l$, the sequence of stationary varifolds Γ_p has a convergent subsequence as $p \to \infty$ by the compactness theorem for integral varifolds. The limit is a stationary integral varifold

Level curves of surface Λ_p

FIGURE 3

Γ which is translationally invariant by $t\mathbf{u}$ for any $t \in \mathbf{R}$, since each Γ_p is invariant by $\mathbf{u}/p$ translation. But $t\mathbf{u}$ translation gives an S^1 action on T. Therefore Γ/S^1 is a stationary 1-dimensional integral varifold in the 2-torus $T^2 = T/S^1$. After regularity of Γ/S^1 has been established, it follows that Γ/S^1 is a collection of geodesic arcs.

Finally monotonicity shows that Γ/S^1 cannot consist of a single geodesic loop; i.e., the subsequence of minimal surfaces Γ_p cannot converge to a minimal 2-torus. Furthermore the homology class of Γ must be (Poincaré) dual to the homology class of the circle α in the direction $\mathbf{v} + \mathbf{w}$; i.e., Γ and α have intersection number 1 modulo 2, since each Γ_p has this property. The shortest collection of geodesic loops with this characteristic is clearly the union of two circles in the directions of $\mathbf{v}$ and $\mathbf{w}$. This pair of loops pulls back to $C \cup D$ in $T = T_1$. In addition, the area of $C \cup D$ is exactly the upper bound l for the area of Γ. This establishes that $\Gamma = C \cup D$ and so the subsequnce of minimal surfaces Γ_p lifts to a sequence of triply periodic minimal surfaces in $\mathbf{R}^3$ which converge to the lattice of planes which pass through the points $m\mathbf{v}$ or $m\mathbf{w}$ for $m \in \mathbf{Z}$ and are spanned by the vectors $\{\mathbf{u}, \mathbf{w}\}$ or $\{\mathbf{u}, \mathbf{v}\}$ respectively.

§2 Complete noncompact hyperbolic 3-manifolds of finite volume.

Our main result in this section is the following (cf. [UK]).

THEOREM 3. *If Σ is a complete noncompact hyperbolic 3-manifold with finite volume, then Σ admits a complete embedded minimal surface of finite area.*

REMARKS. (1) Here Σ has *cusps*; i.e., there is a compact 3-submanifold V of Σ with ∂V consisting of embedded tori and $\Sigma \sim V$ is homeomorphic to a finite disjoint union of copies of $T^2 \times \mathbf{R}$, where T^2 is the torus. Each such $T^2 \times \mathbf{R}$ is called a *cusp* of Σ and the cross-sectional tori have fundamental groups which inject into $\pi_1(\Sigma)$.

(2) A *Heegard surface* Λ of Σ is defined as a closed orientable surface which separates Σ, and the closures of the two components of $\Sigma \sim \Lambda$ are hollow handlebodies; i.e., boundary connected sums of standard handlebodies and copies of $T^2 \times [0,1)$. The $T^2 \times [0,1)$ factors come from the cusps of Σ. Some examples of hyperbolic Σ with Heegard genus 2 (by which we mean that Λ has genus 2 which is the smallest possible) are all once-punctured torus bundles over the circle with hyperbolic metrics (see

[FH]), and hyperbolic knot or link complements in S^3 with one freeing arc. (A *freeing arc* is an embedded arc in S^3 whose endpoints meet the knot or link, and the complement in S^3 of the union of the knot or link and the freeing arc is an open handlebody.) For example, the figure 8 knot and the Whitehead link each have one freeing arc. Therefore their complements with the standard hyperbolic metrics are examples of Σ satisfying the hypotheses of Theorem 3.

PROOF OF THEOREM 3: (Sketch) Choose a Heegard decomposition of Σ into two hollow handlebodies glued along a Heegard surface Λ. Then we can sweep out Σ by a family of surfaces Λ_t, $t \in (0,1)$, where Λ_t is isotopic to Λ for $0 < t < 1$. For t near 0 or 1, Λ_t is constructed by connecting together some tori cross-sections in the cusps by thin tubes. Note that there may be no cusps on one side of Λ, in which case Λ_t will collapse onto a graph. Also there may be several tubes connected to the same torus if Λ has large genus. At any rate, the area of Λ_t converges to zero as t approaches 0 or 1; one can think of Λ_t as approaching a graph which may or may not be compact. Consequently we can use such sweep-outs to perform the minimax construction.

§3 Equivariant minimax in 3-manifolds.

Suppose Σ is a closed connected orientable Riemannian 3-manifold and G is a finite group of isometries of Σ. Let Λ be a Heegard surface for Σ which is G-equivariant; i.e., $g\Lambda = \Lambda$ for all $g \in G$. Assume furthermore that G preserves the two components of $\Sigma \sim \Lambda$. Then there is a sweep-out of Σ by a family Λ_t, $0 \le t \le 1$, so that each Λ_t is G-equivariant and Λ_t is isotopic to Λ for $0 < t < 1$, with Λ_0 and Λ_1 graphs in Σ. We refer to Λ_t as a G-equivariant sweep-out and perform the basic minimax procedure amongst such G-equivariant families.

THEOREM 4. *There are sequences of G-equivariant families Λ_t^i and parameters t_i so that as $i \to \infty$, $\Lambda_{t_i}^i$ converges in the $\mathbf{F}$ metric topology to a smooth closed embedded G-equivariant minimal surface M so that $\text{genus}(M) \le \text{genus}(\Lambda)$, and the G-equivariant index and nullity of M satisfy $\text{index}_G(M) \le 1 \le \text{index}_G(M) + \text{nullity}_G(M)$.*

REMARKS. (1) The G-equivariant index, index$_G(M)$, is the number of negative eigenvalues (counted with multiplicity) for the restriction of the second variation operator for area to G-equivariant normal vectorfields to M. The G-equivariant nullity is the multiplicity of zero eigenvalues for this restricted operator. In general, index(M) will be somewhat larger than index$_G(M)$ (cf. Theorem 1).

(2) Note that M may be disconnected, so the components M_i satisfy $g(M_i) = M_i$ or $g(M_i) = g(M_j)$ with $M_i \cap M_j = \emptyset$, for each $g \in G$. So G-equivariance is a more precise description than G-invariance.

(3) Exactly as in Remark 3 following Theorem 1, M is obtained from Λ by a sequence of compressions and projections of components onto double covers of nonorientable components of M. Here the compressions are G-equivariant; i.e., they are achieved by a family of disks $\{D_i\}$ with the property that $g(D_i) = D_i$ or $g(D_i) = D_j$ for all $g \in G$.

We now give some basic applications of this theorem to specific examples. For a much more detailed account of the usefulness of Theorem 4, see [PR3].

EXAMPLE 1. THE POINCARÉ HOMOLOGY 3-SPHERE. This is a 3-dimensional spherical space form; i.e., a closed orientable Riemannian 3-manifold which is locally isometric to S^3 with the standard spherical metric. In fact the Poincaré homology 3-sphere $\Sigma = S^3/I^*$, where I^* is the binary icosahedral group of order 120. (See [OP] or [TC] for more details on Σ.) Since I^* is a subgroup of $SU(2) \approx S^3$, the Lie group of 2×2 unitary matrices with determinant one, we can view Σ as the left coset space of I^* in $SU(2)$.

The basic (nonequivariant) minimax construction (Theorem 1) gives an embedded minimal surface which is orientable of genus 2 in Σ, as Σ has Heegard genus 2 (see [PR2]). We shall exploit the equivariance to obtain minimal surfaces of higher genus in Σ. (In fact in [PR3] it is shown that there is a sequence of such surfaces with genus $\to \infty$.)

Let us view Σ as obtained by identifying faces of a fundamental domain for the action of I^* on S^3. As is well-known, the regular spherical dodecahedron with dihedral angles $2\pi/3$ is such a fundamental domain; i.e., S^3 is tesselated by 120 such

dodecahedra which are permuted by the action of I^*. (See [WS] for the original reference.) Then Σ is constructed by gluing opposite faces of this dodecahedron with a twist of $\pi/5$ ([WS]).

There is an obvious isometry group action by the dodecahedral (icosahedral) group $I = A_5$ on Σ. Another way of viewing this action is to write $SO(4) = SU(2) \times SU(2)/\mathbf{Z}_2$, given by left and right multipliction of $SU(2)$ on itself, where the $\mathbf{Z}_2$ is generated by simultaneous left and right multiplication by $-e$, e being the identitiy in $SU(2)$. Then the subgroup $I^* \times I^*/\mathbf{Z}_2$ of $SO(4)$ is the lift of the A_5 action on Σ to S^3.

We now describe an A_5-equivariant sweep-out of Σ. One starts with the dual 1-skeleton Λ_0 of the dodecahedron (see Figure 4). This closes up under the face identification to form a bouquet of 6 circles in Σ and is obviously A_5-equivariant. For $0 < t < 1$, one chooses an A_5-equivariant regular neighborhood K_t of Λ_0 and define Λ_t as the boundary ∂K_t of K_t. Finally as $t \to 1-$, K_t enlarges until $K_1 = \Sigma$. Consequently it can be arranged that $\Lambda_t \to \Lambda_1$ as $t \to 1-$, where Λ_1 is the 1-skeleton of the dodecahedron in Σ.

Therefore Theorem 4 can be applied to obtain an A_5-equivariant minimal surface M of total genus at most 6. It is easy to check that the surface Λ_t, for any $0 < t < 1$, can only compress A_5-equivariantly to a collection of minimal 2-spheres. But Σ has no minimal embedded 2-spheres by Frankel's theorem ([FT]). We concluded that Σ admits an embedded minimal closed orientable surface of genus 6.

This surface lifts to a minimal surface of genus 601 in S^3. Notice that the surface in S^3 has the same symmetries and genus as the fourth example in Table 1 of [KPS]. It is likely that these two examples are in fact identical. In [KPS] the minimal surfaces are constructed by a different technique, which is rather explicit. See also [PR3] for development of many new classes of minimal surfaces in S^3, including ones with the same genera and symmetry groups as all the examples in [KPS]. As an interesting case, using $\mathbf{Z}_5$ as the group of isometries acting on the Poincaré homology sphere Σ, we can find a genus 4 embedded orientable minimal surface in Σ. (See example 3 following and [PR3].) This surface lifts to a genus 361 minimal surface in S^3 which

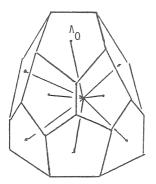is a new example, since it has different symmetries and genus to types previously described in [LB] and [KPS].

EXAMPLE 2. WEBER-SEIFERT HYPERBOLIC DODECAHEDRAL SPACE. (See [WS].) The 3-manifold Σ is now hyperbolic; i.e., Σ has a Riemannian metric which is locally isometric to $\mathbf{H}^3$. A fundamental domain for the action of $\pi_1(\Sigma)$ on $\mathbf{H}^3$ is a regular hyperbolic dodecahedron with all dihedral angles $2\pi/5$. To form Σ, one identifies opposite faces after a twist of $3\pi/5$. As before, Σ can be swept out by A_5-equivariant closed orientable surfaces Λ_t of genus 6, using the symmetry group A_5 of the dodecahedron.
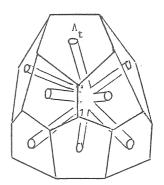
By Theorem 4, there is an embedded closed orientable minimal surface M of genus 6 in Σ. Note that Λ_t again cannot compress A_5-equivariantly to a collection of minimal 2-spheres, this time by the Gauss-Bonnet theorem. Also using $\mathbf{Z}_5$ as the isometry group, we obtain a genus 4 closed orientable minimal surface embedded in Σ. (See example 3 following and [PR3].)

REMARKS. (1) Both these minimal surfaces lift to infinite genus minimal surfaces in $\mathbf{H}^3$ which are analogous to the triply periodic minimal surfaces of §1. In $\mathbf{H}^3$ the surfaces are invariant under the isometric action of $\pi_1(\Sigma)$ and divide $\mathbf{H}^3$ into two handlebodies with infinitely many handles.

(2) Here Σ has Heegard genus 3, so there is no sweep-out of Σ by genus 2 surfaces as for the Poincaré homology sphere (see [PR2]). A sweep-out by genus 3 surfaces using Theorem 1 gives a minimal surface of genus 2 or 3 for the Weber-Seifert dodecahedral space. We are not able to tell which genus occurs.

EXAMPLE 3. BRANCHED COVERS OF THE TREFOIL KNOT. Let C denote the trefoil knot in S^3. (See Figure 5.) As is well known, $\pi_1(S^3 \sim C)$ can be expressed in the form $1 \to \mathbf{Z} \to \pi_1(S^3 \sim C) \to \mathbf{SL}(2, \mathbf{Z}) \to 1$, where $\mathbf{SL}(2, \mathbf{Z})$ is the modular group ([MJ1], [GR], [RV]). Now there are many well known homomorphisms from $\mathbf{SL}(2, \mathbf{Z})$ (and therefore $\pi_1(S^3 \sim C)$) onto finite groups G; for example there is the homomorphism $\mathbf{SL}(2, \mathbf{Z}) \to \mathbf{SL}(2, \mathbf{Z}_p)$ induces by reduction modulo p. These homomorphisms yield *regular branched coverings* Σ of S^3, branched over the trefoil knot, with (branched) covering transformation group G.

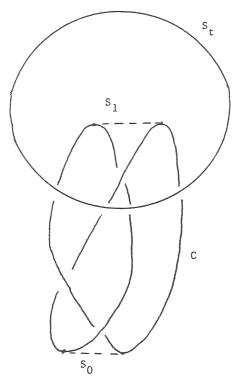Dodecahedron with dual 1-skeleton A_5-equivariant Heegard surface

FIGURE 4

FIGURE 5

There is a simple natural way to construct a sweep-out Λ_t in Σ by Heegard surfaces which are G-equivariant and have smallest genus amongst such surfaces. Observe that we can sweep out S^3 by embedded 2-spheres S_t, for $0 < t < 1$, each of which meets C transversely in exactly four points. This is called a 2-*bridge representation* of the trefoil knot. (See Figure 5 and [RD] for more information on knots.) We can arrange that as $t \to 0$ or 1, $S_t \to S_0$ or S_1 which are arcs with endpoints on C, as in Figure 5. Now it is easy to see that S_t lifts to Λ_t, a G-equivariant Heegard surface for Σ. One observes that Λ_t is a branched cover of S_t, branched over the four points $S_t \cap C$, so it is easy to compute the genus of Λ_t. Also S_0 and S_1 lift to graphs Λ_0 and Λ_1 in Σ, and Λ_t collapses onto these graphs as $t \to 0$ or 1.

Applying Theorem 4 to the sweep-out Λ_t, we obtain a G-equivariant embedded minimal surface M in Σ. Note that we must choose a G-equivariant Riemannian metric on Σ. A natural choice is a geometric structure; i.e., a metric which is locally isometric to the Lie group S^3 or $\widetilde{\mathrm{SL}(2,\mathbf{R})}$. (See [TW] or [SP] for information on such metrics.) In this case, there cannot be a G-equivariant compression of the minimaxing sequence converging to M. The reason is that the compression would descend to the 2-spheres S_t in S^3. But S_t can only nontrivially compress to give a component which is a 2-sphere meeting C in two points. But such a surface lifts to a collection of embedded 2-spheres in Σ; and with a geometric structure, Σ admits no minimal 2-spheres (unless $\Sigma = S^3$ which is excluded because S^3 is not a branched cover of C). We conclude that M is isotopic to Λ_t; i.e., M is a G-equivariant Heegard surface in Σ.

Recall that some examples of Brieskorn homology 3-spheres are cyclic branched covers of the trefoil know (see [MJ2]). As a special case, if $G = \mathbf{Z}_5$, then $\Sigma = S^3/I^*$ is the Poincaré dodecahedral space and M has genus 4, as claimed in Example 1 above. This construction, however, works equally well for all 2-bridge knots and links. For example, if $G = \mathbf{Z}_5$ again and C is replaced by the Whitehead link, then Σ is the Weber-Seifert hyperbolic dodecahedral space [WS]. As in Example 1, again M has genus 4. For many generalizations of this construction, see [PR3]. If knots or links with bridge number greater than two are used, then G-equivariant compressions of Λ_t can occur and the genus of M may drop.

§4 Classifying 3-manifolds of positive Ricci curvature.

An old problem in the topology of 3-manifolds is to classify those with finite fundamental group. If Σ is such a closed 3-manifold, then its universal cover $\tilde{\Sigma}$ is a homotopy 3-sphere on which $\pi_1(\Sigma)$ acts as a group of covering transformations. Let us denote this group be G. We now outline a program for studying this class of 3-manifolds Σ, *assuming that* $\tilde{\Sigma} = S^3$ to avoid obvious difficulties with the Poincaré conjecture.

The *geometrization conjecture* (in this case) is that each closed 3-manifold Σ with finite fundamental group is a spherical space form (see Example 1 above). An equivalent form of the conjecture is that Σ is diffeomorphic to S^3/G', where G' is a subgroup of $SO(4)$ which acts freely on S^3 and is isomorphic to $G = \pi_1(\Sigma)$. Alternatively, given a smooth free action of G on S^3, there is a diffeomorphism $\phi: S^3 \to S^3$ so that $\phi^{-1}G\phi$ is a subgroup G' of $SO(4)$. We summarize this by saying that the action of G is *equivalent* to the linear action of G'.

REMARKS. (1) The fundamental groups G of these spherical space forms can be conveniently listed as follows. $G = G_1 \times G_2$, where G_2 is either the trivial group or is a cyclic group of order relatively prime to the orfer of G_1. The possibilities for G_1 are cyclic, dihedral, binary dihedral, generalized binary tetrahedral, binary octahedral, or binary icosahedral. (See [OP] for a discussion of these groups. See also the monograph [TC] for a summary of progress on the geometrization conjecture.)

(2) There is a class of groups $Q(8mkl)$ which may act freely on S^3, but do not occur as fundamental groups of the spherical space forms. (See [TC] and [DM].)

Our strategy is to give a general procedure for studying many free G-actions on S^3 by Morse theory type arguments in an appropriate space of embedded surfaces. We say that a torus embedded in S^3 is *unknotted* if it is a Heegard surface for S^3; i.e., if it decomposes S^3 into two solid tori. Our procedure, which applies for manifolds with many different fundamental groups, divides into two cases, depending on the group G. We summarize the method.

Suppose G is any finite group acting freely on S^3, other than composite cyclic groups and binary icosahedral by cyclic groups.

(1) If G is cyclic of prime order or binary dihedral by cyclic, we outline a program to show that either the G action is equivalent to a linear action or for any Riemannian metric which is G-invariant on S^3, there are infinitely many distinct embedded minimal surfaces which are either 2-spheres or unknotted tori.

(2) THEOREM 5. Suppose G is a finite solvable noncyclic group acting freely on S^3 and G is not dihedral by cyclic. Then either the G action is equivalent to a linear action or for any Riemannian metric which is G-invariant on S^3, there are infinitely many distinct embedded minimal surfaces which are unknotted tori.

REMARKS. (1) The main tool is an iterated minimum/maximum construction. See [PJ2] for another example of such a construction.

(2) It is not difficult to construct a Riemannian metric on S^3 which is G-invariant and admits infinitely many such minimal surfaces. One conjectures that not *every* G-invariant Riemannian metric will have this property.

(3) The reason that composite cyclic actions ($G = \mathbf{Z}_n$ and n is not prime) must be excluded is because of an unsolved topological problem related to a construction used in our program. The question is to show that the space of diffeomorphisms of a 3-dimensional lens space to itself is (weakly) homotopy equivalent to the space of isometries of the lens space. We call this as the *Smale conjecture for lens spaces*, by analogy with the classical Smale conjecture for S^3, as solved by A. Hatcher [HA].

(4) The result *does* include the possibility of a free action on S^3 by one of the exotic groups $Q(8mkl) = G$. In this case, since no free action of G is equivalent to a linear action, the conclusion would be that for any G-invariant metric on S^3, there are infinitely many unknotted tori.

COROLLARY OF METHOD. *Suppose G is a group as in cases 1 or 2 above, and there is a Riemannian metric of positive Ricci curvature on S^3 for which G acts isometrically. Then the action of G is equivalent to a linear action.*

REMARK. This gives a classification of 3-manifolds of positive Ricci curvature and fundamental groups as above, modulo the Poincaré conjecture. R. Hamilton [HR] has given a complete solution in the case of nonnegative Ricci curvature by a heat

flow technique. One of the points of the present approach is that our program may be applicable with weaker (or no) metric hypotheses.

PROOF OF COROLLARY: By a result of B. White [WB2], one may perturb the metric on S^3 so that any smooth closed minimal surface has nullity zero. In particular such surfaces are isolated. Since S^3 has positive Ricci curvature, the compactness theorem of Choi-Schoen [CS] implies that S^3 contains only a finite number of minimal spheres and tori. Thus G must be equivalent to a linear action.

OUTLINE OF THE PROCEDURE.

Case 1. First we consider $\mathbf{Z}_p$ actions on S^3, for p an odd prime. (The case of free $\mathbf{Z}_2$ actions was solved by Livesay [LR].)

By the solution of the Smale conjecture [HA], we know the space of orientation preserving diffeomorphisms $\text{Diff}_0(S^3)$ has the same homotopy type as $SO(4)$. Let $\mathcal{T}$ denote the space of smooth embedded unknotted tori in S^3. There is an obvious fibration $\text{Diff}(T^2) \to \text{Diff}_0(S^3) \to \mathcal{T}$. The exact homotopy sequence of this fibration yields that $\mathcal{T}$ is homotopy equivalent to $\mathbf{RP}^2 \times \mathbf{RP}^2$.

Let g denote a diffeomorphism of S^3 generating the $\mathbf{Z}_p$ action. First we claim that if there is a fixed point for the induced action of $\mathbf{Z}_p$ on $\mathcal{T}$, then the original $\mathbf{Z}_p$ action on S^3 is equivalent to a linear action. The reason is that a fixed point in $\mathcal{T}$ is precisely an embedded unknotted torus T in S^3 which is $\mathbf{Z}_p$-equivariant. We can then easily conjugate the $\mathbf{Z}_p$ action to inear actions on the solid tori which are the closures of the components of $S^3 \sim T$. Note that if $\mathcal{T}$ was homeomorphic to a finite simplicial complex, then the $\mathbf{Z}_p$ action would have a fixed point by the Lefschetz fixed point theorem, since $\mathcal{T}$ is homotopy equivalent to $\mathbf{RP}^2 \times \mathbf{RP}^2$.

Let $\bar{\mathcal{T}}$ denote the space of embedded 2-spheres and unknotted tori in S^3. Note that $\mathcal{T}$ is dense in $\bar{\mathcal{T}}$, as each 2-sphere is the limit of unknotted tori, where the handle is becoming arbitrarily thin; i.e., a compression is occurring as before. The strategy is to show that if there are not fixed points for $\mathbf{Z}_p$ acting on $\mathcal{T}$, then the area functional on $\bar{\mathcal{T}}$ has infinitely many distinct critical points. Consequently these critical points are the desired collection of minimal tori and 2-spheres.

The first step is to apply the basic minimax method, as in Theorem 1. This

yields an embedded minimal unknotted torus or 2-sphere M_1, where we use Heegard tori Λ_t to sweep out S^3. By our assumption, $\mathbf{Z}_p$ has no fixed point on $\mathcal{T}$. Therefore $\mathbf{Z}_p$ acts with no fixed points on $\bar{\mathcal{T}}$, since there cannot be a 2-sphere in S^3 which is invariant under a free $\mathbf{Z}_p$ action for p odd. Consequently the surfaces $M_1, gM_1, \ldots,$ $g^{p-1}M_1$ are all distinct. We can choose a path λ in $\bar{\mathcal{T}}$ from M_1 to gM_1, using the fact that $\mathcal{T}$ is path connected and dense in $\bar{\mathcal{T}}$. Since $\pi_1(\mathcal{T}) = \mathbf{Z}_2 + \mathbf{Z}_2$, we can also arrange that the loop $l = \lambda \cup g\lambda \cup \ldots \cup g^{p-1}\lambda$ is contractible in $\bar{\mathcal{T}}$. Again this used the denseness of $\mathcal{T}$ in $\bar{\mathcal{T}}$. We shall not keep repeating this idea, which is used throughout Case 1.

The second step is to apply the minimax procedure to the path of surfaces λ. Conceivably no new minimal surface is produced if the area of all the surfaces along λ is not larger than $\text{Area}(M_1) = \text{Area}(gM_1)$. If the area does increase, then we find a second minimal surface M_2. The third step is to span l by a disk D, since l is contractible in $\bar{\mathcal{T}}$. We then apply the 2-parameter minimax argument to D. This gives a new critical point and hence another minimal torus or 2-sphere, say M_3. We continue on. $D \cup gD$ is an integral 2-cycle and so spans a 3-chain in $\bar{\mathcal{T}}$, etc.

Notice that we are really working with loops, disks, chains, etc. in $\bar{\mathcal{T}}$ with area very close to the minimax value, since there may not be such families achieving the smallest possible maximum area. The conclusion is that there is an infinite sequence of distinct embedded minimal 2-spheres or unknotted tori, as desired.

REMARK. There is an important problem to be overcome in this argument. The sequence of minimax surfaces constructed could conceivably consist of a single minimal 2-sphere counted with ever increasing multiplicity. (Much the same problem occurs when seeking many closed geodesics in Riemannian manifolds.) This degeneration cannot occur for a minimal 2-torus by an easy topological argument. It appears plausible that this problem can be solved by generalizations of estimates and constructions in [PR2] and [PR3].

Case 2. We sketch the proof of Theorem 5. Here we assume that $G = G_1 \times G_2$, where G_2 is trivial or cyclic and G_1 is binary polyhedral or a $Q(8mkl)$ group, but not binary icosahedral. The argument is by induction on the length of a composition series for

the solvable group G. Note that free $\mathbf{Z}_2$ actions on 3-dimensional spherical space forms have been completely classified ([MR], [EM], [RJ1], [RJ2]). So it suffices to assume we have a free $\mathbf{Z}_p$ action on a 3-dimensional spherical space form $\Sigma = S^3/G'$, where p is an odd prime and $G' = G_1 \times G_2$. Also the results of [RJ2] can be used to reduce the problem to the case where G_1 is either a dihedral or binary dihedral group.

To be able to use a similar procedure to Case 1, two questions have to be answered. First we need to compute the homotopy of $\mathrm{Diff}_0(\Sigma)$. Results for many of the necessary cases were announced by N. V. Ivanov [IN]. A complete solution for all required spherical space forms is contained in [McR]. Second we have to find appropriate surfaces in Σ to play the role of unknotted 2-tori in Case 1. Fortuitously it turns out [RJ1] that all such Σ possess embedded Klein bottles K, representing non-trivial $\mathbf{Z}_2$ cycles in Σ. These Klein bottles are geometrically incompressible; i.e., there is no embedded 2-disk D in Σ with $D \cap K = \partial D$, where ∂D is a noncontractible loop in K. Therefore if we use a sweep-out in Σ by Klein bottles, then any compressions which occur when using the minimax method yield a Klein bottle and some 2-spheres. So both sweep-outs and limit surfaces can be taken to be Klein bottles and consequently we can avoid the technical difficulties of working in $\bar{\mathcal{T}}$ rather than $\mathcal{T}$ as in Case 1.

Let $\mathcal{K}$ denote the space of embedded Klein bottles in Σ in a fixed isotopy class. (In some cases Σ can have several such isotopy classes.) It is easy to reduce the argument to the case where the $\mathbf{Z}_p$ action preserves this isotopy class and so induces an action on $\mathcal{K}$. Exactly as in Case 1, if there is a fixed point K in $\mathcal{K}$, then the action on Σ is equivalent to a linear one. (Note that $\Sigma \sim K$ is an open solid torus. If K is $\mathbf{Z}_p$-invariant, then so is $\Sigma \sim K$. See [RJ1].) Finally it suffices to suppose the $\mathbf{Z}_p$ action on $\mathcal{K}$ is free. To begin with, by [MSY] there is an embedded minimal Klein bottle M_1 of least area in $\mathcal{K}$. We then use the same reasoning as in Case 1 to find an infinite sequence of embedded minimal Klein bottles in Σ. Since these surfaces lift to embedded minimal unknotted 2-tori in S^3, the proof is complete.

REMARKS. (1) Let us examine the above procedure from the viewpoint of Morse

theory. If the $\mathbf{Z}_p$ action on $\mathcal{T}$ or $\mathcal{K}$ is free, then there is an induced p-fold covering, and there must be Z_p-equivariant cycles in arbitrarily large dimensions in $\mathcal{T}$ or $\mathcal{K}$. For example, in many cases $\mathcal{K}$ is contractible. Then $\mathcal{K}/\mathbf{Z}_p$ is a $K(\mathbf{Z}_p, 1)$ space (Eilenberg-Maclane space) and so has nontrivial cycles with $\mathbf{Z}_p$ coefficients in all dimensions. Finally these $\mathbf{Z}_p$-equivariant cycles give rise to critical points of the area functional (minimal surfaces) as in Morse theory.

(2) The difficulty in carrying through the program for all (nonprime) cyclic groups is that one need to know the homotopy type of $\mathrm{Diff}_0 \, L(m, n)$ for arbitrary 3-dimensional lens spaces $L(m, n)$. This calculation has only been done for $L(4k, 2k-1)$ where $k > 1$ ([IN], [McR]). The situation for the binary icosahedral group looks considerable more difficult, since it is not a solvable group.

§5 Equivariant minimax in manifolds of dimension greater than three.

Let Σ be a closed orientable Riemannian n-manifold and let G be a compact Lie group which acts isometrically on Σ. We recall that a *principal orbit* P of G acting on Σ is a maximal orbit type; i.e., the isotropy subgroup of P is conjugate to a subgroup of every isotropy group of an orbit of the G-action. A *singular orbit* Q of the G-action satisfies $\dim Q < \dim P$. An *exceptional orbit* Q of the G-action has the properties that $\dim Q = \dim P$ and $P \to Q$ is a nontrivial covering map. (See [BG], for example.)

As is well known, the collection Σ^* of all principal orbits forms an open dense set in Σ. Let B denote the set of all singular orbits and let E be the collection of all exceptional orbits in Σ. We assume for simplicity that all the transformations in G are orientation preserving on Σ. From [BG, Chapter IV, §3], we note that $\dim(B \cup E) \leq n - 2$. We shall be interested in the special case that the principal orbits have dimension $n - 3$. Then the orbit space Σ/G is a 3-dimensional complex. For example, if Σ is simply connected, the Σ/G is a simply connected 3-manifold, possibly with boundary [BG, Corollary 4.7, p.190]. We have the following result, analogous to Theorems 1 and 4, for the minimax method using sweep-outs of Σ by hypersurfaces which are G-equivariant.

THEOREM 6. *Let Σ be a closed connected orientable Riemannian n-manifold ($n \geq 4$) and let G be a compact Lie group acting isometrically on Σ with principal orbits of codimension 3. Let Λ_t be a sweep-out of Σ by closed hypersurfaces which are G-equivariant; i.e., $g\Lambda_t = \Lambda_t$ for all $g \in G$. Then there is a closed minimal hypersurface M in Σ which is G-equivariant. M is a smooth embedded submanifold of Σ except perhaps for a compact singular set of Hausdorff dimension at most $n - 8$ which lies in the union B of singular orbits of G. Also genus$(M^*/G) \leq$ genus(Λ_t^*/G), where $M^* = M \cap \Sigma^*$ and $\Lambda_t^* = \Lambda_t \cap \Sigma^*$.*

REMARKS. (1) Note that $\Sigma^* \to \Sigma^*/G$ is a principal G-bundle and hence induces a Riemannian submersion. So M^*/G is a smoothly embedded surface. We count genus(M^*/G) and genus(Λ_t^*/G) after compactifying the surfaces by adding a finite number of points.

(2) Since exceptional orbits have codimension 3, they cannot be contained in the singular set of M. In particular if there are no singular orbits, then the singular set of M is empty.

EXAMPLE. For each $n \geq 4$, we construct a sequence M_i of embedded minimal hypersurfaces in S^n, the standard constant curvature sphere of radius one in $\mathbf{R}^{n+1}$. First we set notation. Let $\mathbf{x} = (x_1, \ldots, x_{n+1})$ denote a point in $\mathbf{R}^{n+1}$ with length $|\mathbf{x}|$. Then $S^n = \{\mathbf{x} : |\mathbf{x}| = 1\}$, and we define embedded minimal hypersurfaces $S^{n-1} = \{\mathbf{x} : x_1 = 0\}$, $S^1 \times S^{n-2} = \{\mathbf{x} : x_2^2 + x_3^2 = 1/(n-1), x_1^2 + x_4^2 + \cdots + x_{n+1}^2 = (n-2)/(n-1)\}$, and $S^2 \times S^{n-3} = \{\mathbf{x} : x_1^2 + x_2^2 + x_3^2 = 2/(n-1), x_4^2 + \cdots + x_{n+1}^2 = (n-3)/(n-1)\}$. One notes that S^{n-1} intersects both $S^1 \times S^{n-2}$ and $S^2 \times S^{n-3}$ in (different) copies of $S^1 \times S^{n-3} = W$.
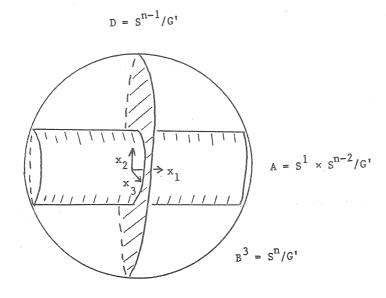
Our sequence M_i has the following properties. As $i \to \infty$, M_i converges as a sequence of varifolds (in the $\mathbf{F}$ metric) to $(S^2 \times S^{n-3}) \cup S^{n-1}$. For suitable i sufficiently large, M_i is smooth everywhere (M_i has no singular set) and is diffeomorphic to $\natural_{2i}S^1 \times S^{n-2}$. The closure of the components of the complement of M_i are handlebodies; i.e., these minimal surfaces are unknotted in S^n. Furthermore, the convergence of M_i to $(S^2 \times S^{n-3}) \cup S^{n-1}$ is smooth away from W. (When $n = 4$, there is also a second possibility; namely, that M_i is diffeomorphic to $\natural_{(2i+2)}S^1 \times S^{n-2}$,

and that the sequence of M_i converges as varifolds to $(S^1 \times S^{n-2}) \cup S^{n-1}$.)

REMARK. These examples have properties which resemble those of Lawson [LB] in S^3. It is well known that there is a sequence of minimal surfaces in S^3 constructed in [LB] which converges (as varifolds) to the finite union of equatorial 2-spheres embedded in S^3, all of which meet along a common great circle. The convergence of these surfaces is smooth away from this circle of intersection, and they are unknotted in S^3.
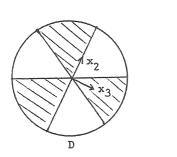
CONSTRUCTION. Let $G' = SO(n-2)$. G' acts on the second factor of the decomposition $\mathbf{R}^{n+1} = \mathbf{R}^3 \times \mathbf{R}^{n-2}$ in the usual way and acts trivially on the first factor. The induced action on S^n yields codimension 3 principal orbits with $S^n/G' = B^3$, the 3-ball. We slightly abuse notation in B^3 and use coordinates (x_1, x_2, x_3), where $x_1^2 + x_2^2 + x_3^2 \leq 1$. Both S^{n-1} and $S^1 \times S^{n-2}$ are G'-invariant with $S^{n-1}/G' = \{(x_1, x_2, x_3) : x_1 = 0\}$ and $S^1 \times S^{n-2}/G' = \{(x_1, x_2, x_3) : x_2^2 + x_3^2 = 1/(n-1), x_1^2 = (n-2)/(n-1)\}$. We denote this disk by D and this annulus by A in Figure 6.
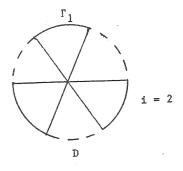
We picture the equivariant sweep-out in S^n by drawing the induced sweep-out in B^3. For fixed $i \geq 1$ and $0 < t < 1$, $\Lambda_t/G' = \Lambda_t'$ is a properly embedded compact orientable surface in B^3 with genus i and three boundary curves on ∂B^3. As $t \to 0+$, Λ_t' converges to D, and as $t \to 1-$, Λ_t' approaches $-D$ (by which we mean D with its opposite orientation). Also the Lie group G of isometries which we require actually is disconnected, with G' as its identity component. G is obtained by extending G' by a finite cyclic group $Z_{2(i+1)} = \langle \alpha | \alpha^{2(i+1)} = 1 \rangle$. This latter group acts on the orbit space B^3 of G' by $\alpha(x_1, x_2, x_3) = (-x_1, x_2 \cos \theta + x_3 \sin \theta, x_2 \sin \theta - x_3 \cos \theta)$, where $\theta = \pi/(i+1)$.

The surface Λ_t' is obtained by joining together two "pinched" annuli. Suppose D is divided into $2i$ sectors which are alternately colored black and white (see Figure 7). Let Γ_1, Γ_2 be the graphs (bouquets of circles) obtained as the boundaries of the black (white) region on D (see Figure 7). Let C_1, C_2 be simple closed curves on ∂B^3 which are disjoint from ∂D and satisfy $C_2 = \alpha C_1$. Notice that $\Gamma_2 = \alpha \Gamma_1$ as well. Finally we connect Γ_1 and C_1 by a tightly pinched annulus A_1. Then $A_2 = \alpha A_1$ has boundary $\Gamma_2 \cup C_2$ and we let $\Lambda_t' = A_1 \cup A_2$ (see Figure 8).
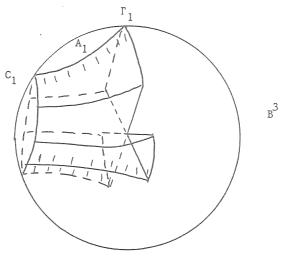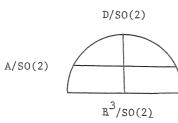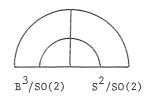
$D = S^{n-1}/G'$

$A = S^1 \times S^{n-2}/G'$

$B^3 = S^n/G'$

FIGURE 6

Γ_1

$i = 2$

D

D

FIGURE 7

FIGURE 8

FIGURE 9

Next, we describe the sweep-out. As $t \to 0+$, C_1 and C_2 shrink to the points $(-1,0,0)$, $(1,0,0)$. Then A_1 looks like the black region on D (see Figure 7) with a spike at the center $(0,0,0)$ out to C_1 (and similarly for A_2 and the white region on D). Consequently, $\Lambda'_t \to D$ as $t \to 0+$. As $t \to 1-$, both C_1 and C_2 approach ∂D. One notes that the induced orientations on C_1 and C_2 from the surface Λ'_t are opposite to that of ∂D. It can be checked that A_1 approaches the white region on D, A_2 converges to the black region, and therefore that $\Lambda'_t \to -D$ as $t \to 1-$.

Theorem 6 is now applicable. Let M_i be the minimal hypersurface obtained by the minimax procedure, and let M'_i denote M'_i/G', for fixed i. Since $\mathrm{Int}\, B^3$ is the part of the orbit space corresponding to the principal orbits of the G' action, it follows that M'_i is regular in $\mathrm{Int}\, B^3$. Just as Theorems 1 and 4, $M'_i \cap \mathrm{Int}\, B^3$ is obtained from $\Lambda'_t \cap \mathrm{Int}\, B^3$ by compressions. There is only one such compression which is $Z_{2(i+1)}$-equivariant; namely, the compression along two disks which are parallel to the disks in $\partial B^3 \sim \partial D$ bounded by C_1 and C_2. Whether M'_i is obtained from Λ'_t by this compression or $M'_i \simeq \Lambda'_t$, $\mathrm{genus}(M'_i) = \mathrm{genus}(\Lambda'_t) = i$.

It is not difficult to see explicitly that the sweep-out above can be done so as to guarantee that $\mathrm{Area}(M_i) \leq \mathrm{Area}(S^{n-1}) + \mathrm{Area}(S^2 \times S^{n-3})$, independent of i. If we now vary i, by the compactness theorem for integral varifolds, a subsequence of $\{M_i\}$ converges to a stationary varifold V. We shall denote this subsequence by $\{M_i\}$ again. Since $Z_{2(i+1)}$ converges to the infinite dihedral group D_∞ (i.e., $SO(2)$ extended by Z_2), we see that V is D_∞-equivariant. V has nonempty singular set, but one can show that if V' denotes V/G', then $V'/SO(2)$ is a union of properly embedded geodesic arcs in the disk $B^3/SO(2)$.

There are exactly two possibilities for $V'/SO(2)$, since the a priori volume bound above gives a length bound on $V'/SO(2)$ in the induced metric on $B^3/SO(2)$. Either $V'/SO(2) = D/SO(2) \cup A/SO(2)$ or $V'/SO(2) = D/SO(2) \cup S^2/SO(2)$, where $S^2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1/(n-1)\}$ in B^3. (See Figure 9.) The latter case arises from the possible compression of Λ'_t to give M'_i. (Note that M_i must be connected by Frankel's theorem [FT]. So if the compression occurs, the disk components containing the curves C_1 and C_2 in B^3 converge to varifolds with zero mass.) In this case,

$V = S^{n-1} \cup (S^2 \times S^{n-3})$. It is straightforward to check that $S^2 \times S^{n-3}$ has smaller area than $S^1 \times S^{n-2}$ in S^n, provided $n \geq 5$; hence the case $V = S^{n-1} \cup (S^1 \times S^{n-2})$ cannot occcur. For $n = 4$, the compression yields a sequence of surfaces converging to $D \cup S^2$ in B^3 and each M_i is diffeomorphic to $\natural_{2i} S^1 \times S^{n-2}$, once regularity has been confirmed. If no compression occurs when $n = 4$, the M_i is diffeomorphic to $\natural_{(2i+2)} S^1 \times S^{n-2}$. It is not clear whether either case can be excluded.

There is one unusual feature in controlling the topology of the critical surfaces M_i. One notes that since ∂B^3 is the orbit space of the singular set B' of the G'-action, any region of M'_i which touches ∂B^3 contributes zero mass to M_i and is therefore invisible to the minimum/maximum construction. We must be careful therefore to preclude the possibility that filligrees of the surface M'_i can push up against ∂B^3 (see Figure 10), adding more connected summands of copies ot $S^1 \times S^{n-2}$. Our methods do not directly preclude filligrees for small numbers i, but the filligrees cannot occur for i sufficiently large by Allard regularity [AW]. Since a varifold V to which the sequence $\{M_i\}$ converges is regular with multiplicity one near B' (whether or not compression occurs), the manifolds M_i converge smoothly to V near B' in S^n.

§6 Complete minimal surfaces in $\mathbf{R}^3$ of genus ≥ 1 and three ends.

THEOREM 7. (Hoffman and Meeks) *For every positive integer i, there exists a complete embedded minimal surface in $\mathbf{R}^3$ having genus i and three ends.*

REMARK. D. Hoffman and W. Meeks [HM1] discovered a complete minimal surface in $\mathbf{R}^3$ having genus one and three ends. Later they discovered similar surfaces of arbitrary genus ≥ 1 [HM2], as well as several different existence proofs for these surfaces [HM3]. Here we give yet another existence proof which we offer because of its simplicity and because it is a nice application of the equivariant minimax construction. It is a pleasure to acknowledge useful conversations with W. Meeks, who explained to us in detail the arguments which are to appear in [HM3].

PROOF BY EQUIVARIANT MINIMAX: We proceed in two steps. First we construct families of bounded minimal surfaces, using essentially the same equivariant minimax construction as in §5. Here all of the surfaces have the same boundary (three

A filligree of M_1
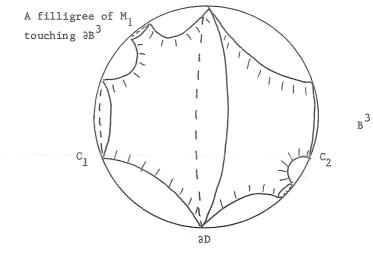touching ∂B^3

C_1

C_2

B^3

∂D

FIGURE 10

circles) and are useful themselves as counterexamples (see remark below). Second we obtain the complete minimal surfaces by an easy blow-up of the bounded surfaces by homothetic transformations.

For any positive real number d, define in $\mathbf{R}^3$ an oriented boundary $B_d = \{(s, x_2, x_3) : x_2^2 + x_3^2 = 1, s \in \{-d, 0, d\}\}$. B_d consists of three parallel circles, and orientation of B_d is chosen so that the middle circle is oriented oppositely to the outer ones. One notes that B_d is D_∞-invariant (notation as in §5).

If d is sufficiently small (which we assume), then there are exactly three D_∞-equivariant (bounded) minimal surfaces with boundary B_d. These are: D_d, which consists of three disks; S_d, which is the union of the area-minimizing catenoid spanning the outer circles and the disk spanning the center circle; and U_d, which is similarly the union of an unstable catenoid and a disk. S_d and U_d have a singular circle where the catenoid and the disk intersect. Orientations are chosen carefully so that $\partial D_d = \partial S_d = \partial U_d = B_d$ as currents. Evidently we have Area$(S_d) \to \pi$ and Area$(U_d) \to$ Area$(D_d) = 3\pi$ as $d \to 0+$.

For the moment we fix d and a positive integer i, and we construct a $Z_{2(i+1)}$-equivariant sweep-out (notation as in §5). Let $\Lambda_{d,t}^i$ be an embedded submanifold of $\mathbf{R}^3$ with boundary B_d which is $Z_{2(i+1)}$-equivariant and has genus i. ($\Lambda_{d,t}^i$ may be constructed exactly as was Λ_t' in §5.) Minimizing $\Lambda_{d,t}^i$ $Z_{2(i+1)}$-equivariantly in its isotopy class, we obtain a $Z_{2(i+1)}$-equivariantly area minimizing surface T_d^i which has genus i and boundary B_d. One checks that Area$(T_d^i) <$ Area(S_d), and that $\Lambda_{d,t}^i$ and T_d^i are isotopic. In our sweep-out, we require that $\Lambda_{d,t}^i \to T_d^i$ as $t \to 0+$.

Just as in §5, as $t \to 1-$ we may use compression to collapse $\Lambda_{d,t}^i$ onto D_d $Z_{2(i+1)}$-equivariantly. It is straightforward to check that this may be done so that Area$(\Lambda_{d,t}^i) <$ Area(U_d) for all $0 \le t \le 1$. (In doing this, it is useful to notice that there is an explicit D_∞-equivariant minimaxing sweep-out Λ_t from S_d to D_d for which the critical surface is exactly U_d. We realize area savings in the $Z_{2(i+1)}$-equivariant case because we can "cut" along the singular circles in the sweep-out while preserving $Z_{2(i+1)}$-equivariance. For example, see the analogous construction in the second drawing in Figure 3.)

We have now constructed a more or less explicit $Z_{2(i+1)}$-equivariant sweep-out $\Lambda_{d,t}^i$. Applying the minimax construction, we obtain a $Z_{2(i+1)}$-equivariant minimal surface M_d^i with $\partial M_d^i = B_d$. Now we list various elementary properties of the surfaces M_d^i.

First, genus$(M_d^i) = 0$ or i. The case genus$(M_d^i) = 0$ is excluded, however, because this could only have occurred through a $Z_{2(i+1)}$-equivariant compression of $\Lambda_{d,t}^i$, in which case we could have $M_d^i = D_d$, a contradiction. Furthermore, similar arguments show that M_d^i is connected, and that $M_d^i \cap P_s = \emptyset$ for $-d < s < d$, where P_s is the plane $\{(x_1, x_2, x_3) : x_1 = s\}$.

Next, one notes that Area$(D_d) \leq$ Area$(M_d^i) \leq$ Area(U_d). Furthermore, since the circles constituting B_d have multiplicity one, it is straightforward to check that M_d^i occurs with multiplicity one (as a varifold). Also, the $Z_{2(i+1)}$-equivariance forces that M_d^i contains the origin 0; in fact, 0 is the only point on M_d^i which intersects the x_1-axis because compression cannot occur. Finally, it follows from the constancy theorem that as $d \to 0+$, M_d^i converges in the $\mathbf{F}$ metric to the disk D_0 (counted with multiplicity three).

We are now able to describe a simple blow-up procedure to obtain a $Z_{2(i+1)}$-equivariant complete embedded minimal surface S^i in $\mathbf{R}^3$ of genus i with three ends. For all numbers d sufficiently small, one uses monotonicity of density ratios to choose a number $0 < r_d < 1$ such that Area$(M_d^i \cap \mathbf{B}(0, r_d)) = (5/2)\pi r_d^2$. Evidently $r_d \to 0$ as $d \to 0+$. We define S_d^i to be the homothetic expansion of M_d^i by the factor r_d^{-1}. We may choose a suitable sequence $\{d_j\}$ decreasing to 0 such that $\lim_{j \to \infty} S_{d_j}^i = S^i$. It is standard that S^i is regular because the surfaces $S_{d_j}^i$ have strong local stability properties and uniformly bounded genus (cf. [PR3]). Since S^i is clearly a complete minimal surface in $\mathbf{R}^3$, it remains only to show that no degeneration has occurred. S^i cannot be the surface which is the $x_2 x_3$-coordinate plane (possibly counted with multiplicity) because Area$(S^i \cap \mathbf{B}(0, 1)) = $ Area$(S_{d_j}^i) = (5/2)\pi$ for all j. Nor can S^i consist of three parallel planes whose total area in $\mathbf{B}(0, 1)$ is $(5/2)\pi$; in that case a neighborhood of any point on the x_1-axis between the planes intersects $S_{d_j}^i$ for j large, violating the monotonicity of density ratios. Finally, S^i cannot be a $Z_{2(i+1)}$-

equivariant catenoid because $0 \in S^i$. No other degeneration is possible because of the $Z_{2(i+1)}$-equivariance of S^i. Thus genus(S^i) = i, which completes the proof.

REMARKS. (1) If we fix d small and vary the genus i, then the minimal surfaces T_d^i and M_d^i are themselves interesting examples. Evidently neither T_d^i nor M_d^i preserves the rotational symmetry of B_d for any positive integer i. (A minimal surface of genus zero analogous to T_d^i and having similar asymmetry has been exhibited in [GH].) Furthermore, there are various *a priori* curvature estimates ([AM], [WB1]) for minimal surfaces having bounded genus and extreme boundary. (By enlarging the center circle slightly, the boundary B_d may be assumed to lie on the boundary of a uniformly convex set.) Since a suitably chosen subsequence of the surfaces T_d^i converges (as varifolds, in the **F** metric) to S_d as the genus $i \to \infty$, these surfaces are counterexamples to various plausible generalizations of these estimates.

(2) The boundary B_d supports an infinite number of noncongruent minimal spanning surfaces. This phenomenon can be generalized substantially to many boundaries with no rotational symmetries and to higher dimensions. See [PJ2].

REFERENCES

[AW]   W. Allard, *On the first variation of a varifold*, Ann. of Math. **95** (1972), 417-491.

[AM]   M. Anderson, *Curvature estimates for minimal surfaces in 3-manifolds*, Ann. Scient. Éc. Norm. Sup. **18** (1985), 89-105.

[BG]   G. E. Bredon, Introduction to compact transformation groups, Academic Press 1972.

[CS]   H. I. Choi and R. Schoen, *The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature*, Invent. Math., **81** (1985), 387-394.

[DM]   J. R. Davis and R. J. Milgram, A survey of the spherical space form problem, Harwood, London 1985.

[EM]   B. Evans and J. Maxwell, *Quaternion actions on S^3*, Am. J. Math. **101** (1979), 1123-1130.

[FT]   T. Frankel, *On the fundamental group of a compact minimal submanifold*, Ann. of Math. **83** (1976), 68-73.

[GR]   C. Gardener and J. H. Rubinstein, *A note on 3-dimensional homogeneous space*, Comp. Math. **39** (1979), 297-299.

[GH]   R. Gulliver and S. Hildebrandt, *Boundary configurations spanning continua of minimal surfaces*, Manuscripta Math. **54** (1986), 323-348.

[HR]   R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255-306.

[HPR]   J. Hass, J. Pitts, and J. Rubinstein, *Triply periodic minimal surfaces in Euclidean space*, in preparation.

[HA]   A. Hatcher, *A proof of the Smale conjecture*, $\text{Diff}(S^3) \simeq O(4)$, Ann. of Math. **117** (1983), 553-607.

[HM1]   D. Hoffman and W. Meeks, *A complete minimal surface in $\mathbf{R}^3$ with genus 1 and 3 ends*, J. Dif. Geo. **21** (1985), 109-127.

[HM2]   D. Hoffman and W. Meeks, *Complete embedded minimal surfaces of finite total curvature*, Bul. of Amer. Math. Soc. **12** (1985), 134-136.

[HM3]   D. Hoffman and W. Meeks, preprint.

[IN]   N. V. Ivanov, *Homotopy of spaces of automorphisms of some 3 dimensional manifolds*, Soviet Math. Dokl. **20** (1979), 47-50.

[KPS]   H. Karcher, U. Pinkall, and I. Sterling, *New minmal surfaces in S^3*, Bonn (1986), preprint.

[LB]   H. B. Lawson, *Complete minimal surfaces in S^3*, Ann. of Math. **92** (1970), 335-374.

[LR]   R. Livesay, *Fixed point free involutions on the 3-sphere*, Ann. of Math. **72** (1960), 603-611.

[McR]   D. McCullough and J. H. Rubinstein, *On the homotopy type of the diffeomorphism spaces of prism manifolds*, in preparation.

[MW1]   W. Meeks, *The geometry and conformal structure of triply periodic minimal surfaces in $\mathbf{R}^3$*, PhD thesis, Cal. 1975.

[MW2] W. Meeks, *The topological uniqueness of minimal surfaces in 3 dimensional Euclidean space*, Topology **20** (1981), 399-410.

[MW3] W. Meeks, Lectures on Plateau's problem, IMPA, Rio de Janiero, 1978.

[MW4] W. Meeks, *Recent progress on the geometry of surfaces in $\mathbf{R}^3$ and on the use of computer graphics as a research tool*, preprint.

[MSY] W. Meeks, L. Simon, and S. T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. **116** (1982), 621-659.

[MJ1] J. Milnor, Introduction to algebraic K-theory, Princeton Univ. Press, Princeton 1971.

[MJ2] J. Milnor, *On the 3-dimensional Brieskorn manifolds $M(p,q,r)$*, Ann. of Math. Studies, Princeton Univ. Press **75** (1975), 175-226.

[MR] R. Myers, *Free involutions on lens spaces*, Topology **20**, (1981) 313-318.

[OP] P. Orlik, *Seifert manifolds*, Springer-Verlag Lecture Notes 291, New York 1972.

[PJ1] J. Pitts, Existence and regularity of minimal surfaces on riemannian manifolds, Princeton University Press, Princeton, New Jersey 1981.

[PJ2] J. Pitts, *Boundaries spanned by an infinite number of minimal surfaces*, in preparation.

[PR1] J. Pitts and J. H. Rubinstein, *Existence of minimal surfaces of bounded topological type in three-manifolds*, Proceedings of the Centre for Mathematical Analysis **10**, Canberra, 1986, 163-176.

[PR2] J. Pitts and J. H. Rubinstein, *Minimal surfaces of bounded topological type in three-manifolds*, preprint.

[PR3] J. Pitts and J. H. Rubinstein, *Equivariant minimax and minimal surfaces in geometric three-manifolds*, preprint.

[RD] D. Rolfson, Knots and links, Publish or Perish, Berkeley 1976.

[RJ1] J. H. Rubinstein, *On 3-manifolds that have finite fundamental group and contain Klein bottles*, Trans. Amer. Math. Soc. **251** (1979), 129-137.

[RJ2]   J. H. Rubinstein, *Free actions of some finite groups on S^3*, Math. Ann. **240** (1979), 165-175.

[RV]   F. Raymond and A. T. Vasquez, *3-manifolds whose universal coverings are Lie groups*, Topology and its Applic. **12** (1981), 161-179.

[SP]   P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), 401-487.

[SS]   L. Simon and R. Smith, *On the existence of embedded minimal 2-spheres in the 3-sphere, endowed with an arbitrary metric*, preprint.

[TC]   C. Thomas, Elliptic structures on 3-manifolds, Cambridge Univ. Press, Cambridge 1986.

[TW]   W. P. Thurston, *The geometry and topology of 3-manifolds*, Notes at Princeton University.

[UK]   K. Uhlenbeck, *Closed minimal surfaces in hyperbolic 3-manifolds*, Seminar on Minimal Submanifolds, Princeton Univ. Press, 1983, 147-168.

[WS]   C. Weber and H. Seifert, *Die beiden Dodekaederräume*, Math. Zeit. **37** (1933), 237-253.

[WB1]   B. White, *Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals*, Invent. Math., to appear.

[WB2]   B. White, *The space of minimal submanifolds of a manifold with varying riemannian metrics*, preprint.

DEPARTMENT OF MATHEMATICS

TEXAS A&M UNIVERSITY

DEPARTMENT OF MATHEMATICS

THE UNIVERSITY OF MELBOURNE