16. METHODS FOR INTEGRAL OPERATORS

In this section we describe some methods of approximating an integral operator of the following kind. Let X denote either $L^2([a,b])$ or $C([a,b])$, and accordingly, let $\tilde{X}$ denote either $L^2([a,b]\times[a,b])$ or $C([a,b]\times[a,b])$; we shall denote by $\| \ \|$ the L^2-norm $\| \ \|_2$ in the first case and the supremum norm $\| \ \|_\infty$ in the second. Let $k \in \tilde{X}$, and consider the <u>Fredholm integral operator</u> $T : X \to X$ <u>with kernel</u> k given by

$$(16.1) \qquad Tx(s) = \int_a^b k(s,t)x(t)dt \ , \ x \in X \ , \ s \in [a,b] \ .$$

It is well known that T is a compact operator and

$$(16.2) \qquad \|T\|_2 \leq \|k\|_2 \ , \quad \|T\|_\infty \leq (b-a)\|k\|_\infty \ .$$

(cf. [L], 17.5(d).) We shall also compare the methods introduced in this section with those related to projections, as described in Section 15.

Degenerate kernel method

A kernel $k \in \tilde{X}$ is said to be <u>degenerate</u> if

$$(16.3) \qquad k(s,t) = \sum_{i=1}^{m} x_i(s)y_i(t) \ , \ s,t \in [a,b] \ ,$$

where x_i and y_i belong to X, $i = 1,\dots,m$. Notice that an integral operator with a degenerate kernel is a bounded operator of finite rank. For the kernel given by (16.3), we have for $x \in X$ and $s \in [a,b]$,

$$Tx(s) = \int_a^b \left[\sum_{i=1}^m x_i(s)y_i(t) \right] x(t)dt$$
$$= \sum_{i=1}^m \left[\int_a^b y_i(t)x(t)dt \right] x_i(s) .$$

Thus, the range of T is contained in the linear span of $\{x_1,\ldots,x_m\}$.

THEOREM 16.1 Let T be given by (16.1) and let (k_n) be a sequence of degenerate kernels such that $\|k_n - k\| \to 0$. Let

(16.4) $$T_n^D x(s) = \int_a^b k_n(s,t)x(t)dt \ , \ x \in X \ , \ s \in [a,b] .$$

Then $T_n^D \xrightarrow{\|\ \|} T$.

Proof We note that $T_n^D - T$ is an integral operator with kernel $k_n - k$. Hence by (16.2),

$$\|T_n^D - T\| \leq \|k_n - k\| \max\{1,b-a\} .$$

Since $\|k_n - k\| \to 0$, we see that $T_n^D \xrightarrow{\|\ \|} T$. //

We now describe some ways of constructing a sequence of degenerate kernels which converges to a given kernel k in $\tilde{X}$.

First, let $X = L^2([a,b])$, and $k \in \tilde{X} = L^2([a,b]\times[a,b])$. Let $u_1,u_2,\ldots$ be an orthonormal basis of X , and consider

$$k_{i,j} = \int_a^b \int_a^b k(s,t)u_j(t)\overline{u_i(s)}\, dtds \ , \quad i,j = 1,2,\ldots .$$

If we let

$$k_n(s,t) = \sum_{i,j\leq n} k_{i,j}u_i(s)\overline{u_j(t)} \ , \quad n = 1,2,\ldots ,$$

then $\|k_n - k\|_2 \to 0$ in $\tilde{X}$. This follows by noting that

$$\int_a^b \int_a^b |k(s,t)|^2 ds dt = \sum_{i,j\geq 1} |k_{i,j}|^2 < \infty ,$$

$$\int_a^b \int_a^b |k(s,t) - k_n(s,t)|^2 ds dt = \sum_{i,j\geq n} |k_{i,j}|^2 .$$

(cf. [L], p.267.) It can be easily seen that in this case

$T_n^D = T_n^G = \pi_n T \pi_n$, where $\pi_n x = \sum_{j=1}^{n} \langle x,u_j \rangle u_j$.

Various other degenerate kernels in $L^2([a,b] \times [a,b])$ are considered in [SN].

Next, let $X = C([a,b])$. We approximate the kernel $k(s,t)$ by interpolation in the second variable. Let

$$a = t_0^{(n)} \leq t_1^{(n)} < \ldots < t_n^{(n)} \leq t_{n+1}^{(n)} = b ,$$

and $u_i^{(n)} \in C([a,b])$ be such that $u_i^{(n)}(t_j^n) = \delta_{i,j}$. Consider

$$k_n(s,t) = \sum_{i=1}^{n} k(s,t_i^{(n)}) u_i^{(n)}(t) .$$

Assume that $u_i^{(n)}(t) \geq 0$ and $\sum_{i=1}^{n} u_i^{(n)}(t) = 1$ for all $t \in [a,b]$.

We recall that this is the case for the piecewise linear hat functions.

Now for $s,t \in [a.b]$,

$$|k(s,t) - k_n(s,t)| = \left| \sum_{i=1}^{n} \left[k(s,t) - k(s,t_i^{(n)})\right] u_i^{(n)}(t) \right|$$

$$\leq \max\{ |k(s,t) - k(s,t_i^{(n)})| : s,t \in [a,b]\} .$$

The uniform continuity of k shows that $\|k - k_n\|_\infty \to 0$ if $h_n \to 0$, where h_n is the mesh of the partition. If the kernel $k(s,t)$ is approximated by interpolation in the first variable, then it can be noticed that $T_n^D = T_n^P = \pi_n T$, where π_n is the interpolatory projection..

Another way to approximate a continuous kernel k is to consider the Bernstein polynomials

$$k_n(s,t) = \sum_{i,j=0}^{n} k(\tfrac{i}{n},\tfrac{j}{n}) \binom{n}{i} \binom{n}{j} s^i(1-s)^{n-i} t^j (1-t)^{n-j} ,$$

where for simplicity we have taken $a = 0$ and $b = 1$. Then $\|k_n - k\|_\infty \to 0$ by a proof analogous to Korovkin's classical theorem. (See [L], 3.18 and 3.19.)

In case the function k is real analytic, and has a uniformly and absolutely convergent double Taylor series expansion

$$k(s,t) = \sum_{i,j=0}^{\infty} k_{i,j}(s-s_0)^i(t-t_0)^j$$

for $s,t \in [a,b]$, then we can consider the truncations

$$k_n(s,t) = \sum_{i,j=0}^{n} k_{i,j}(s-s_0)^i(t-t_0)^j$$

so that $\|k_n - k\|_\infty \to 0$. A simple example is given by $k(s,t) = e^{st}$. Often $k(s,t)$ has an expansion of the type $\sum_{i=0}^{\infty} s^i y_i(t)$ or $\sum_{i=0}^{\infty} x_i(s)t^i$, where x_i and y_i are polynomials.

We remark that the degenerate kernel method can be employed in conjunction with methods related to projections, thus giving rise to additional approximations: If $\pi_n \xrightarrow{p} I$, and $T_n = \pi_n T_n^D$, then it is easy to see that $T_n \xrightarrow{\|\ \|} T$ (cf. Problem 13.4), while if either $T_n = T_n^D \pi_n$ or $T_n = \pi_n T_n^D \pi_n$, then $T_n \xrightarrow{cc} T$ by (13.4)

Quadrature methods

First we briefly discuss approximate quadrature rules. Let $X = C([a,b])$ and consider the *nodes*

$$a = t_0^{(n)} \le t_1^{(n)} \le \dots \le t_{n-1}^{(n)} \le t_n^{(n)} \le t_{n+1}^{(n)} = b$$

and correspondingly, the *weights* $w_i^{(n)}$, $i = 1,\dots,n$. We assume that $t_i^{(n)} = t_j^{(n)}$ implies $w_i^{(n)} = w_j^{(n)}$. For $n = 1,2,\dots$ consider the <u>quadrature formula</u>

$$(16.5) \qquad f_n(x) = \sum_{j=1}^{n} w_j^{(n)} x(t_j^{(n)}) \ , \ x \in X \ ;$$

$f_n(x)$ is supposed to approximate $\int_a^b x(t)dt$. A famous theorem of Polya says that $f_n(x) \to \int_a^b x(t)dt$ for every $x \in C([a,b])$ if and only if

(i) $\sup\{\|f_n\| : n = 1,2,\dots\} = \sup\{ \sum_{j=1}^{n} |w_j^{(n)}| : n = 1,2,\dots\} < \infty$

and

(ii) $f_n(y) \to \int_a^b y(t)dt$ for every y in a dense subset of X .

For example, one can consider the dense subset $\text{span}\{1,t,t^2,\dots\}$ of $C([a,b])$ in the condition (ii) above. In case the weights $w_j^{(n)}$ are all nonnegative, then

$$\sum_{j=1}^{n} |w_j^{(n)}| = \sum_{j=1}^{n} w_j^{(n)} = f_n(1) \ .$$

Hence it follows that the conditions (i) and (ii) can be replaced by the condition (cf. [L], 9.5.)

' $f_n(t^j) \to (b^{j+1}-a^{j+1}) / (j+1)$ for $j = 0,1,2,\dots$.'

We now describe two methods of approximating an integral operator T given by (16.1), which are based on an approximate quadrature rule. Let a quadrature formula be given by (16.5). The most natural approximating operator

$$(16.6) \qquad T_n^N x(s) = \sum_{j=1}^{n} w_j^{(n)} k(s,t_j^{(n)}) x(t_j^{(n)}) \ , \ x \in C([a,b]) \ , \ s \in [a,b]$$

gives the <u>Nyström</u> <u>method</u> for approximating T. Note that if we let $k_j(s) = k(s,t_j^{(n)})$, $s \in [a,b]$, then the range of T_n^N is contained in the linear span of $\{k_1,\ldots,k_n\}$. Thus, T_n^N is of finite rank.

Let $\pi_n : C([a,b]) \to C([a,b])$ be a (bounded) projection for $n = 1,2,\ldots$. Then the operator

$$T_n^F = \pi_n T_n^N \tag{16.7}$$

gives the <u>Fredholm</u> <u>method</u> for approximating T. Since T_n^N is of finite rank, so is T_n^F. If $\pi_n x = \sum_{i=1}^{n} \langle x,e_i^*\rangle e_i$, with $e_i \in X$, $e_i^* \in X^*$ such that $\langle e_j,e_i^*\rangle = \delta_{i,j}$, then

$$\begin{aligned} T_n^F x &= \sum_{i=1}^{n} \langle T_n^N x,e_i^*\rangle e_i \\ &= \sum_{i=1}^{n} \left[\sum_{j=1}^{n} w_j^{(n)} x(t_i^{(n)}) \langle k(\cdot,t_j^{(n)}),e_i^*\rangle\right] e_i \,. \end{aligned}$$

THEOREM 16.2 (Anselone) Let $f_n(x) \to \int_a^b x(t)dt$ for every $x \in C([a,b])$. Then $T_n^N \xrightarrow{cc} T$.

If (π_n) is a sequence of projections such that $\pi_n \xrightarrow{p} I$, then $T_n^F = \pi_n T_n^N \xrightarrow{cc} T$.

Proof Let $x \in C([a,b])$. For fixed $s \in [a,b]$, let

$$y_s(t) = k(s,t)x(t)\,, \quad a \le t \le b\,.$$

Then

$$\begin{aligned} f_n(y_s) &= \sum_{j=1}^{n} w_j^{(n)} y_s(t_j^{(n)}) \\ &= \sum_{j=1}^{n} w_j^{(n)} k(s,t_j^{(n)}) x(t_j^{(n)}) \\ &= T_n^N x(s)\,. \end{aligned}$$

Since $f_n(y_s) \to \int_a^b y_s(t)dt = \int_a^b k(s,t)x(t)dt$, we see that for each fixed x ,

$$T_n^N x(s) \to Tx(s) .$$

We must show that this convergence is uniform for $s \in [a,b]$ to conclude $T_n^N \xrightarrow{p} T$. For this purpose, consider the set

$$Y = \{y_s : s \in [a,b]\} \subset C([a,b]) .$$

Y is uniformly bounded, since

$$\|y_s\|_\infty \leq \|k\|_\infty \|x\|_\infty \text{ for all } s \in [a,b] .$$

Also, for t_1 and t_2 in $[a,b]$, we have

$$\begin{aligned} |y_s(t_1) - y_s(t_2)| &\leq |k(s,t_1)x(t_1) - k(s,t_2)x(t_1)| \\ &\quad + |k(s,t_2)x(t_1) - k(s,t_2)x(t_2)| \\ &\leq \|x\|_\infty \sup_{s\in[a,b]} |k(s,t_1) - k(s,t_2)| \\ &\quad + |x(t_1) - x(t_2)| \sup_{s\in[a,b]} |k(s,t)| . \end{aligned}$$

By the uniform continuity of k and x , we see that for every $\epsilon > 0$, there is $\delta > 0$ such that $|t_1-t_2| < \delta$ implies $|y_s(t_1) - y_s(t_2)| < \epsilon$ for all $s \in [a,b]$. This shows that the set Y equicontinuous. Now by Ascoli's theorem ([L], 3.17), E is totally bounded. Hence the pointwise convergence of the continuous linear functionals f_n is uniform on E . Thus, $\|T_n^N x - Tx\| \to 0$ for every $x \in C([a,b])$.

To show $T_n^N \xrightarrow{cc} T$, it is enough to prove that the set

$$E = \bigcup_{n=1}^{\infty} \{T_n^N x : \|x\|_\infty \leq 1\}$$

is totally bounded, since T is compact. The set E is uniformly bounded since $\|T_n^N\| \leq \alpha < \infty$ by the uniform boundedness principle. Also, for s_1 and s_2 in $[a,b]$,

$$|T_n^N x(s_1) - T_n^N x(s_2)| \leq \sum_{j=1}^{n} |w_j^{(n)}| \; |k(s_1, t_j^{(n)}) - k(s_2, t_j^{(n)})| \; |x(t_j^{(n)})|$$

$$\leq \max_{t \in [a,b]} |k(s_1,t) - k(s_2,t)| \; \|x\|_\infty \sum_{j=1}^{n} |w_j^{(n)}| \,.$$

But by Polya's theorem, $\sum_{j=1}^{n} |w_j^{(n)}| \leq \beta < \infty$; also, k is uniformly continuous. Hence the set E is equicontinuous. Again, by Ascoli's theorem we see that E is totally bounded. This completes the proof of $T_n^N \xrightarrow{cc} T$.

Let, now, $\pi_n \xrightarrow{p} I$, in addition. Then letting $A_n = \pi_n$, $A = I$, $B_n = T_n^N$ and $B = T$ in (13.4),

$$T_n^F = \pi_n T_n^N = A_n B_n \xrightarrow{cc} AB = T \,. \qquad //$$

We now prove a negative result regarding the norm convergence of the Nyström approximation (T_n^N) to T .

PROPOSITION 16.3 $2\|T\| \leq \underline{\lim}_{n\to\infty} \|T_n^N - T\|$.

Proof Let $\epsilon > 0$. Then there exist $x \in C([a,b])$ and $s \in [a,b]$ such that $\|x\| = 1$ and

$$|Tx(s)| > \|T\| - \epsilon \,.$$

As $T_n^N x(s) \to Tx(s)$, there is n_0 such that for all $n \geq n_0$, we have

$$|T_n^N x(s) - Tx(s)| < \epsilon \,.$$

Now, by altering the function x only on small nighbourhoods of $t_1^{(n)},\dots,t_n^{(n)}$, we can construct, for each $n \geq n_0$, a function x_n such that $\|x_n\| = 1$,

$$x_n(t_j^{(n)}) = -x(t_j^{(n)}) \; , \; j = 1,\dots,n \; ,$$
$$|Tx_n(s) - Tx(s)| < \epsilon \; .$$

Then $T_n^N x_n(s) = -T_n^N x(s)$. Hence

$$\begin{aligned} |(T-T_n^N)x_n(s)| &= |Tx_n(s) + T_n^N x(s)| \\ &\geq 2|Tx(s)| - 2\epsilon \; . \\ &\geq 2\|T\| - 4\epsilon \; . \end{aligned}$$

Since $\|x_n\| = 1$, we see that

$$\|(T-T_n^N)\| \geq \|(T-T_n^N)x_n\| \geq 2\|T\| - 4\epsilon \; .$$

Thus, $\underline{\lim}_{n\to\infty} \|(T-T_n^N)\| \geq 2\|T\| - 4\epsilon$. But as $\epsilon > 0$ is arbitrary, the proof is complete. //

The above result shows that the Nyström approximation (T_n^N) does not converge to T in the norm except in the trivial case $T = 0$. It was for this reason, that the theory of collectively compact approximation was developed (cf.[AN]), and has proved to be very useful. In case the kernel k of the integral operator T is smooth and f_n is a repeated quadrature formula, then we do have $|||T_n^N - T||| \to 0$, where the underlying space $C^1([a,b])$ is equipped with the norm

$$|||x||| = \|x\|_\infty + \|x'\|_\infty \; .$$

(Cf. [B], p.109 and 112.)

On the other hand, if the kernel k is discontinuous but satisfies some regularity conditions, then by considering the underlying space to be the set of all Riemann-integrable functions, a partial extension of

Theorem 16.2 regarding the convergence of the Nyström approximation can be obtained. (See [AN], Theorem 2.13.)

It can be easily observed that Proposition 16.3 (along with its proof which is due to Anselone) holds for any sequence (T_n) in place of the Nyström approximation (T_n^N), provided $T_n x(s) \to Tx(s)$ for every $s \in [a,b]$ and $x \in C([a,b])$, and $T_n x = T_n y$ whenever $x(t_j^{(n)}) = y(t_j^{(n)})$, $j = 1,\ldots,n$, $n = 1,2,\ldots$. In particular, it holds for $T_n = T_n^F$, and if π_n is an interpolatory projection then for $T_n = T_n^S = T\pi_n$ as well as for $T_n = T_n^G = \pi_n T \pi_n$.

We now give examples of some well known quadrature formulae which can be used while employing the Nyström or the Fredholm approximations. Many of these arise from *interpolatory projections*. As in Section 15, consider the *nodes*

$$a = t_0^{(n)} \le t_1^{(n)} < \ldots < t_n^{(n)} \le t_{n+1}^{(n)} = b ,$$

and let $u_i^{(n)} \in C([a,b])$ be such that $u_i^{(n)}(t_j^{(n)}) = \delta_{i,j}$, $1 \le i,j \le n$. Using the interpolatory projection

$$\pi_n x = \sum_{i=1}^{n} x(t_i^{(n)}) u_i^{(n)} , \quad x \in C([a,b]) ,$$

we define the quadrature formula

$$f_n(x) = \int_a^b \pi_n x(t)dt = \sum_{i=1}^{n} \left[\int_a^b u_i^{(n)}(t)dt \right] x(t_i^{(n)}) , \tag{16.8}$$

so that the weights are

$$w_i^{(n)} = \int_a^b u_i^{(n)}(t)dt .$$

Note that for $i = 1,\dots,n$,

$$f_n(u_i^{(n)}) = \int_a^b u_i^{(n)}(t)dt \ .$$

Thus, the quadrature formula f_n is exact on the linear span of $\{u_1^{(n)},\dots,u_n^{(n)}\}$. Also, for $x \in C([a,b])$ and $s \in [a,b]$, we have

$$\begin{aligned} T_n^N(\pi_n x)(s) &= \sum_{j=1}^{n} w_j^{(n)} k(s,t_j^{(n)})(\pi_n x)(t_j^{(n)}) \\ &= \sum_{j=1}^{n} w_j^{(n)} k(s,t_j^{(n)}) x(t_j^{(n)}) \\ &= T_n^N x(s) \ . \end{aligned}$$

Hence

$$T_n^N \pi_n = T_n^N \ , \tag{16.9}$$

when π_n is an interpolatory projection and the quadrature formula f_n is induced by π_n . If we employ an interpolatory projection $\tilde{\pi}_n$ with nodes at $\tilde{t}_i^{(n)}$, $i = 1,\dots,n$, while considering the Fredholm approximation $T_n^F = \tilde{\pi}_n T_n^N$, then for $x \in C([a,b])$ and $s \in [a,b]$,

$$\begin{aligned} T_n^F x(s) &= \tilde{\pi}_n T_n^N x(s) \\ &= \sum_{i=1}^{n} (T_n^N x)(\tilde{t}_i^{(n)}) u_i^{(n)}(s) \\ &= \sum_{i=1}^{n} \left[\sum_{j=1}^{n} w_j^{(n)} k(\tilde{t}_i^{(n)},t_j^{(n)}) x(t_j^{(n)}) \right] u_i^{(n)}(s) \end{aligned} \tag{16.10}$$

Observe that in the Nyström approximation, the kernel $k(s,t)$ is discretized only in the second variable, while in the Fredholm approximation it is discretized in both the variables.

If $\pi_n \xrightarrow{p} I$, then clearly $f_n(x) = \int_a^b \pi_n x(t)dt \to \int_a^b x(t)dt$, i.e., the quadrature formula is convergent. Thus, Theorem 16.2 becomes applicable. But the quadrature formula f_n may be convergent although (π_n) is not a pointwise approximation of I .

Various interpolatory projections discussed in Section 15 yield interesting quadrature formulae.

(i) <u>Lagrange interpolation</u>. In this case, $\text{span}\{u_1^{(n)},\ldots,u_n^{(n)}\}$ is the set of all polynomials of degree at most $n-1$, so that the quadrature formula f_n is exact on $1,t,\ldots,t^{n-1}$. Hence by Polya's theorem, we see that $f_n(x) \to \int_a^b x(t)dt$ for every $x \in C([a,b])$ if and only if $\sum_{i=1}^{n} |\int_a^b u_i^{(n)}(t)dt| \leq \alpha < \infty$. If the weights $w_i^{(n)} = \int_a^b u_i^{(n)}(t)dt$ are nonnegative, then this condition is automatically satisfied.

If $a = -1$, $b = 1$, and the nodes $t_i^{(n)}$ are the Gauss *points* (i.e., the roots of the Legendre polynomial of degree $n-1$), or the *Tchebychev points* (i.e., the roots of the Tchebychev polynomial of degree $n-1$ (of the first, or of the second kind), then the weights are positive, and the corresponding quadrature formulae are convergent. In the case of Gauss points, the quadrature formula f_n is, in fact, exact on all polynomials of degree at most $2n-1$. (See [D], 2.5.5 and 2.7.)

If the nodes are equidistant, i.e., $t_i^{(n)} = a + (i-1)(b-a)/(n-1)$, $i = 1,\ldots,n$, then the corresponding quadrature formula is known as the <u>Newton-Cotes rule</u>. The weights are of mixed signs and it was shown by Polya that for some $x \in C([a,b])$, this rule does not converge to $\int_a^b x(t)dt$.

(ii) <u>Piecewise linear interpolation.</u> If the mesh $h_n = \max\{t_i^{(n)}-t_{i-1}^{(n)} : i = 1,\ldots,n+1\} \to 0$ as $n \to \infty$, then we have seen in Section 15 that $\pi_n \xrightarrow{p} I$, and consequently the corresponding quadrature formula is convergent. In this case, f_n is exact on the linear span of the hat functions $e_1^{(n)},\ldots,e_n^{(n)}$. In particular, this is so for any $x(t) = ct + d$, where c and d are constants, since then $x(t) = \sum_{i=1}^{n} (ct_i^{(n)} + d)e_i^{(n)}(t)$. The weight $w_i^{(n)} = \int_a^b e_i^{(n)}(t)dt$ can easily be calculated by considering the area under the graph of the hat function $e_i^{(n)}$. In fact, we have

$$w_i^{(n)} = \begin{cases} t_1^{(n)} - a + (t_2^{(n)} - t_1^{(n)})/2 \ , & \text{if} \quad i = 1 \\ (t_{j+1}^{(n)} - t_{j-1}^{(n)})/2 & , \text{if} \quad i = 2,\ldots,n-1 \\ b - t_n^{(n)} + (t_n^{(n)} - t_{n-1}^{(n)})/2 \ , & \text{if} \quad i = n \ . \end{cases}$$

For various choices of the nodes considered in Section 15, we obtain the following weights and the corresponding quadrature formulae:

1. $t_i^{(n)} = i/n$, $i = 1,\ldots,n$: $w_1^{(n)} = 3/2n$, $w_i^{(n)} = 1/n$ for $i = 2,\ldots,n-1$, and $w_n^{(n)} = 1/2n$, so that

$$f_n(x) = \frac{1}{n}\left[\frac{3x(1/n) + x(1)}{2} + \sum_{i=2}^{n-1} x(\tfrac{i}{n})\right] .$$

2. $t_i^{(n)} = (2i-1)/2n$, $i = 1,\ldots,n$: $w_i^{(n)} = 1/n$ for i , and we have the <u>compound mid-point rule</u>

$$f_n(x) = \frac{1}{n} \sum_{i=1}^{n} x(\tfrac{2i-1}{2n}) .$$

3. $t_i^{(n)} = (i-1)/(n-1)$, $i = 1,\ldots,n$: $w_1^{(n)} = 1/2(n-1) = w_n^{(n)}$, and $w_i^{(n)} = 1/(n-1)$ for $i = 2,\ldots,n-1$; this gives the <u>compound trapezium rule</u>

$$f_n(x) = \frac{1}{(n-1)}\left[\frac{x(0) + x(1)}{2} + \sum_{i=2}^{n-1} x(\tfrac{i-1}{n-1})\right] .$$

4.

$$t_i^{(n)} = \begin{cases} (i+r_1)/n & , \quad \text{if} \quad i = 1,3,\dots,n-1 \\ (i-1+r_2)/n & , \quad \text{if} \quad i = 2,4,\dots,n \end{cases} ,$$

where n is even, and $-1 < r_1 < r_2 < 1$. Then $w_1^{(n)} = \frac{1}{n} + \frac{r_1+r_2}{2n}$, $w_i^{(n)} = \frac{1}{n}$ for $i = 2,\dots,n-1$, and $w_n^{(n)} = \frac{1}{n} - \frac{r_1+r_2}{2n}$. In case $r_1 + r_2 = 0$, as is the case for the compound Gauss two point rule $(r_1 = -1/\sqrt{3}$, $r_2 = 1/\sqrt{3})$ and the compound Tchebychev two point rule $(r_1 = -1/\sqrt{2}$, $r_2 = 1/\sqrt{2})$, we have,

$$f_n(x) = \frac{1}{n}\left[\sum_{\substack{i=1 \\ i \text{ odd}}}^{n} x\left[\frac{i+r_1}{n}\right] + \sum_{\substack{i=2 \\ i \text{ even}}}^{n} x\left[\frac{i-1+r_2}{n}\right]\right] .$$

There are several other convergent quadrature formulae such as the <u>compound Simpson rule</u> : n odd, $n \geq 3$; $t_i^{(n)} = \frac{i-1}{n-1}$, $i = 1,\dots,n$, the weights being

$$w_i^{(n)} = \begin{cases} 1/3(n-1) , & \text{if} \quad i = i,n \\ 4/3(n-1) , & \text{if} \quad i = 2,4,\dots,n-1 \\ 2/3(n-1) , & \text{if} \quad i = 3,5,\dots,n-2 , \end{cases}$$

so that

$$f_n(x) = \frac{1}{3(n-1)}\left[x(0) + x(1) + 4 \sum_{\substack{i=1 \\ i \text{ odd}}}^{n-2} x(\tfrac{i}{n-1}) + 2 \sum_{\substack{i=2 \\ i \text{ even}}}^{n-3} x(\tfrac{i}{n-1})\right] .$$

Then $f_n(x) \to \int_0^1 x(t)dt$ for every $x \in C([a,b])$. (See Problem 15.4 with $s_i^{(n)} = (t_{i-1}^{(n)} + t_i^{(n)})/2$.)

We conclude this section by comparing methods related to projections discussed in Section 15 with methods introduced in the present section. Let $X = C([a,b])$ and (π_n) be a sequence of interpolatory projections:

$$\pi_n x = \sum_{i=1}^{n} x(t_i^{(n)})u_i^{(n)} \quad , \; x \in X \; .$$

Let the quadrature formula f_n be induced by π_n . Then we have

$$T_n^N = T_n^N \pi_n \; , \; T_n^S = T\pi_n \; ;$$

$$T_n^F = \pi_n T_n^N \pi_n \; , \quad T_n^G = \pi_n T \pi_n \; .$$

THEOREM 16.4 Let $\pi_n \xrightarrow{p} I$, and assume that the functions $u_1^{(n)},\dots,u_n^{(n)}$ satisfy

$$\sup\{|t-t_j^{(n)}| \; : \; u_j^{(n)}(t) \neq 0 \; , \quad j = 1,\dots,n\} \to 0 \; .$$

Then $\|T_n^N - T_n^S\| \to 0$ and $\|T_n^F - T_n^G\| \to 0$.

Proof For $x \in C([a,b])$, we have

$$\begin{aligned} T_n^N x(s) &= \sum_{j=1}^{n} w_j^{(n)} k(s,t_j^{(n)})x(t_j^{(n)}) \\ &= \sum_{j=1}^{n} \left[\int_a^b u_j^{(n)}(t)dt \right] k(s,t_j^{(n)})x(t_j^{(n)}) \; , \end{aligned}$$

and

$$\begin{aligned} T_n^S x(s) &= \int_a^b k(s,t)\pi_n x(t)dt \\ &= \sum_{j=1}^{n} \left[\int_a^b k(s,t)u_j^{(n)}(t)dt \right] x(t_j^{(n)}) \; . \end{aligned}$$

Let $E_{n,j} = \{t \in [a,b] \; : \; u_j^{(n)}(t) \neq 0\}$, and

$$\alpha_n(s) = \sup\{|k(s,t_j^{(n)}) - k(s,t)| \; : \; t \in E_{n,j} \; , \; j = 1,\dots,n\}$$

Then for $\|x\|_\infty \le 1$, we have

$$|T_n^N x(s) - T_n^S x(s)| \leq \sum_{j=1}^{n} \int_{E_{n,j}} |k(s,t_j^{(n)}) - k(s,t)| \; |u_j^{(n)}(t)|dt$$
$$\leq \alpha_n(s) \sum_{j=1}^{n} \int_a^b |u_j^{(n)}(t)|dt \; .$$

Let $\epsilon > 0$, and find $\delta > 0$ such that the conditions $s \in [a,b]$ and $|t_1-t_2| < \delta$ imply $|k(s,t_1) - k(s,t_2)| < \delta$. By our assumption on the functions $u_j^{(n)}$, $j = 1,\ldots,n$, we can choose n_0 such that for all $n \geq n_0$, we have $\sup\{|t-t_j^{(n)}| : u_j^{(n)}(t) \neq 0 , \; j = 1,\ldots,n\} < \delta$. Then $\alpha_n(s) \leq \epsilon$ for all $n \geq n_0$ and $s \in [a,b]$. Also, by (15.6)

$$\sum_{j=1}^{n} \int_a^b |u_j^{(n)}(t)|dt = \|\pi_n\| \leq \alpha < \infty ,$$

since $\pi_n \xrightarrow{p} I$. Hence for all $n \geq n_0$ we have

$$\|T_n^N - T_n^S\| \leq \epsilon\alpha \; .$$

Thus, $\|T_n^N - T_n^S\| \to 0$. Also,

$$\|T_n^F - T_n^G\| = \|\pi_n(T_n^N - T_n^S)\| \leq \alpha\|T_n^N - T_n^S\| \; .$$

Hence $\|T_n^F - T_n^G\| \to 0$, as well. //

Note that the hypothesis of the above theorem is satisfied if $u_1^{(n)},\ldots,u_n^{(n)}$ are the piecewise linear hat functions, and the mesh of the partition tends to zero.

To sum up, we list several ways of approximating the integral operator

$$Tx(s) = \int_a^b k(s,t)x(t)dt \; , \; x \in C([a,b]) \; , \; s \in [a,b] \; ,$$

by considering the nodes $a \leq t_1^{(n)} < \ldots < t_n^{(n)} \leq b$, and the functions $u_i^{(n)} \in C([a,b])$ such that $u_i^{(n)}(t_j^{(n)}) = \delta_{i,j}$.

(16.11)

$$T_n^P x(s) = \sum_{i=1}^{n} \left[\int_a^b k(t_i^{(n)}, t)x(t)dt \right] u_i^{(n)}(s)$$

$$T_n^S x(s) = \sum_{j=1}^{n} x(t_j^{(n)}) \int_a^b k(s,t)u_j^{(n)}(t)dt$$

$$T_n^G x(s) = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} x(t_j^{(n)}) \int_a^b k(t_i^{(n)}, t)u_j^{(n)}(t)dt \right] u_i^{(n)}(s)$$

$$T_n^D x(s) = \sum_{j=1}^{n} \left[\int_a^b x(t)u_j^{(n)}(t)dt \right] k(s, t_j^{(n)})$$

$$T_n^N x(s) = \sum_{j=1}^{n} \left[x(t_j^{(n)}) \int_a^b u_i^{(n)}(t)dt \right] k(s, t_j^{(n)})$$

$$T_n^F x(s) = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} x(t_j^{(n)})k(t_i^{(n)}, t_j^{(n)}) \int_a^b u_j^{(n)}(t)dt \right] u_i^{(n)}(s) .$$

Problems

16.1 Let T be an integral operator with a degenerate kernel given by (16.3), and assume that $x_1,\ldots,x_n$ are linearly independent in X. Then the operator $T|_{\text{span}\{x_1,\ldots x_n\}}$ is represented by the matrix $\left[\int_a^b x_j(t)y_i(t)dt \right]$, $i,j = 1,\ldots,n$, with respect to the basis $x_1,\ldots,x_n$. The nonzero eigenvalues of T are obtained by solving this matrix eigenvalue problem.

16.2 Let T be a Fredholm integral operator on $C([a,b])$ with a continuous kernel $k(s,t)$. Let $a \le t_1^{(n)} < \ldots < t_n^{(n)} \le b$, and $u_i^{(n)} \in C([a,b])$ be such that $u_i^{(n)}(t_j^{(n)}) = \delta_{i,j}$. For $x \in C([a,b])$ and $s \in [a,b]$, let

$$T_n x(s) = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \left[\int_a^b x(t) u_j^{(n)}(t) dt \right] k(t_i^{(n)}, t_j^{(n)}) \right] u_i^{(n)}(s)$$

$$\tilde{T}_n x(s) = \sum_{j=1}^{n} \left[\sum_{i=1}^{n} \left[\int_a^b u_i^{(n)}(t) u_j^{(n)}(t) dt \right] x(t_i^{(n)}) \right] k(s, t_j^{(n)}) ,$$

$$T_n^{\#} x(s) = \sum_{i=1}^{n} \left[\sum_{m=1}^{n} x(t_m^{(n)}) \sum_{j=1}^{n} \left[\int_a^b u_m^{(n)}(t) u_j^{(n)}(t) dt \right] k(t_i^{(n)}, t_j^{(n)}) \right] u_i^{(n)}(s) .$$

Then $T_n \xrightarrow{\|\ \|} T$, $\tilde{T}_n \xrightarrow{cc} T$ and $T_n^{\#} \xrightarrow{cc} T$ if the mesh h_n of the partition tends to zero.

16.3 Consider the piecewise constant interpolatory projection π_n given in Problem 15.3. The quadrature formula induced by π_n is

$$f_n(x) = \sum_{j=1}^{n} (t_j^{(n)} - t_{j-1}^{(n)}) x(s_j^{(n)}) ,$$

where $s_j^{(n)} \in (t_{j-1}^{(n)}, t_j^{(n)}]$, $j = 1,\dots,n$. The Riemann sum $f_n(x)$ gives a rectangular rule and converges to $\int_a^b x(t)dt$ for every Riemann integrable function x on $[a,b]$.

16.4 If we approximate the integrals appearing in $T_n^S x(s)$ and $T_n^D x(s)$ of (16.11) by the quadrature formula induced by π_n , then we obtain $T_n^N x(s)$. If we do this for $T_n^P x(s)$ and $T_n^G x(s)$, we obtain $T_n^F x(s)$.