5. RESOLVENT OPERATORS

In this section we define the spectrum and the resolvent set of $T \in BL(X)$. The analyticity and the power series expansion of the resolvent operator are the main considerations. We also obtain the spectral radius formula. This section lays the basis of the spectral theory.

Let $T \in BL(X)$. The <u>resolvent set of</u> T is defined and denoted as follows:

$$\rho(T) = \{z \in \mathbb{C} : T - zI \text{ is invertible in } BL(X)\} .$$

The <u>spectrum of</u> T is the complement of $\rho(T)$ in $\mathbb{C}$, and is denoted by $\sigma(T)$. It follows by the *open mapping theorem* ([L], 11.1) that $\lambda \in \sigma(T)$ if and only if either $T - \lambda I$ is not one to one, or it is not onto X . For $z \in \rho(T)$, the operator

$$R(T,z) = (T-zI)^{-1}$$

is called the <u>resolvent operator of</u> T <u>at</u> z . When there is no confusion possible, we shall denote it simply by $R(z)$.

It can be observed immediately that for any $z_0 \in \mathbb{C}$,

$$\sigma(T+z_0I) = \{\lambda + z_0 : \lambda \in \sigma(T)\} ,$$

and if $0 \notin \sigma(T)$, i.e., if T is invertible, then

$$\sigma(T^{-1}) = \{1/\lambda : \lambda \in \sigma(T)\} .$$

It then follows that for $z_0 \in \rho(T)$,

$$\sigma(R(z_0)) = \{1/(\lambda-z_0) : \lambda \in \sigma(T)\} . \tag{5.1}$$

In fact, it can be readily verified that for $z \in \rho(T)$ and $z \neq z_0$,

$$\left[R(z_0) - \frac{I}{z-z_0}\right]^{-1} = -(z-z_0)I - (z-z_0)^2R(z) \ . \tag{5.2}$$

In general, it is very difficult to find $R(z)$ for $z \in \rho(T)$. Only in some specific cases this is possible. For example, if $P \in BL(X)$ is a projection, then for all $z \neq 0$ or 1, $z \in \rho(T)$, and

$$R(z) = -\frac{I}{z} - \frac{P}{z(z-1)} \ . \tag{5.3}$$

Of course, if $P = 0$, then $\sigma(P) = \{0\}$, while if $P = I$, then $\sigma(P) = \{1\}$; in all other cases, $\sigma(P) = \{0,1\}$.

If X is finite dimensional, then it is theoretically possible to describe a construction of the resolvent operators. Let T be represented by a matrix $M = (t_{i,j})$ with respect to some ordered basis. Then $z \in \rho(T)$ if and only if the determinant of $M - zI$ is not zero, and in this case $R(z)$ is represented by the matrix $(s_{i,j})/\det(M-zI)$, where $s_{i,j}$ is the co-factor of the element $t_{j,i}$ in M. These results are proved in the first course on linear algebra.

Let us now consider some properties of the resolvent operators. We have the important relation

$$TR(z) = I + zR(z) = R(z)T \ , \quad z \in \rho(T) \ . \tag{5.4}$$

Also, for $z,w \in \rho(T)$, the <u>first resolvent identity</u>

$$R(z) - R(w) = (z-w)R(z)R(w) \tag{5.5}$$

follows by noting that $(z-w)I = (T-wI) - (T-zI)$ and applying $R(z)$ on the left and $R(w)$ on the right. From (5.4) and (5.5) it follows that $R(z)$ commutes with T and with $R(w)$.

The concept of the spectral radius of an operator plays an important role in spectral theory. The <u>spectral radius of</u> T is defined by

$$r_\sigma(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} .$$

We shall see that $r_\sigma(T)$ is finite and the supremum is attained at some $\lambda \in \sigma(T)$. We shall also give an expression for $r_\sigma(T)$ in terms of the norms of the operators T^k, $k = 1,2,\ldots$. For the present, we note that if $z \in \rho(T)$, then by (5.1),

$$\begin{aligned} r_\sigma(R(z)) &= \sup\{|\lambda| : \lambda \in \sigma(R(z))\} \\ &= \sup\{1/|\lambda-z| : \lambda \in \sigma(T)\} , \\ &= 1/\mathrm{dist}(z,\sigma(T)) . \end{aligned} \tag{5.6}$$

We now prove the analyticity of the resolvent operator.

THEOREM 5.1 (First Neumann expansions) Let $T \in BL(X)$. The resolvent set $\rho(T)$ is open, and the map $z \mapsto R(z) \in BL(X)$ is analytic in $\rho(T)$ as well as at infinity.

In fact, for $z_0 \in \rho(T)$ and $|z-z_0| < \mathrm{dist}(z_0,\sigma(T))$, we have

$$R(z) = \sum_{k=0}^{\infty} [R(z_0)]^{k+1}(z-z_0)^k . \tag{5.7}$$

Also, for $|z| > r_\sigma(T)$, we have

$$R(z) = - \sum_{k=0}^{\infty} T^k z^{-(k+1)} . \tag{5.8}$$

In fact, $r_\sigma(T) = \overline{\lim_{k\to\infty}} \|T^k\|^{1/k}$.

If $|z| > \|T\|$, then

$$\|R(z)\| \le \frac{1}{|z| - \|T\|} . \tag{5.9}$$

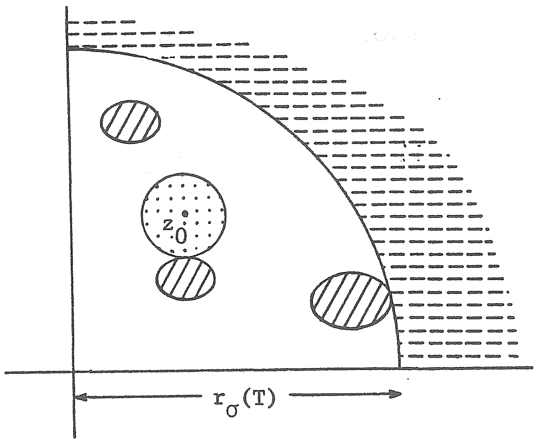

Figure 5.1

/// $\sigma(T)$

⋮⋮ the expansion (5.7) holds here

☰ the expansion (5.8) holds here

Proof We first note that if for some $z \in \mathbb{C}$, the series on the right side of either (5.7) or (5.8) converges in BL(X), then $z \in \rho(T)$ and the sum of the series is $R(z)$. To see this, let $z_0 \in \rho(T)$ and let the sum of the series on the right side of (5.7) be $f(z)$. If $a_k = [R(z_0)]^{k+1}$, $k = 0,1,\ldots,$ and $f_n(z) = \sum_{k=0}^{n} a_k(z-z_0)^k$, then since a_k commutes with $(T-zI)$, we have

$$\begin{aligned} f_n(z)(T-zI) &= (T-zI)f_n(z) \\ &= [(T-z_0I)-(z-z_0)]f_n(z) \\ &= I + \sum_{k=1}^{n} a_{k-1}(z-z_0)^k - \sum_{k=0}^{n} a_k(z-z_0)^{k+1} \\ &= I - a_n(z-z_0)^{n+1} . \end{aligned}$$

Since the n-th term of a convergent series tends to zero as $n \to \infty$, we see that $f(z)(T-zI) = (T-zI)f(z) = I$, i.e., $z \in \rho(T)$ and $f(z) = R(z)$. In an exactly similar manner one proves the result for the series on the right side of (5.8).

Now, by Theorem 4.8, the series (5.7) converges if

$$\begin{aligned} |z-z_0| &< 1 \,/\, \overline{\lim_{k\to\infty}} \, \|[R(z_0)]^{k+1}\|^{1/k} \\ &= 1 \,/\, \overline{\lim_{k\to\infty}} \, \|[R(z_0)]^{k}\|^{1/k} \,, \end{aligned}$$

so that every such z is in $\rho(T)$. Thus, $\rho(T)$ is open in $\mathbb{C}$ and it follows by Theorem 4.8 that $R(z)$ is an analytic function on $\rho(T)$. Since $\sigma(T)$ is then a closed set and the open disk

$$D = \{z \,:\, |z-z_0| < \operatorname{dist}(z_0,\sigma(T))\}$$

is clearly contained in $\rho(T)$, $R(z)$ is analytic on D and the Taylor expansion (5.7) of $R(z)$ is valid for $z \in D$.

Next, it follows similarly (cf. (4.14)) that the series (5.8) converges if

$$|z| > \overline{\lim_{k\to\infty}} \, \|T^{k-1}\|^{1/k} = \overline{\lim_{k\to\infty}} \, \|T^{k}\|^{1/k} = r \,, \text{ say} \,.$$

Hence $|z| > r$ implies $z \in \rho(T)$, i.e., $r_\sigma(T) \le r$. On the other hand, since $R(z)$ is analytic on $\tilde{D} = \{z : |z| > r_\sigma(T)\}$, the Laurent expansion (5.8) of $R(z)$ is valid in $\tilde{D}$ (Theorem 4.9). In particular, the series (5.8) converges if $|z| > r_\sigma(T)$. But by Corollary 4.7, it diverges if $|z| < r$. This shows that $r \le r_\sigma(T)$. Thus,

$$r_\sigma(T) = \overline{\lim_{k\to\infty}} \, \|T^{k}\|^{1/k} \,.$$

Since for $|z| > r_\sigma(T)$, and $w = 1/z$,

$$R(\tfrac{1}{w}) = - \sum_{k=0}^{\infty} T^k w^{k+1} \,,$$

we see that the function g defined by $g(w) = R(1/w)$ for $0 < |w| < 1/r_\sigma(T)$ and $g(0) = 0$, is analytic at 0 , i.e., $R(z)$ is analytic at $z = \infty$.

Finally, if $|z| > \|T\|$, then by (5.8),

$$\|R(z)\| \le \sum_{k=0}^{\infty} \frac{1}{|z|}\left[\frac{\|T\|}{|z|}\right]^k = \frac{1}{z - \|T\|} .$$

This proves (5.9). //

THEOREM 5.2 (Spectral radius formula) Let $X \neq \{0\}$. The spectrum $\sigma(T)$ of $T \in BL(X)$ is a nonempty compact subset of $\mathbb{C}$ and

$$\text{(5.10)} \qquad r_\sigma(T) = \lim_{k\to\infty} \|T^k\|^{1/k} = \inf\{\|T^k\|^{1/k} : k = 1,2,\dots\} .$$

Proof Since $\rho(T)$ is open (Theorem 5.1), its complement $\sigma(T)$ is closed in $\mathbb{C}$. Also, it is bounded since by Theorem 5.1,

$$r_\sigma(T) = \overline{\lim_{k\to\infty}} \|T^k\|^{1/k} \le \|T\| .$$

Thus, $\sigma(T)$ is a compact subset of $\mathbb{C}$. If $\sigma(T) = \emptyset$, then $R(z)$ would be analytic for all z in $\mathbb{C}$. Also, by (5.9) we see that $R(z) \to 0$ as $z \to \infty$. Thus, $R(z)$ would be a bounded function on $\mathbb{C}$, and by Liouville's theorem (Proposition 4.1(a)) it would reduce to a constant function, the constant being zero. This is clearly impossible, since the inverse of an operator on X cannot be zero unless $X = \{0\}$. Thus, $\sigma(T) \neq \emptyset$.

To prove (5.10), we note that if $\lambda \in \sigma(T)$, then $\lambda^k \in \sigma(T^k)$ for $k = 1,2,\dots,$ so that $|\lambda^k| \le \|T^k\|$, or $|\lambda| \le \|T^k\|^{1/k}$. Thus, by Theorem 5.1,

$$r_\sigma(T) \le \inf_{k=1,2,\dots} \|T^k\|^{1/k} \le \underline{\lim}_{k\to\infty} \|T^k\|^{1/k} \le \overline{\lim_{k\to\infty}} \|T^k\|^{1/k} = r_\sigma(T) .$$

Hence the limit exists and equals the infimum. //

COROLLARY 5.3 Let $T \in BL(X)$, and let Y be a closed subspace of X with $R(T) \subset Y$. Then

$$(5.11) \qquad r_\sigma(T_Y) \le r_\sigma(T) .$$

Proof By (5.10),

$$r_\sigma(T_Y) = \lim_{k\to\infty} \|(T_Y)^k\|^{1/k} = \lim_{k\to\infty} \|(T^k)_Y)\|^{1/k} \le \lim_{k\to\infty} \|T^k\|^{1/k} = r_\sigma(T) . \qquad //$$

We remark, however, that $\sigma(T_Y)$ and $\sigma(T)$ may not be comparable, as the following examples show.

Let $X = \mathbb{C}^2$, $T[x(1),x(2)]^t = [x(1),2x(2)]^t$ and $Y = \{[x(1),0]^t : x(1) \in \mathbb{C}\}$. Then

$$\sigma(T) = \{1,2\} \text{ and } \sigma(T_Y) = \{1\} .$$

Thus, $\sigma(T_Y)$ is a proper subset of $\sigma(T)$. Let $\tilde{X} = \ell^2(\mathbb{Z})$, the set of all doubly infinite square-summable sequences of complex numbers. Consider the right shift operator

$$\tilde{T}x(i) = x(i-1) , \quad i = 0,\pm1,\pm2,\ldots, \quad x \in \tilde{X} .$$

and let

$$\tilde{Y} = \{x \in X : x(n) = 0 \text{ for all } n = 0,-1,-2,\ldots\} .$$

Then $\sigma(\tilde{T}) = \{z \in \mathbb{C} : |z| = 1\}$ since $r_\sigma(\tilde{T}) \le \|\tilde{T}\| = 1$, $r_\sigma(\tilde{T}^{-1}) \le \|\tilde{T}^{-1}\| = 1$ and if $|z| = 1$, then $T - zI$ is not onto (the vector y defined by $y(0) = 1$, $y(i) = 0$ for $i \ne 0$ is not in its range). But $\sigma(\tilde{T}_{\tilde{Y}}) = \{z \in \mathbb{C} : |z| \le 1\}$ (cf. [L], 12.6). Thus, $\sigma(\tilde{T})$ is properly contained in $\sigma(\tilde{T}_{\tilde{Y}})$. Finally, let $X^{\#} = X \oplus \tilde{X}$, $T^{\#} = T \oplus \tilde{T}$ and $Y^{\#} = Y \oplus \tilde{Y}$. Then

$$\{2\} \subset \sigma(T^{\#}) \subset \{2\} \cup \{z \in \mathbb{C} : |z| = 1\} , \quad \sigma(T^{\#}_{Y^{\#}}) = \{z \in \mathbb{C}\ |z| \le 1\} .$$

Thus, neither $\sigma(T^{\#})$ nor $\sigma(T_Y^{\#}\#)$ is contained in the other.

COROLLARY 5.4 Let T_1 , $T_2 \in BL(X)$. Then

(5.12) $$r_\sigma(T_1T_2) = r_\sigma(T_2T_1) .$$

Proof Since for $k = 1,2,\ldots$

$$(T_1T_2)^k = T_1(T_2T_1)^{k-1}T_2 \quad \text{and} \quad (T_2T_1)^k = T_2(T_1T_2)^{k-1}T_1 ,$$

the desired result follows from the spectral radius formula (5.10). //

COROLLARY 5.5 Let U be an open subset of $\mathbb{C}$ and $f : U \to BL(X)$ be a continuous function. Then the real-valued function $z \mapsto r_\sigma(f(z))$ is upper semicontinuous on U , and as such it is bounded above and *attains* its maximum on each compact subset of U .

Proof For $k = 1,2,\ldots,$ the function

$$h_k(z) = \|[f(z)]^k\|^{1/k}$$

is upper semicontinuous (in fact, continuous) for $z \in U$. Now, by the spectral radius formula (5.10),

$$r_\sigma(f(z)) = \inf\{h_k(z) : k = 1,2,\ldots\} , \quad z \in U .$$

Let $z_0 \in U$ and $\epsilon > 0$. Then there exists an integer k such that

$$h_k(z_0) < r_\sigma(f(z_0)) + \epsilon/2 .$$

Since h_k is upper semicontinuous at z_0 , there is $\delta > 0$ such that $|z-z_0| < \delta$ implies $h_k(z) \leq h_k(z_0) + \epsilon/2$, so that

$$r_\sigma(f(z)) \leq h_k(z) < r_\sigma(f(z_0)) + \epsilon .$$

Thus, the function $z \mapsto r_\sigma(f(z))$ is upper semicontinuous. The remaining part is easy. //

The above result will be useful when we consider a perturbation V_0 of a given operator T_0 and let $f(z) = V_0R_0(z)$, where $R_0(z) = (T_0-zI)^{-1}$ for $z \in \rho(T_0)$. (See (9.14).)

Before we end this section, we consider operators whose spectra reduce to a single point; by means of a translation, we can take this point to be 0.

An operator $T \in BL(X)$ is said to be <u>quasi-nilpotent</u> if $\sigma(T) = \{0\}$. The spectral radius formula shows that T is quasi-nilpotent if and only if $\|T^k\|^{1/k} \to 0$ as $k \to \infty$. In particular, if $T^m = 0$ for some nonnegative integer m, then this condition is satisfied; such operators are called <u>nilpotent</u>.

PROPOSITION 5.6 Let $T \in BL(X)$ be quasi-nilpotent. If T is of finite rank m, then $T^{m+1} = 0$, so that T is nilpotent. In particular, if X is of dimension m, then $T^m = 0$.

Proof For $k = 1,2,\ldots,$ let $Y_k = R(T^k)$. Then $T(Y_k) \subset Y_k$, and

$$Y_1 \supset Y_2 \supset \ldots . \tag{5.13}$$

If T is of rank m, then $\dim Y_1 = m$; since each Y_k is then finite dimensional, it is closed in X (Proposition 3.1). Also, by Corollary 5.3

$$r_\sigma(T_{Y_k}) \leq r_\sigma(T) = 0 ,$$

so that $r_\sigma(T_{Y_k}) = 0$. Since $\sigma(T_{Y_k}) \neq \emptyset$ by Theorem 5.2, we see that $\sigma(T_{Y_k}) = \{0\}$. As Y_k is finite dimensional, T_{Y_k} cannot be onto. Thus, the inclusions in the descending chain (5.13) are proper at each stage. Hence $\dim Y_k \leq m - (k-1)$.

In particular,

$$\dim R(T^{m+1}) = \dim Y_{m+1} = 0 ,$$

i.e., $T^{m+1} = 0$.

If X is of dimension m , then $T : X \to X$ is not onto since $0 \in \sigma(T)$. Hence the rank of T is at most $m - 1$, so that $T^m = 0$ by what we have already proved. //

As an example, let $X = \mathbb{C}^n$, and for $x = [x(1),\dots,x(n)]^t \in \mathbb{C}^n$,

$$Tx = [x(2),\dots,x(k),0,\dots,0]^t$$

for some $2 \leq k \leq n$. The matrix representing T with respect to the standard basis is

$$\begin{bmatrix} 0 & 1 & & & & 0 & \\ & 0 & 1 & & & & \\ & & & \ddots & & & \\ & & & 0 & 1 & & \\ & & & & 0 & 0 & \\ & & 0 & & & \ddots & 0 \\ & & & & & & 0 \end{bmatrix},$$

where the 1's occur k times consecutively. Then we see that $T^{k+1} = 0$, but $T^k \neq 0$, $1 \leq k \leq n-1$.

In fact, if T is a nilpotent operator on an m dimensional space X , and if ℓ is the smallest positive integer with $T^{\ell} = 0$, then T can be represented, with respect to a suitable basis of X , in the Jordan canonical form as follows. (See [K], p.22.)

$$\left[\begin{array}{ccc:ccc:c:ccc}
J_1 & & & & & & & & & \\
& \ddots & & & & & & & & \\
& & J_{p_1} & & & & & & & \\ \hdashline
& & & J_{p_1+1} & & & & & & \\
& & & & \ddots & & & & & \\
& & & & & J_{p_2} & & & & \\ \hdashline
& & & & & & \ddots & & & \\ \hdashline
& & & & & & & J_{p_{\ell-1}+1} & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & J_{p_\ell}
\end{array}\right]$$

The integers $p_1, \ldots, p_\ell$ satisfy

$$1 \leq p_1 \leq \ldots \leq p_\ell \; , \tag{5.14}$$
$$p_1 + \ldots + p_\ell = m \; ;$$

each submatrix or block J_k is of the form

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} ,$$

and for each k satisfying $p_i + 1 \leq k \leq p_{i+1}$, $i = 0, \ldots, \ell-1$ ($p_0 = 0$) , the size of J_k is $\ell - i$. It follows by (5.14) that

$$\ell - 1 + p_\ell \leq m \leq \ell \, p_\ell \; . \tag{5.15}$$

Note that the total number of blocks of the form J_k is p_ℓ , and the size of the largest such block is ℓ .

Problems

5.1 For $A, B \in BL(X)$, $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$. In fact, if $0 \neq z \in \rho(AB)$, then $z \in \rho(BA)$ and $(BA-zI)^{-1} = [B(AB-zI)^{-1}A - I]/z$.

5.2 The map $z \mapsto r_\sigma(R(z))$ is continuous for $z \in \rho(T)$.

5.3 For $z_0 \in \rho(T)$, the k-th derivative of $R(z)$ at z_0 is $k![R(z_0)]^{k+1}$, $k = 1,2,\ldots$.

5.4 The radius of convergence of the power series (5.7) is $\mathrm{dist}(z_0,\sigma(T))$, so that $\rho(T)$ is the <u>natural domain of analyticity</u> of $R(z)$, i.e., for every $z_0 \in \rho(T)$, the radius of convergence of the Taylor expansion of $R(z)$ at z_0 equals $\mathrm{dist}(z_0,\mathbb{C}\setminus\rho(T))$. The series (5.8) diverges if $|z| < r_\sigma(T)$.

5.5 The sequence $(\|T^k\|^{1/k})$ is not always monotonically decreasing. (Let $X = \mathbb{C}^2$ and $T[x(1),x(2)]^t = [a^2x(2),b^2x(1)]^t$, where $a > b > 0$. Then $T^{2n} = (ab)^{2n}I$, $T^{2n+1} = (ab)^{2n}T$ for $n = 1,2,\ldots$, so that $\|T^{2n}\|_2^{1/2n} = ab$, $\|T\|_2 = a^2$, $\|T^{2n+1}\|_2^{1/(2n+1)} = ab(a/b)^{1/(2n+1)} > ab$.)

5.6 Let $r_\sigma(T) < 1$, and $y \in X$. Define $x_0 = y \neq 0$, and for $n = 1,2,\ldots$, let $x_n = y + Tx_{n-1}$. Then (x_n) converges to the unique solution x of $(I-T)x = y$. Let $\epsilon > 0$ be such that $r_\sigma(T) + \epsilon < 1$. Then there is n_0 such that for $n = 0,1,\ldots$,

$$\frac{\|x_n-x\|}{\|x\|} \leq \max_{k=0,\ldots,n_0} \left\{\frac{\|T^k\|}{[r_\sigma(T)+\epsilon]^k}\right\} [r_\sigma(T)+\epsilon]^{n+1} .$$

If $\|T\| < 1$, then for $n = 0,1,\ldots$, $\dfrac{\|x_n-x\|}{\|y\|} \leq \dfrac{\|T\|^{n+1}}{1 - \|T\|}$.

5.7 Let $T \in BL(X)$. The series $\sum_{k=0}^{\infty} T^k$ converges in $BL(X)$ if and only if $\|T^k\| < 1$ for some positive integer k.