1. ADJOINT CONSIDERATIONS

A useful way of studying a complex Banach space X and a bounded linear operator T on X is to consider the <u>adjoint space</u>

$$X^* = \{x^* : X \to \mathbb{C} \text{ , } x^* \text{ is conjugate linear and continuous}\}$$

of X and the *adjoint operator* T^* associated with T. In this section we develop these concepts. This is done in such a way as to make the well-known Hilbert space situation a particular case of our development.

For $x^* \in X^*$ and $x \in X$, we denote the value of x^* at x by

$$\langle x^*,x\rangle .$$

Then we easily see that for x^* and y^* in X^*, x and y in X and $t \in \mathbb{C}$,

$$\begin{aligned} \langle x^*,x+y\rangle &= \langle x^*,x\rangle + \langle x^*,y\rangle , \\ \langle x^*,tx\rangle &= \bar{t}\langle x^*,x\rangle , \\ \langle x^*+y^*,x\rangle &= \langle x^*,x\rangle + \langle y^*,x\rangle , \\ \langle tx^*,x\rangle &= t\langle x^*,x\rangle . \end{aligned} \tag{1.1}$$

We say that $\langle\ ,\ \rangle$ is the <u>scalar product</u> on $X^* \times X$. For the sake of convenience, we introduce the following notation:

(1.2) $\langle x,x^*\rangle = \overline{\langle x^*,x\rangle}$, x in X and x^* in X^*.

For x^* in X^*, let

$$\|x^*\| = \sup\{ |\langle x^*,x\rangle| : x \text{ in } X , \|x\| \le 1\} .$$

This defines a norm on X^* and makes it a Banach space. We have the *fundamental inequality*:

(1.3) $|\langle x^*,x\rangle| = |\langle x,x^*\rangle| \leq \|x^*\| \, \|x\|$, x^* in X^* and x in X .

Many books on functional analysis consider the *dual space*

$$X' = \{x' : X \to \mathbb{C} : x' \text{ is linear and continuous}\}$$

of X instead of the adjoint space X^* . We prefer the framework of the adjoint space because in case X is a Hilbert space, X^* can be *linearly* identified with X itself, as we shall see later. In any event, we remark that $x' \in X'$ iff its complex conjugate $\overline{x'} \in X^*$. This allows us to transfer many well-known results about X' to X^* , such as the following basic extension result.

PROPOSITION 1.1 (Hahn-Banach theorem) Let Y be a subspace of X and $y^* \in Y^*$. Then there is $x^* \in X^*$ such that $x^*|_Y = y^*$ and $\|x^*\| = \|y^*\|$.

Proof Since $\overline{y^*} \in Y'$, there is $x' \in X'$ with $x'|_Y = \overline{y^*}$ and $\|x'\| = \|\overline{y^*}\| = \|y^*\|$, by the Hahn-Banach extension theorem ([L], 7.6). The proof is complete if we let $x^* = \overline{x'}$. //

COROLLARY 1.2 If $0 \neq a \in X$, then there is $x^* \in X^*$ with $\langle x^*,a\rangle = \|a\|$ and $\|x^*\| = 1$. More generally, if Y is a closed subspace of X and $a \notin Y$, then there is $x^* \in X^*$ such that $\langle x^*,a\rangle = \mathrm{dist}(a,Y)$, $\|x^*\| = 1$ and $x^*|_Y \equiv 0$.

Proof The first result follows by letting $Y = \text{span}\{a\}$ and $\langle y^*, ta\rangle = \bar{t}\|a\|$ in Proposition 1.1. The second part can be proved by considering the quotient space X / Y with the quotient norm $|||x+y||| = \inf\{\|x+y\| : y \in Y\} = \text{dist}(x,Y)$ for $x \in X$, and then using the first part. //

The above result is useful in expressing the duality between X and X^*: Just as $\langle x^*,x\rangle = 0$ for all $x \in X$ implies, by definition, that $x^* = 0$, we see that $\langle x,x^*\rangle = 0$ for all $x^* \in X^*$ implies, by the above corollary, that $x = 0$. Moreover, just as we have by definition, for $x^* \in X^*$,

$$\|x^*\| = \sup\{|\langle x^*,x\rangle| : x \in X , \|x\| \leq 1\} ,$$

we see by (1.3) and the above corollary that for $x \in X$,

$$\|x\| = \sup\{|\langle x,x^*\rangle| : x^* \in X^* , \|x^*\| \leq 1\} .$$

For a subset E of X, we define the <u>annihilator</u> $E^\perp$ <u>of</u> E to be the following subset of X^*:

$$E^\perp = \{x^* \in X^* : \langle x^*,x\rangle = 0 \text{ for all } x \in E\} .$$

It is easy to see that $E^\perp$ is, in fact, a closed subspace of X^*. The concept of an annihilator will be used later in relating the range of a bounded linear map to the zero space of its adjoint.

Let X and Y be complex Banach spaces, and let $BL(X,Y)$ denote the set of all <u>bounded linear maps from</u> X <u>to</u> Y. For $T \in BL(X)$, the operator norm of T is defined as follows:

$$\|T\| = \sup\{\|Tx\| : x \in X , \|x\| \leq 1\} .$$

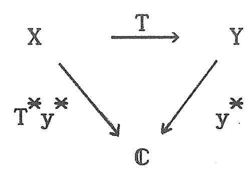
Two important subspaces related to T are the <u>null space of</u> T :

$$Z(T) = \{x \in X : Tx = 0\} ,$$

and the <u>range of</u> T :

$$R(T) = \{y \in Y : y = Tx \text{ for some } x \in X\} .$$

For $T \in BL(X,Y)$ and $y^* \in Y^*$, we see that $y^*T \in X^*$. We denote this element of X^* by T^*y^* . Thus, the following diagram commutes:



The <u>adjoint</u> T^* <u>of</u> T is the map from Y^* to X^* defined by

$$\langle T^*y^*,x\rangle = \langle y^*,Tx\rangle \quad \text{for } y^* \in Y^* , x \in X .$$

Taking conjugates, and using the notation (1.2), we have

(1.4) $$\langle Tx,y^*\rangle = \langle x,T^*y^*\rangle \quad \text{for } x \in X , y^* \in Y^* .$$

Proposition 1.3 (a) For $T \in BL(X,Y)$, we have $T^* \in BL(Y^*,X^*)$ and $\|T^*\| = \|T\|$.

(b) For $T , S \in BL(X,Y)$ and $t \in \mathbb{C}$, we have

$$(T + S)^* = T^* + S^* \quad \text{and} \quad (tT)^* = \bar{t}T^* .$$

Thus, $T \mapsto T^*$ is a conjugate linear isometry of $BL(X,Y)$ into $BL(Y^*,X^*)$.

(c) The null space of T^* equals the annihilator of the range of T :

$$Z(T^*) = R(T)^{\perp} .$$

(d) Let Z be a complex Banach space, and $U \in BL(Y,Z)$. Then

$$(UT)^* = T^*U^* .$$

Proof (a) T^* is clearly linear. Also,

$$\begin{aligned} \|T^*\| &= \sup\{\|T^*y^*\| : y^* \in Y^* , \|y^*\| \leq 1\} \\ &= \sup\{|\langle y^*,Tx\rangle| : y^* \in Y^* , \|y^*\| \leq 1 , x \in X , \|x\| \leq 1\} \\ &= \sup\{\|Tx\| : x \in X , \|x\| \leq 1\} \\ &= \|T\| . \end{aligned}$$

(b) The proof of this part is easy. For example, one quickly shows that for every $y^* \in Y^*$, and $x \in X$,

$$\langle (T+S)^*y^*,x\rangle = \langle (T^*+S^*)y^*,x\rangle .$$

(c) We have $y^* \in Z(T^*)$ if and only if $\langle y^*,Tx\rangle = \langle T^*y^*,x\rangle = 0$ for every $x \in X$ if and only if $y^* \in R(T)^{\perp}$.

(d) For $z^* \in Z^*$ and $x \in X$, we have

$$\langle (UT)^*z^*,x\rangle = \langle z^*,UTx\rangle = \langle U^*z^*,Tx\rangle = \langle T^*U^*z^*,x\rangle .$$

Hence the result. //

Special Case of a Hilbert Space.

Let X be a Hilbert space with the inner product $\langle\ ,\ \rangle_X$, and let $\|x\| = (\langle x,x\rangle_X)^{1/2}$ for $x \in X$. Given $x^* \in X^*$, define $f : X \to \mathbb{C}$ by

$$f(x) = \langle x,x^*\rangle , x \in X .$$

Then f is a continuous linear functional on X of norm $\|x^*\|$. The *Riesz representation theorem* ([L], 24.2) shows that there is unique $y \in X$ such that

$$\langle x,x^*\rangle = f(x) = \langle x,y\rangle_X$$

for all $x \in X$; moreover, $\|y\| = \|f\| = \|x^*\|$. The correspondence $x^* \mapsto y$ of X^* with X is, thus, a linear isometry onto. Whenever X is a Hilbert space, we shall, from now on, identify X^* with X via the above correspondence, and drop the suffix X in the inner product notation $\langle\ ,\ \rangle_X$ without any ambiguity.

Let $A : X \to X$ be a linear map. The generalized polarization identity

$$4\langle Ax,y\rangle = \langle A(x+y),x+y\rangle - \langle A(x-y),x-y\rangle$$
$$+ i\langle A(x+iy),x+iy\rangle - i\langle A(x-iy),x-iy\rangle ,$$

where x and y belong to X , is often useful.

For a subset E of the Hilbert space X , the *annihilator*

$$E^{\perp} = \{y \in X : \langle x,y\rangle = 0 \text{ for all } x \in E\}$$

consists of all elements of X which are orthogonal to E . The double annihilator $E^{\perp\perp}$ has a nice characterization: If F denotes the closure of the linear span of E , then

(1.5) $$E^{\perp\perp} = F .$$

It is easy to check that F is contained in $E^{\perp\perp}$. On the other hand, suppose for a moment that there is some a in $E^{\perp\perp}$, but not in F . Then by Corollary 1.2, there is $x^* \in X^*$ such that $x^*|_F = 0$ but $\langle x^*,a\rangle = 1$, i.e., there is $y \in X$ such that $\langle z,y\rangle = 0$ for all $z \in F$, but $\langle a,y\rangle = 1$. This is impossible since $y \in E^{\perp}$ and $a \in E^{\perp\perp}$ so that $\langle a,y\rangle = 0$.

For $T \in BL(X)$, the adjoint operator $T^* \in BL(X)$ is characterized by

$$\langle Tx,y \rangle = \langle x,T^*y \rangle \quad \text{for all } x \text{ and } y \text{ in } X .$$

In addition to $Z(T^*) = R(T)^\perp$, as noted in Proposition 1.3(c), we also have

$$Z(T) = R(T^*)^\perp , \tag{1.6}$$

when X is a Hilbert space. This follows since $x \in Z(T)$ if and only if $0 = \langle Tx,y \rangle = \langle x,T^*y \rangle$ for all $y \in X$ if and only if $x \in R(T^*)^\perp$. Thus, T (resp., T^*) is one to one if and only if the range of T^* (resp., T) is dense in X .

The norms of the operators T and T^* are related by the B^*-algebra condition

$$\|T^*T\| = \|T\|^2 . \tag{1.7}$$

This can be proved as follows.

$$\begin{aligned} \|T^*T\| &\leq \|T^*\| \, \|T\| \\ &= \|T\|^2 \\ &= \sup\{\|Tx\|^2 : x \in X , \|x\| \leq 1\} \\ &= \sup\{\langle Tx,Tx \rangle : x \in X , \|x\| \leq 1\} \\ &= \sup\{\langle T^*Tx,x \rangle : x \in X , \|x\| \leq 1\} \\ &\leq \|T^*T\| . \end{aligned}$$

If T^* commutes with T , i.e., $T^*T = TT^*$, we say that T is <u>normal</u>, and if $T^* = T$, we say that T is <u>self-adjoint</u>. It is clear that every self-adjoint operator is normal.

For $x \in X$ and $T \in BL(X)$, we have

$$\|Tx\|^2 - \|T^*x\|^2 = \langle Tx,Tx\rangle - \langle T^*x,T^*x\rangle$$
$$= \langle (T^*T-TT^*)x,x\rangle .$$

Hence it follows by using the generalized polarization identity that

(1.8) $T \in BL(X)$ is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in X$.

For a self-adjoint operator T, we have

$$\langle Tx,x\rangle = \langle x,T^*x\rangle$$
$$= \langle x,Tx\rangle$$
$$= \overline{\langle Tx,x\rangle}$$

for all $x \in X$, so that $\langle Tx,x\rangle$ is real. Conversely, let $\langle Tx,x\rangle$ be real for all $x \in X$. Then for $x,y \in X$, the generalized polarization identity shows that,

$$\begin{aligned} 4\langle Tx,y\rangle &= \langle T(x+y),x+y\rangle - \langle T(x-y),x-y\rangle \\ &\quad + i\langle T(x+iy),x+iy\rangle - i\langle T(x-iy),x-iy\rangle \\ &= \langle x+y,T(x+y)\rangle - \langle x-y,T(x-y)\rangle \\ &\quad + i\langle x+iy,T(x+iy)\rangle - i\langle x-iy,T(x-iy)\rangle \\ &\qquad\qquad (\text{since } \langle Tz,z\rangle \text{ is real for all } z \in X) \\ &= \langle T^*(x+y),x+y\rangle - \langle T^*(x-y),x-y\rangle \\ &\quad + i\langle T^*(x+iy),x+iy\rangle - i\langle T^*(x-iy),x-iy\rangle \\ &= 4\langle T^*x,y\rangle . \end{aligned}$$

Hence $T^* = T$, i.e., T is self-adjoint. Thus,

(1.9) $T \in BL(X)$ is self-adjoint if and only if $\langle Tx,x\rangle$ is real for all $x \in X$.

Examples of adjoint spaces and operators

(i) Let X be an n dimensional space with $1 \leq n < \infty$, and let $x_1,\dots,x_n$ be an ordered basis for X. Then for x in X, we have

$$x = t_1(x)x_1 + \dots + t_n(x)x_n ,$$

where $t_j(x) \in \mathbb{C}$, $j = 1,\dots,n$, is uniquely determined by x. If we let

$$\langle x_j^*, x\rangle = \overline{t_j(x)} , \ j = 1,\dots,n ,$$

then $x_1^*,\dots,x_n^*$ is an ordered basis for X^* and we have

$$\langle x_i^*, x_j\rangle = \delta_{i,j} , \quad i,j = 1,\dots,n ,$$

where $\delta_{i,j}$ is the <u>Kronecker symbol</u> : $\delta_{i,j}$ equals 0 if $i \neq j$, and equals 1 if $i = j$. This basis is called the <u>basis of</u> X^* <u>which is adjoint to the given basis</u> $x_1,\dots,x_n$ <u>of</u> X.

For x in X and x^* in X^*, we have

$$x = \langle x,x_1^*\rangle x_1 + \dots + \langle x,x_n^*\rangle x_n ,$$

$$\text{(1.9)} \qquad x^* = \langle x^*,x_1\rangle x_1^* + \dots + \langle x^*,x_n\rangle x_n^* ,$$

$$\langle x^*,x\rangle = \langle x^*,x_1\rangle\langle x_1^*,x\rangle + \dots + \langle x^*,x_n\rangle\langle x_n^*,x\rangle .$$

Let, now, Y be an m-dimensional space with $1 \leq m < \infty$. Let $y_1,\dots,y_m$ be an ordered basis for Y, and $y_1^*,\dots,y_m^*$ be the corresponding adjoint basis for Y^*. If $T : X \to Y$ is linear, and we let

$$t_{i,j} = \langle Tx_j, y_i^*\rangle , \qquad i,j = 1,\dots,n ,$$

then we see that for $j = 1,\dots,n$,

$$Tx_j = \langle Tx_j, y_1^* \rangle y_1 + \dots + \langle Tx_j, y_m^* \rangle y_m$$
$$= \sum_{i=1}^{m} t_{i,j}\, y_i \;.$$

Thus, for x in X ,

$$Tx = \sum_{j=1}^{n} \langle x, x_j^* \rangle Tx_j$$
$$= \sum_{i=1}^{m} \left[\sum_{j=1}^{n} t_{i,j} \langle x, x_j^* \rangle \right] y_i \;.$$

The operator T can be represented by the $m \times n$ matrix $A = [t_{i,j}]$, with respect to the bases $x_1,\dots,x_n$ and $y_1,\dots,y_m$ of X and Y respectively, in the following sense:

$$\begin{bmatrix} t_{1,1} & \cdots & t_{1,n} \\ \vdots & & \vdots \\ t_{m,1} & \cdots & t_{m,n} \end{bmatrix} \begin{bmatrix} \langle x, x_1^* \rangle \\ \vdots \\ \langle x, x_n^* \rangle \end{bmatrix} = \begin{bmatrix} \langle Tx, y_1^* \rangle \\ \vdots \\ \langle Tx, y_m^* \rangle \end{bmatrix} .$$

Now consider the adjoint operator $T^* : Y^* \to X^*$. It can be easily seen that $(X^*)^*$ can be identified with X , and we can regard $x_1,\dots,x_n$ as the basis of $(X^*)^*$ which is adjoint to the basis $x_1^*,\dots,x_n^*$ of X^* . Since

$$\langle T^* y_j^*, x_i \rangle = \langle y_j^*, Tx_i \rangle = \overline{\langle Tx_i, y_j^* \rangle} = \bar{t}_{j,i}$$

for $i = 1,\dots,m$ and $j = 1,\dots,n$, we see that the adjoint operator T^* is represented by the <u>conjugate transpose matrix</u> $A^H = [\bar{t}_{j,i}]$, with respect to the bases $y_1^*,\dots,y_m^*$ and $x_1^*,\dots,x_n^*$ of Y^* and X^* respectively.

A commonly occurring situation is when $X = \mathbb{C}^n$, the set of all column vectors with n entries of complex numbers. Let $e_j^{(n)}$ denote the column vector whose i-th entry $e_j^{(n)}(i)$ equals $\delta_{i,j}$. To save space, let $x = \begin{bmatrix} x(1) \\ \vdots \\ x(n) \end{bmatrix}$ in $\mathbb{C}^n$ be denoted by $[x(1),\ldots,x(n)]^t$, where the superscript t denotes the transpose. Note that x^H denotes the conjugate transpose of x, i.e., the row vector $[\overline{x(1)},\ldots,\overline{x(n)}]$. For $x \in \mathbb{C}^n$, we have

$$x = \sum_{j=1}^{n} x(j)e_j^{(n)} .$$

so that $e_1^{(n)},\ldots,e_n^{(n)}$ is a basis of X, the so called <u>standard basis</u>. If $x^* \in X^*$ and we let

$$\langle x^*, e_j^{(n)} \rangle = y(j) , \quad j = 1,\ldots,n ,$$

then

$$\langle x^*, x \rangle = \sum_{j=1}^{n} \overline{x(j)}\, y(j) ,$$

so that X^* can be identified again with the set $\mathbb{C}^n$ of column vectors $[y(1),\ldots,y(n)]^t$, and we can consider $x_j^* = e_j^{(n)}$, $j = 1,\ldots,n$, as the corresponding adjoint basis. Then we have for all $x \in X$ and $y \in X^*$,

$$\langle y, x \rangle = \sum_{j=1}^{n} \overline{x(j)}\, y(j) = x^H y .$$

If $Y = \mathbb{C}^m$, and $T : \mathbb{C}^n \to \mathbb{C}^m$ is linear, then

$$t_{i,j} = \langle Te_j^{(n)}, e_i^{(m)} \rangle = (e_i^{(m)})^H Te_j^{(n)}$$

is simply the i-th entry of the m-vector $Te_j^{(n)}$ for $j = 1,\ldots,n$ and $i = 1,\ldots,m$. Thus, Tx is given by the product of the $m \times n$ matrix $[\langle Te_j^{(n)}, e_i^{(m)}\rangle]$ with the $n \times 1$ matrix $x \in \mathbb{C}^n$. Conversely, an $m \times n$ matrix defines a linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$ in a natural way. We shall denote an operator and the corresponding matrix by the same letter T.

The i-th entry of the n-vector $T^* e_j^{(m)}$ is

$$\langle T^* e_j^{(m)}, e_i^{(n)}\rangle = \langle e_j^{(m)}, Te_i^{(n)}\rangle = \bar{t}_{j,i}$$

for $i = 1,\ldots,n$ and $j = 1,\ldots,m$. Thus, the adjoint T^* of an operator T is given by the conjugate transpose T^H of the corresponding matrix T.

(ii) Let $X = \ell^p$, $1 \leq p < \infty$, the space of all p-summable sequences of complex numbers, with the norm

$$\|[x(1),x(2),\ldots]^t\| = \left[\sum_{j=1}^{\infty} |x(j)|^p\right]^{1/p},$$

for $x = [x(1),x(2),\ldots]^t$ in X. Then X^* can be identified with ℓ^q, where $1/p + 1/q = 1$, via the map $x^* \mapsto y$ with

$$\langle x^*, e_j\rangle = y(j),$$

where $e_j = [0,\ldots,0,1,0,\ldots]^t$, the entry 1 occurring only in the j-th place ([L], 13.4(b)). Now, for $x = \sum_{j=1}^{\infty} x(j)e_j$ in X we have

$$\langle x^*, x\rangle = \sum_{j=1}^{\infty} \overline{x(j)}\, y(j).$$

Let $T \in BL(\ell^p, \ell^q)$, and

$$Te_j = [t_{1,j}, t_{2,j}, \dots]^t ,$$

so that $\langle Te_j, e_i \rangle = t_{i,j}$. Since

$$Tx = \sum_{j=1}^{\infty} x(j) Te_j ,$$

we have for $i = 1,2,\dots,$

$$Tx(i) = \sum_{j=1}^{\infty} x(j) t_{i,j} .$$

Now, $T^* \in BL(\ell^p, \ell^q)$, and

$$T^* e_j = [\bar{t}_{j,1}, \bar{t}_{j,2}, \dots]^t,$$

since $\langle T^* e_j, e_i \rangle = \langle e_j, Te_i \rangle = \bar{t}_{j,i}$. We note that T and T^* are thus given by the infinite matrices $[t_{i,j}]$ and $[\bar{t}_{j,i}]$, $i,j = 1,2,\dots,$ respectively.

(iii) Let $X = L^p([a,b])$, $1 \leq p < \infty$, the set of all p-integrable complex-valued functions on $[a,b]$ with the norm

$$\|x\|_p = \left[\int_a^b |x(t)|^p dm(t)\right]^{1/p} ,$$

where m is the Lebesgue measure. Then X^* can be identified with $L^q([a,b])$, where $1/p + 1/q = 1$, since for every $x^* \in X^*$, there is a unique $y \in L^q([a,b])$ such that

$$\langle x^*, x \rangle = \int_a^b \overline{x(t)} y(t) dm(t) , \quad x \in X .$$

(See [L], 14.3.)

Consider, for simplicity, $p = 2 = q$, and let $T \in BL(L^2([a,b]))$ be the integral operator

$$Tx(s) = \int_a^b k(s,t)x(t)dm(t) \ , \quad x \in X \ ,$$

where $\int_a^b\int_a^b |k(s,t)|^2 dm(s)dm(t) < \infty$. Then for all $x,y \in X$, we have

$$\begin{aligned} \langle T^* y, x\rangle &= \langle y, Tx\rangle \\ &= \int_a^b \overline{Tx(t)}y(t)dm(t) \\ &= \int_a^b \left[\int_a^b \overline{k(t,s)}\ \overline{x(s)}\ dm(s)\right] y(t)dm(t) \\ &= \int_a^b \overline{x(s)}\left[\int_a^b \overline{k(t,s)}\ y(t)dm(t)\right] dm(s) \end{aligned}$$

so that for $a \leq s \leq b$,

$$T^* y(s) = \int_a^b \overline{k(t,s)}\ y(t)dm(t) \ .$$

Thus, T^* is again an integral operator with kernel $k^*(s,t) = \overline{k(t,s)}$.

(iv) Let $X = C([a,b])$, the set of all complex-valued continuous functions on the closed and bounded interval $[a,b]$ of the real line, with the supremum norm. Then for every $x^* \in X^*$, there is a unique normalized function of bounded variation, say y, such that

$$\langle x^*, x\rangle = \int_a^b \overline{x(t)}dy(t) \quad \text{for all} \quad x \in X \ .$$

(See [L], 14.6).

Let T be an integral operator as in (iii) above, with $k(s,t)$ continuous for $s,t \in [a,b]$. Then for every $x \in C([a,b])$ and every normalized function y of bounded variation on $[a,b]$, we have, as earlier,

$$\langle T^* y,x\rangle = \int_a^b \overline{x(s)}\left[\int_a^b \overline{k(t,s)}dy(t)\right]ds$$
$$= \int_a^b \overline{x(s)}dz(s) \ ,$$

where

$$z(s) = \int_a^s \left[\int_a^b \overline{k(t,u)}dy(t)\right]du, \quad a \le s \le b \ .$$

Since this is true for every $x \in X$, we see that for $a \le s \le b$,

$$T^* y(s) = z(s)$$
$$= \int_a^b \left[\int_a^s \overline{k(t,u)}du\right]dy(t)$$

Problems

1.1 Let Y be a closed subspace of X, and $x_0 \in X$ but $x_0 \notin Y$. Then there is $x^* \in X^*$ such that

$$\langle x^*,y\rangle = 0 \text{ for all } y \in Y \ , \ \langle x^*,x_0\rangle = 1 \ , \text{ and } \|x^*\| = 1/\text{dist}(x_0,Y) \ .$$

1.2 For fixed $x \in X$, define $f_x : X^* \to \mathbb{C}$ by $f_x(x^*) = \langle x,x^*\rangle$. Then $f_x \in X^{**}$. Identify x with f_x, so that $X \subset X^{**}$. Let $E \subset X$. Then

$$E^{\perp\perp} \cap X = \text{the closure of span}\{E\} \text{ in } X \ .$$

If $T \in BL(X,Y)$, then

$$Z(T) = X \cap R(T^*)^{\perp} \ ,$$

(1.10)

$$\text{the closure of } R(T) \text{ in } X = X \cap Z(T^*)^{\perp} \ .$$

If $R(T)$ is closed, then

$$R(T^*) = Z(T)^{\perp} . \tag{1.11}$$

In general, does the closure of $R(T^*)$ in X^* equal $Z(T)^{\perp}$?

1.3 Let X and Y be Hilbert spaces and $T \in BL(X,Y)$. Then the closure of $R(T)$ (resp., $R(T^*)$) equals $Z(T^*)^{\perp}$ (resp., $Z(T)^{\perp}$) .

Also, $Z(T^*T) = Z(T)$ and the closure of $R(T^*T)$ equals the closure of $R(T^*)$. If $R(T)$ is closed, then $R(T^*)$ is closed and $R(T^*T) = R(T^*) = Z(T)^{\perp}$. Further, T^*T is invertible if and only if T is one to one and $R(T)$ is closed. (Hint: $R(T)$ is closed if and only if $\nu(T) = \inf\{\|Tx\| : x \in Z(T)^{\perp} , \|x\| = 1\} > 0$)

1.4 If $T \in BL(X,Y)$ is invertible, then $T^* \in BL(Y^*,X^*)$ is invertible and $(T^{-1})^* = (T^*)^{-1}$. The converse also holds.
(See (8.1).)

1.5 Let $X = L^2([a,b])$ or $C([a,b])$, and

$$Tx(s) = \int_a^b e^{st}x(t)dm(t) , \quad x \in X , \quad a \leq s \leq b .$$

If $X = L^2([a,b])$, then $T^* = T$, while if $X = C([a,b])$, then

$$T^*y(s) = \int_a^b \frac{e^{ts}-e^{ta}}{t}\, dy(t)$$

for every normalized function y of bounded variation on $[a,b]$; in particular, if $y \in C^1([a,b])$, then

$$T^*y(s) = \int_a^b \frac{e^{ts}-e^{ta}}{t}\, y'(t)dt .$$