19. NUMERICAL EXAMPLES

In this section we report the results of our numerical calculations and illustrate the use of the algorithms given in Section 17.

We consider the space $X = C([a,b])$ of all complex-valued continuous functions on the interval $[a,b]$. Given a positive integer M, let $t_1^{(M)},\dots,t_M^{(M)}$ be points in $[0,1]$ and let

$$\pi_M x = \sum_{i=1}^{M} \langle x, f_i^* \rangle f_i \ , \tag{19.1}$$

where $\langle x, f_i^* \rangle = x(t_i^{(M)})$, and $f_i \in C([0,1])$ is such that $f_i(t_j^{(M)}) = \delta_{i,j}$, $i,j = 1,\dots,M$. An element x of $C([a,b])$ is discretized by $\pi_M x$.

Let T be a Fredholm integral operator on $C([a,b])$ given by

$$Tx(s) = \int_a^b k(s,t)x(t)dt \ , \quad s \in [a,b] \ , \quad x \in C([0,1]) \ ,$$

where k is a continuous complex-valued function on $[a,b] \times [a,b]$. Note that T is a compact operator on $C([0,1])$.

Given a convergent quadrature formula (cf.(16.5))

$$\sum_{j=1}^{M} w_j^{(M)} x(t_j^{(M)}) \ , \quad x \in C([a,b]) \ ,$$

with nodes at $t_j^{(M)}$, $j = 1,\dots,M$, we replace the operator T by its Nyström approximation

$$\tilde{T}x(s) = \sum_{j=1}^{M} w_j^{(M)} k(s, t_j^{(M)}) x(t_j^{(M)}) \ , \quad s \in [a,b] \ , \quad x \in C([a,b]) \ , \tag{19.2}$$

which is easier to handle numerically. From now on we do not make any distinction between T and $\tilde{T}$.

A finite rank operator T_0 which is near T and for which the eigenvalue problem can be solved easily is chosen from a sequence (T_n) which approximates T in the norm or in a collectively compact manner. The approximating sequence of operators determines the 'method' of approximation: projection, Sloan, Galerkin (1 and 2), Nyström, or Fredholm (1 and 2). (See Remark 17.12 and Table 17.1.) Let

$$T_0x = \sum_{i=1}^{n} \langle x, x_i^* \rangle x_i \ , \quad x \in C([a,b]) \ .$$

For the projection, Sloan, Galerkin and Fredholm methods we employ the projection

$$\pi_0 x = \sum_{i=1}^{n} x(t_i^{(n)}) e_i \ , \tag{19.3}$$

where $e_i \in C([0,1])$ satisfies $e_i(t_j^{(n)}) = \delta_{i,j}$, $i,j = 1,\ldots,n$.

As we have discussed in Section 18, we need the following matrices to implement the algorithms of Section 17:

$$A = [\langle x_j, x_i^* \rangle]_{n\times n} \ , \quad [AV] = [\langle x_j, f_i^* \rangle]_{M\times n} \ ,$$
$$[TAH] = [\langle Tf_j, x_i^* \rangle]_{n\times M} \ , \quad [T2AH] = [\langle T^2 f_j, x_i^* \rangle]_{n\times M} \ ,$$
$$[TM] = [\langle Tf_j, f_i^* \rangle]_{M\times M} \ , \quad [T2M] = [\langle T^2 f_j, f_i^* \rangle]_{M\times M} \ .$$

To characterize these matrices for each of the methods listed above, we introduce the following auxiliary matrices:

$$[KN] = [w_j^{(n)} k(t_i^{(n)}, t_j^{(n)})]_{n\times n} \ , \quad [KM] = [w_j^{(M)} k(t_i^{(M)}, t_j^{(M)})]_{M\times M} \ ,$$
$$[KH] = [w_j^{(M)} k(t_i^{(n)}, t_j^{(M)})]_{n\times M} \ , \quad [KV] = [w_j^{(n)} k(t_i^{(M)}, t_j^{(n)})]_{M\times n} \ .$$

$$[IV] = [\langle e_j, f_i^* \rangle]_{M\times n} \ ,$$

where $w_j^{(n)} = \int_a^b e_j(t)dt$, $j = 1,\ldots,n$.

In the following table, the entries in the first column refer to those in the first column of Table 17.1.

T_0	A	[AV]	[TAH]	[T2AH]	[TM]	[T2M]
T_0^P	[KH][IV]	[IV]	[KH][KM]	[TAH][KM]	[KM]	$[KM]^2$
T_0^S	[KH][IV]	[KM][IV]	[KH]	[TAH][KM]	[KM]	$[KM]^2$
T_0^G 1	[KH][IV]	[IV]	[KH][IV][KH]	[TAH][KM]	[KM]	$[KM]^2$
T_0^G 2	[KH][IV]	[IV][KH][IV]	[KH]	[TAH][KM]	[KM]	$[KM]^2$
T_0^N	[KN]	[KV]	[KH]	[TAH][KM]	[KM]	$[KM]^2$
T_0^F 1	[KN]	[IV]	[KN][KH]	[TAH][KM]	[KM]	$[KM]^2$
T_0^F 2	[KN]	[IV][KN]	[KH]	[TAH][KM]	[KM]	$[KM]^2$

Table 19.1

We remark that the matrix [KV] appears only once as the matrix [AV] for the Nyström method. Also, in all cases, the matrix [TM] equals the matrix [KM], and the matrices [T2AH] and [T2M] are obtained by multiplying the matrices [TAH] and [TM], respectively on the right by the matrix [KM]. In case the kernel k is conjugate symmetric (i.e., satisfies $k(t,s) = \overline{k(s,t)}$ for all s and t) as well as all the weights are real and equal, the matrices [KN] and [KM] are self-adjoint; note that the matrix A (= [KH][IV]) may still not be self-adjoint for the projection, Sloan and Galerkin methods.

Employing the matrices given in the above table, we can use the discretization procedure outlined in Section 18 to implement the

algorithms of Section 17. Starting with a nonzero simple eigenvalue λ_0 of A and a corresponding eigenvector $\underset{\sim}{u}$ such that $\underset{\sim}{y}^H\underset{\sim}{u} = 1/\lambda_0$, where $\underset{\sim}{y}$ is an eigenvector of A^H corresponding to $\bar{\lambda}_0$, these algorithms iteratively generate approximations λ_j and $\underset{\sim}{c}_j$ $(j = 0,1,\ldots)$ of a simple nonzero eigenvalue $\lambda^{(M)}$ and a corresponding eigenvector $\underset{\sim}{c}^{(M)}$ of $[TM]$. The elements $\lambda^{(M)}$ and $\underset{\sim}{c}^{(M)}$ are supposed to *represent* eigenelements λ and φ of T. For this reason, we denote $\underset{\sim}{c}_j$, $\underset{\sim}{c}^{(M)}$ and $\lambda^{(M)}$, by φ_j, φ and λ in the numerical tables.

For the numerical experiments reported here, we have taken $[a,b] = [0,1]$. Since we replace the operator T by its Nyström approximation $\tilde{T}$ (given by (19.2)), the choice of the functions f_i, $i = 1,\ldots,M$, used for discretizing a continuous function $x \in C([0,1])$ by $\pi_M x$ (as in (19.1)), is immaterial as long as $f_j(t_i^{(M)}) = \delta_{ij}$, $i,j = 1,\ldots,M$. The functions e_i, $i = 1,\ldots,n$ appearing in the expression for the projection π_0 (employed in the projection, Sloan, Galerkin and Fredholm methods) are the piecewise linear hat functions with nodes at $t_i^{(n)}$. The exact value of e_i at $t \in [0,1]$ can be found from the explicit formulae given in Section 3 for e_i, $i = 1,\ldots,n$ ..

We consider two different kinds of nodes:

(i) Equidistant points: $t_i^{(M)} = \frac{i-1}{M-1}$, $i = 1,\ldots,M$;
$t_i^{(n)} = \frac{i-1}{n-1}$, $i = 1,\ldots,n$;

in this case the compound trapezium rule Q_1 is applied for which

$$w_i^{(M)} = \begin{cases} \frac{1}{2(M-1)}, & \text{if } i = 1 \text{ or } M \\ \frac{1}{M-1}, & \text{otherwise} \end{cases} \qquad w_i^{(n)} = \begin{cases} \frac{1}{2(n-1)}, & \text{if } i = 1 \text{ or } n \\ \frac{1}{n-1}, & \text{otherwise} \end{cases}.$$

(ii) Gauss two points (repeated):

$$t_i^{(M)} = \left[i - \frac{1}{\sqrt{3}}\right]/M \quad \text{and} \quad t_i^{(n)} = \left[i - \frac{1}{\sqrt{3}}\right]/n \ , \quad \text{if } i \text{ is odd}$$

$$t_i^{(M)} = \left[i - 1 + \frac{1}{\sqrt{3}}\right]/M \quad \text{and} \quad t_i^{(n)} = \left[i - 1 + \frac{1}{\sqrt{3}}\right]/n \ , \quad \text{if } i \text{ is even} ;$$

in this case the compound Gauss two point rule Q_2 is applied for which

$$w_i^{(M)} = 1/M \ , \quad i = 1, \ldots, M \ , \quad \text{and} \quad w_i^{(n)} = 1/n \ , \quad i = 1, \ldots, n \ .$$

We have often chosen $M = 100$ and $n = 10$; since $n - 1$ divides $M - 1$, the grid with n equidistant points is contained in the grid with M equidistant points. This is not the case for the (repeated) Gauss two points, however.

We consider integral operators on $C([0,1])$ with the following two kernels:

$$k_1(s,t) = \exp(st) \ , \quad 0 \le s \ , \ t \le 1 \ ,$$

$$k_2(s,t) = \begin{cases} s(1-t) \ , & \text{if } 0 \le s \le t \le 1 \\ t(1-s) \ , & \text{if } 0 \le t \le s \le 1 \end{cases} .$$

Both k_1 and k_2 are nonnegative and symmetric. Note that k_1 is smooth (in fact, real analytic in s and t) , but k_2 is not (the partial derivatives are discontinuous on the diagonl of the square $[0,1] \times [0,1]$) .

The 4 largest eigenvalues of the integral operator with kernel k_1 are simple and have the decimal expansions

$$1.3530301647.., \ (1.0598322..)10^{-1}, \ (3.560749..)10^{-3}, \ (7.6379..)10^{-5}.$$

The nonzero eigenvalues of the integral operator with kernel k_2 are all simple; they are

$$\frac{1}{\pi^2}\ ,\quad \frac{1}{4\pi^2}\ ,\quad \frac{1}{9\pi^2}\ ,\quad \frac{1}{16\pi^2}\ ,\ \dots\ ,$$

and the corresponding eigenfunctions are $\sin k\pi t$, $t \in [0,1]$, $k = 1,2,\dots$.

Since the computed iterates converge to eigenelements of a discretized version of T , it is important to be able to choose an appropriate discretization scheme to start with. We illustrate this point by comparing the largest eigenvalue obtained by discretizing the integral operator with kernel k_1 by various nodes $t_i^{(n)} \in [0,1]$, but always with weights $w_i^{(n)} = 1/n$.

$t_i^{(n)} \rightarrow$ $n \downarrow$	$\begin{cases}(i-1/\sqrt{3})/n\ , & i \text{ odd} \\ (i-1+1/\sqrt{3})/n\ , & i \text{ even}\end{cases}$	$\frac{2i-1}{2n}$	$\frac{i-1}{n-1}$
2	1.352080...		
8	1.353026...	1.351644...	1.379129...
10	1.353028...		1.373071...
100	1.353030...	1.353021...	1.354777...

Table 19.2

The above table shows that with the choice $M = 100$, $w_i^{(M)} = \frac{1}{M}$, and $t_i^{(M)} = \frac{i-1}{M-1}$, the largest eigenvalue $\lambda^{(M)}$ of $[TM]$ agrees with the largest eigenvalue λ of T only in the first two decimal places and hence we cannot even hope to approximate λ up to more than 2 decimal places by any of the iteration schemes.

Our computations were performed on a Cyber 170 Model 840 (Network Operating System 2.4.3,647/642) in single precision, for which the floating-point arithmetic gives 14 reliable decimal digits. Thus, 10^{-15}(=1.0 E-015) is considered to be equal to 0 .

The iteration process is stopped, unless otherwise specified, when the residual norm RESID = $\|T\varphi_j - \lambda_j \varphi_j\|_\infty$ as well as the relative increment RELIN = $\|\varphi_{j+1} - \varphi_j\|_\infty / \|\varphi_j\|_\infty$ are both less than 10^{-12} (=1.0 E-12), or if 30 iterates have already been calculated.

With the kernel k_1, quadrature formula Q_2 and the Fredholm method(2), the time taken (in CP seconds) for implementing the Rayleigh-Schrödinger iteration scheme is listed below.

n	M	Compilation	Execution
10	100	1.150	2.337
10	150	1.169	4.261

These times include the calculation of all the 10 eigenvalues and eigenvectors of A. They should be compared with the execution time of 14.436 CP seconds for solving the 100×100 matrix eigenvalue problem, and of 48.802 CP seconds for the 150×150 matrix eigenvalue problem.

Comments on the numerical results

Tables 19.3, 19.4 and 19.5

Performance of the four iteration schemes discussed in Section 17 is compared in the case of two specific examples. In Table 19.3, the Rayleigh-Schrödinger scheme gives a slow improvement of accuracy, while the fixed point scheme gives substantially better results. The Ahués scheme is only marginally better than the fixed point scheme, while the modified fixed point scheme gives a spectacular rate of convergence. The values for some subsequent iterates of the Rayleigh-Schrödinger scheme are as follows:

j	8	10	12	14
$\lambda - \lambda_j$	-7.7 E-10	2.4 E-11	-7.1 E-13	1.6 E-14
$\|\varphi-\varphi_j\|_\infty$	2.1 E-11	3.4 E-13	4.2 E-14	7.1 E-15

In Table 19.4, the behaviour of the Rayleigh-Schrödinger Scheme and the fixed point scheme is similar; one can note the semigeometric convergence of the eigenvalue iterates as expected under the collectively compact convergence (cf. (11.30) and Problem 11.1) but, unexpectedly, the convergence of the eigenvector iterates is geometric. The modified fixed point scheme and the Ahués scheme give very fast convergence.

Table 19.5 compares the number of iterations needed for satisfying the stopping criteria for the four iteration schemes in various examples. The modified fixed point scheme gives the best results.

Table 19.6

The results for the power iteration scheme (11.36) with various pairs of starting vectors exhibit the expected linear rate of convergence for the eigenvector iterates, although the eigenvalue iterates occasionally give an improved accuracy. (See (12.6).) Note that in this case, the ratio of the second largest eigenvalue to the largest eigenvalue is $(1.05983)10^{-1}/1.353030 = .078$, approximately.) The phenomenal accuracy of the starting vector φ_0^N and the consequent accuracy of the first eigenvalue iterate are noteworthy.

Table 19.7

Performance of the five methods: projection, Sloan, Galerkin(2), Nyström and Fredholm(2) in approximating the second largest eigenvalue and a corresponding eigenvector of the integral operator with kernel k_2 is compared. The Sloan method gives the fastest convergence.

Table 19.8

The four largest eigenvalues of the integral operator T with kernel k_1 are approximated. The convergence is almost immediate for the largest eigenvalue $\lambda(1)$ and becomes progressively delayed for the next three eigenvalues $\lambda(2)$, $\lambda(3)$ and $\lambda(4)$. In the case of $\lambda(3)$ and $\lambda(4)$, the norm of the relative increment does *not* become less than 10^{-12} even up to 30 iterates. Note that

$\text{dist}(\lambda(1), \sigma(T)\backslash\{\lambda(1)\}) = 1.2$, $\text{dist}(\lambda(2), \sigma(T)\backslash\{\lambda(2)\}) = 1.0$ E-01, $\text{dist}(\lambda(3), \sigma(T)\backslash\{(3)\}) = 3.4$ E-03, $\text{dist}(\lambda(4), \sigma(T)\backslash\{\lambda(4)\}) = 7.5$ E-05,

approximately.

Tables 19.9 and 19.10

The effect of lowering the size n of the initial eigenvalue problem is considered in these tables. As the size n decreases from 10 to 2, the number of iterations needed to satisfy the stopping criteria increases. This increase is marginal in Table 19.9, but substantial in Table 19.10. Even the 30th iterates do not satisfy the stopping criteria for the size $n = 2$ in Table 19.10.

Comparison of various iteration schemes

Method: Fredholm (2), Kernel: k_2 , Quadrature: Q_1

M = 100, n = 10, λ : the largest eigenvalue

	Rayleigh-Schrödinger		Fixed point		Ahués		Modified fixed point	
j	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$
0	-1.0 E-03	7.3 E-04	-1.0 E-03	7.3 E-04	-1.0 E-03	7.3 E-04	-1.0 E-03	7.3 E-04
1	1.0 E-03	1.6 E-05	1.0 E-03	1.6 E-05	1.0 E-03	7.1 E-06	1.0 E-03	2.7 E-08
2	2.2 E-05	6.7 E-06	2.2 E-05	2.1 E-07	-1.0 E-05	7.0 E-08	3.0 E-08	1.1 E-12
3	-9.3 E-06	5.4 E-07	2.8 E-07	2.8 E-09	9.8 E-08	6.9 E-10	1.2 E-12	7.3 E-15
4	-7.4 E-07	1.1 E-07	3.5 E-09	3.6 E-11	-9.7 E-10	6.8 E-12	-2.7 E-15	7.3 E-15
5	1.5 E-07	1.7 E-08	4.4 E-11	4.7 E-13	9.5 E-12	6.9 E-14		
6	2.4 E-08	1.8 E-09	5.6 E-13	7.1 E-15	-9.7 E-14	7.2 E-15		
7	-2.6 E-09	5.6 E-10	4.9 E-15	7.2 E-15	-1.3 E-15	7.3 E-15		

Table 19.3

Comparison of various iteration schemes

Method: Fredholm (2), Kernel: k_1 , Quadrature: Q_2

M = 100, n = 10, λ : the largest eigenvalue

	Rayleigh-Schrödinger		Fixed point		Ahués		Modified fixed point	
j	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$
0	1.7 E-06	1.3 E-02	1.7 E-06	1.3 E-02	1.7 E-06	1.3 E-02	1.7 E-06	1.3 E-02
1	4.2 E-04	1.3 E-04	4.2 E-04	1.3 E-04	4.2 E-04	3.0 E-06	4.2 E-04	1.0 E-06
2	9.9 E-08	2.9 E-06	9.9 E-08	1.1 E-06	-1.0 E-07	7.6 E-10	2.8 E-08	6.9 E-11
3	-9.7 E-08	7.3 E-08	3.1 E-08	9.5 E-09	2.5 E-11	1.9 E-13	1.9 E-12	3.9 E-14
4	-8.6 E-11	1.1 E-09	5.2 E-12	8.2 E-11	-1.4 E-14	5.0 E-14	-1.4 E-14	3.9 E-14
5	4.2 E-11	4.8 E-11	2.3 E-12	6.8 E-13				
6	5.7 E-14	5.1 E-13	-1.4 E-14	4.3 E-14				
7	-2.1 E-14	7.5 E-14	-1.4 E-14	4.3 E-14				

Table 19.4

Number of iterations (i) needed to satisfy the stopping criteria

Method: Fredholm(2), M = 100, n = 10

Iteration scheme	Kernel	Quadrature	i	$\lvert\lambda-\lambda_i\rvert$	$\lVert\varphi-\varphi_i\rVert_\infty$
	k_1	Q_1	7	2.2 E-13	8.2 E-14
Rayleigh	k_1	Q_2	7	2.1 E-14	7.4 E-14
-Schrödinger	k_2	Q_1	14	1.6 E-14	7.1 E-15
	k_2	Q_2	14	4.4 E-16	5.1 E-15
	k_1	Q_1	5	2.1 E-14	8.3 E-14
Fixed point	k_1	Q_2	7	1.4 E-14	4.2 E-14
	k_2	Q_1	7	4.9 E-15	7.2 E-15
	k_2	Q_2	8	5.7 E-15	2.4 E-15
	k_1	Q_1	4	1.4 E-14	8.3 E-14
Ahués	k_1	Q_2	4	1.4 E-14	4.9 E-14
	k_2	Q_1	7	1.3 E-15	7.2 E-15
	k_2	Q_2	7	4.8 E-15	2.6 E-15
	k_1	Q_1	4	3.6 E-14	8.7 E-14
Modified	k_1	Q_2	4	1.4 E-14	3.9 E-14
fixed point	k_2	Q_1	4	2.6 E-15	7.2 E-15
	k_2	Q_2	5	4.8 E-15	2.8 E-15

Table 19.5

Power iteration with various starting vectors

Kernel: k_1 , Quadrature: Q_2

M = 100, n = 10, λ : the largest eigenvalue

	$x_0 = \varphi_0^G$, $x_0^* = \varphi_0^{*G}$		$x_0 = \varphi_0^N$, $x_0^* = \varphi_0^{*N}$		$x_0 = \varphi_0^F$, $x_0^* = \varphi_0^{*F}$	
j	$\lambda - \lambda_j$	$\|\varphi - x_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi - x_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi - x_j\|_\infty$
0		1.3 E-02		9.3 E-07		1.3 E-02
1	4.2 E-04	1.3 E-04	8.0 E-11	5.3 E-08	4.2 E-04	1.2 E-04
2	3.9 E-07	9.6 E-06	9.7 E-13	4.1 E-09	2.3 E-09	8.9 E-06
3	3.1 E-08	7.5 E-07	5.0 E-14	3.2 E-10	1.5 E-10	7.0 E-07
4	2.4 E-09	5.9 E-08	-7.1 E-15	2.5 E-11	1.2 E-11	5.4 E-08
5	1.9 E-10	4.6 E-09	-7.1 E-15	1.9 E-12	9.2 E-13	4.3 E-09
6	1.5 E-11	3.6 E-10	-1.4 E-14	1.5 E-13	6.4 E-14	3.3 E-10
7	1.1 E-12	2.8 E-11	-1.4 E-14	3.2 E-14	-1.4 E-14	2.6 E-11
8	7.1 E-14	2.2 E-12			-7.1 E-15	2.0 E-12
9	0.0	1.7 E-13			-7.1 E-15	1.5 E-13
10	-1.4 E-14	2.8 E-14			-2.1 E-14	3.6 E-14

Table 19.6

Comparison of various methods

Scheme: Fixed point

Kernel: k_2 , Quadrature: Q_2

M = 100, n = 10, λ : the 2nd largest eigenvalue

	Projection method		Sloan method		Galerkin method(2)		Nyström method		Fredholm method(2)	
j	$\lambda-\lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda-\lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda-\lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda-\lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda-\lambda_j$	$\|\varphi-\varphi_j\|_\infty$
0	8.2 E-04	1.1 E-01	8.2 E-04	2.7 E-05	8.2 E-04	2.6 E-03	-9.1 E-04	7.6 E-04	-9.1 E-04	2.7 E-03
1	1.2 E-05	4.2 E-03	1.2 E-05	1.0 E-06	8.2 E-04	2.7 E-05	8.8 E-04	6.8 E-05	8.4 E-04	2.0 E-04
2	-1.1 E-07	1.7 E-04	-1.1 E-07	3.7 E-08	1.2 E-05	3.0 E-06	7.1 E-05	4.3 E-06	6.6 E-05	5.4 E-06
3	4.4 E-09	6.6 E-06	4.4 E-09	1.3 E-09	-8.2 E-08	2.2 E-07	3.6 E-06	2.8 E-07	3.5 E-06	4.6 E-07
4	-1.0 E-10	2.6 E-07	-1.0 E-10	4.8 E-11	2.7 E-09	2.9 E-08	1.9 E-07	1.9 E-08	1.9 E-07	4.3 E-08
5	3.0 E-12	9.8 E-09	3.0 E-12	1.7 E-12	5.2 E-10	2.2 E-09	1.1 E-08	1.4 E-09	1.1 E-08	4.8 E-09
6	-8.6 E-14	3.7 E-10	-8.6 E-14	6.4 E-14	-4.0 E-11	2.9 E-10	7.3 E-10	1.0 E-10	7.3 E-10	6.4 E-10
7	2.6 E-15	1.4 E-11	1.7 E-15	2.2 E-15	6.4 E-12	2.7 E-11	5.0 E-11	7.9 E-12	5.3 E-11	9.0 E-11
8	-1.1 E-15	5.2 E-13	-6.7 E-16	1.1 E-15	-5.5 E-13	3.3 E-12	3.6 E-12	6.1 E-13	4.2 E-12	1.2 E-11
9	-7.8 E-16	8.2 E-14			7.2 E-14	3.4 E-13	2.7 E-13	4.7 E-14	3.7 E-13	1.6 E-12
10	-2.0 E-15	6.3 E-14			-7.8 E-15	4.1 E-14	2.0 E-14	3.2 E-15	3.4 E-14	2.1 E-13
11					-2.2 E-16	4.4 E-15	1.1 E-15	1.1 E-15	2.6 E-15	2.8 E-14
12					-1.2 E-15	1.3 E-15			-6.7 E-16	4.1 E-15
13									-7.8 E-16	1.1 E-15

Table 19.7

Results for the four largest eigenvalues

Scheme: Fixed point, Method: Nyström

Kernel: k_1, Quadrature: Q_2

M = 100, n = 10

Stopping criterion: RESID < 10^{-12}

	$\lambda = \lambda(1)$		$\lambda = \lambda(2)$		$\lambda = \lambda(3)$		$\lambda = \lambda(4)$	
j	$\lambda-\lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda-\lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda-\lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda-\lambda_j$	$\|\varphi-\varphi_j\|_\infty$
0	1.7 E-06	9.3 E-07	7.6 E-06	1.0 E-06	8.3 E-06	9.2 E-07	2.0 E-06	4.1 E-07
1	8.0 E-11	2.1 E-11	1.4 E-09	1.3 E-10	2.5 E-09	1.7 E-09	-1.8 E-09	1.1 E-09
2	0.0	5.3 E-14	1.6 E-14	1.4 E-14	-7.9 E-12	3.3 E-12	-3.9 E-11	5.3 E-11
3			-4.4 E-15	1.3 E-14	1.7 E-14	7.5 E-15	1.2 E-12	1.4 E-12
4							-2.7 E-14	3.2 E-14

Table 19.8

Results for varying values of n

Scheme: Rayleigh-Schrödinger, Method: Nyström

Kernel: k_1 , Quadrature: Q_2

M = 100, λ : the largest eigenvalue

n →	10		8		6		4		2	
j ↓	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$
0	1.7 E-06	9.3 E-07	4.1 E-06	2.5 E-06	1.3 E-05	9.2 E-06	6.4 E-05	5.6 E-05	9.5 E-04	1.2 E-03
1	8.0 E-11	2.1 E-11	4.7 E-10	1.4 E-10	4.6 E-09	1.5 E-09	1.1 E-07	4.5 E-08	2.1 E-05	1.1 E-05
2	0.0	5.3 E-14	7.1 E-15	4.3 E-14	3.4 E-13	1.1 E-13	4.1 E-11	1.6 E-11	8.0 E-08	4.0 E-08
3	-2.1 E-14		-7.1 E-15		-1.4 E-14	5.3 E-14	1.4 E-14	8.2 E-14	3.0 E-10	2.2 E-11
4					-2.1 E-14		-2.1 E-14		-1.4 E-13	1.7 E-12
5									-2.1 E-14	9.2 E-14
6									-7.1 E-15	

Table 19.9

Results for varying values of n

Scheme: Rayleigh-Schrödinger, Method: Nyström

Kernel: k_2 , Quadrature: Q_2

M = 100, λ : the largest eigenvalue

n → / j ↓	10		8		6		4		2	
	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$	$\lambda - \lambda_j$	$\|\varphi-\varphi_j\|_\infty$
0	-8.9 E-04	7.6 E-04	-1.4 E-03	1.2 E-03	-2.5 E-03	2.8 E-03	-5.9 E-03	5.8 E-03	-4.3 E-03	4.7 E-03
1	8.9 E-04	1.7 E-05	1.4 E-03	4.3 E-05	2.5 E-03	1.7 E-04	5.5 E-03	8.1 E-04	2.5 E-02	1.7 E-02
5	1.1 E-07	1.3 E-08	3.4 E-07	7.6 E-08	1.1 E-06	7.8 E-07	-6.4 E-06	9.1 E-06	-2.9 E-03	8.8 E-04
10	1.2 E-11	1.4 E-13	1.3 E-10	1.2 E-11	1.9 E-09	8.7 E-10	-1.3 E-07	5.0 E-08	-5.0 E-04	1.7 E-03
15	-4.0 E-15		4.4 E-14	5.3 E-15	3.9 E-12	1.2 E-12	-2.1 E-09	3.6 E-10	5.2 E-04	2.7 E-04
16			-6.7 E-15		9.9 E-13	1.6 E-13	2.3 E-10	4.6 E-10	1.2 E-04	5.0 E-04
20					2.7 E-15	3.2 E-15	-3.2 E-11	4.7 E-12	1.1 E-04	5.4 E-04
21					-1.8 E-15		-2.4 E-12	4.9 E-12	2.9 E-04	4.9 E-04
29							-2.7 E-15	9.8 E-15	1.2 E-04	2.9 E-05
30							-9.3 E-15		-2.2 E-05	

Table 19.10