8. SPECTRUM OF THE ADJOINT OPERATOR

In this section we investigate resolvent operators, spectral projections, reduced resolvents and quasi-nilpotent operators associated with the adjoint T^* of an operator $T \in BL(X)$. The underlying story behind these results is that the operation of taking an adjoint of an operator in $BL(X)$ is like the operation of taking the complex conjugate of a complex number. We shall see that points in the discrete spectrum of T^* correspond to points in the discrete spectrum of T ; thus the situation here is analogous to the finite dimensional case. The concept of a Rayleigh quotient is introduced and used to obtain estimates for an eigenvalue. We conclude this section by proving the spectral theorem for compact normal operators, and by pointing out some special results for self-adjoint operators.

THEOREM 8.1 Let $T \in BL(X)$. Then

(a) $$\rho(T^*) = \{\bar{z} : z \in \rho(T)\} ,$$

(8.1) $$\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\} ,$$

and for $z \in \rho(T)$, we have

(8.2) $$[R(T,z)]^* = R(T^*,\bar{z}) .$$

(b) Let Γ be a (positively oriented simple rectifiable closed) curve in $\rho(T)$, and let $\bar{\Gamma}$ be the conjugate curve. Then

(8.3) $$[P_\Gamma(T)]^* = P_{\bar{\Gamma}}(T^*) ,$$

(8.4) $$[S_\Gamma(T,z)]^* = S_{\bar{\Gamma}}(T^*,\bar{z}) \quad \text{for} \quad z \notin \Gamma ,$$

(8.5) $$[D_\Gamma(T,z)]^* = D_{\bar{\Gamma}}(T^*,\bar{z}) \quad \text{for} \quad z \in \mathbb{C} .$$

Proof (a) Let $z \in \rho(T)$. Then

$$(T-zI)R(T,z) = I = R(T,z)(T-zI) .$$

Taking adjoints on both sides, we have

$$[R(T,z)]^*(T^*-\bar{z}I) = I = (T^*-\bar{z}I)[R(T,z)]^* .$$

This shows that $\bar{z} \in \rho(T^*)$ and

$$R(T^*,\bar{z}) = [R(T,z)]^* .$$

Conversely, let $\bar{z} \in \rho(T^*)$. Then by Proposition 1.3(c),

$$[(T-zI)(X)]^{\perp} = Z(T^*-\bar{z}I) = \{0\} ,$$

so that the range of $(T-zI)$ is dense in X . That it is also closed in X (and hence equals X) can be seen as follows. Let $x \in X$ and find $x^* \in X^*$ such that

$$\langle x^*,x\rangle = \|x\| \quad \text{and} \quad \|x^*\| = 1$$

by Corollary 1.2. Then

$$\begin{aligned}\langle x^*,x\rangle &= \langle (T^*-\bar{z}I)(T^*-\bar{z}I)^{-1}x^*,x\rangle \\ &= \langle (T^*-\bar{z}I)^{-1}x^*,(T-zI)x\rangle .\end{aligned}$$

Thus,

$$\|x\| \leq \|(T^*-\bar{z}I)^{-1}\| \, \|(T-zI)x\| \tag{8.6}$$

by the fundamental inequality (1.3). Since X is complete, (8.6) implies that the range of $(T-zI)$ is closed in X . Thus, $(T-zI)$ is onto. The inequality (8.6) also shows that $(T-zI)$ is one to one, and that its inverse is bounded by $\|(T^*-\bar{z}I)^{-1}\|$. Hence $z \in \rho(T)$. Now, (8.1) follows.

(b) By (4.19), we have

$$
\begin{aligned}
[P_\Gamma(T)]^* &= \left[\frac{-1}{2\pi i}\int_\Gamma R(T,z)dz\right]^* \\
&= \frac{1}{2\pi i}\left[-\int_{\bar\Gamma}[R(T,\bar w)]^* dw\right] \\
&= \frac{-1}{2\pi i}\int_{\bar\Gamma} R(T^*,w)dw \text{ , by (8.2)} \\
&= P_{\bar\Gamma}(T^*) \text{ .}
\end{aligned}
$$

Similarly, for $z \notin \Gamma$,

$$
\begin{aligned}
[S_\Gamma(T,z)]^* &= \left[\frac{1}{2\pi i}\int_\Gamma \frac{R(T,w)}{w-z}\,dw\right]^* \\
&= \frac{1}{2\pi i}\int_{\bar\Gamma}\left[\frac{R(T,\bar w)}{\bar w - z}\right]^* dw \\
&= \frac{1}{2\pi i}\int_{\bar\Gamma}\frac{R(T^*,w)}{w-\bar z}\,dw \text{ , by (8.2)} \\
&= S_{\bar\Gamma}(T^*,\bar z) \text{ .}
\end{aligned}
$$

Finally, for $z \in \mathbb{C}$,

$$
\begin{aligned}
[D_\Gamma(T,z)]^* &= [(T-zI)P_\Gamma(T)]^* = [P_\Gamma(T)^*(T^*-\bar zI) \\
&= P_{\bar\Gamma}(T^*)(T^*-\bar zI) \text{ , by (8.3)} \\
&= (T^*-\bar zI)P_{\bar\Gamma}(T^*) \\
&= D_{\bar\Gamma}(T^*,\bar z) \text{ .} \qquad //
\end{aligned}
$$

COROLLARY 8.2 (a) λ is an isolated point of $\sigma(T)$ if and only if $\bar\lambda$ is an isolated point of $\sigma(T^*)$.

(b) λ is a pole of $R(T,z)$ of order ℓ if and only if $\bar\lambda$ is a pole of $R(T^*,z)$ of order ℓ .

(c) λ is a discrete spectral value of T if and only if $\bar\lambda$ is a discrete spectral value of T^* ; the algebraic (resp., geometric) multiplicity of λ as an eigenvalue of T equals the algebraic (resp., geometric) multiplicity of $\bar\lambda$ as an eigenvalue of T^* .

Proof (a) is a direct consequence of (8.1).

(b) λ is a pole of order ℓ of $R(T,z)$ if and only if λ is an isolated point of $\sigma(T)$ and

$$[D_\lambda(T,\lambda)]^\ell = 0 \ , \ [D_\lambda(T,\lambda)]^{\ell-1} \neq 0 \ .$$

But this happens if and only if $\bar{\lambda}$ is an isolated point of $\sigma(T^*)$ and

$$[D_{\bar{\lambda}}(T^*,\bar{\lambda})]^\ell = 0 \ , \ [D_{\bar{\lambda}}(T^*,\bar{\lambda})]^{\ell-1} \neq 0 \ ,$$

since by (8.5), we have $D_{\bar{\lambda}}(T^*,\bar{\lambda}) = [D_\lambda(T,\lambda)]^*$. (Recall Proposition 1.3(a).)

(c) We have $\lambda \in \sigma_d(T)$ if and only if λ is an isolated point of $\sigma(T)$ and rank $P_\lambda(T) < \infty$. By (8.1), (8.3) and Theorem 3.7,

$$\text{rank } P_{\bar{\lambda}}(T^*) = \text{rank}[P_\lambda(T)]^* = \text{rank } P_\lambda(T) \ .$$

Also, $\bar{\lambda} \in \sigma_d(T^*)$ if and only if $\bar{\lambda}$ is an isolated point of $\sigma(T^*)$ and rank $P_{\bar{\lambda}}(T^*) < \infty$. Thus,

$$\sigma_d(T^*) = \{\bar{\lambda} : \lambda \in \sigma_d(T)\} \ ,$$

and the corresponding algebraic multiplicities of λ and $\bar{\lambda}$ are equal. Finally, let $Y = R(P_\lambda(T))$ and $Z = Z(P_\lambda(T))$. Then by (2.2), $Z^\perp = R([P_\lambda(T)]^*) = R([P_{\bar{\lambda}}(T^*)]$, which is the spectral subspace associated with T^* and $\bar{\lambda}$. The map $A = (T-\lambda I)^*|_{Z^\perp}$ can be identified with the map B^* , where $B = (T-\lambda I)|_Y$, by Proposition 2.2. Since $Z((T-\lambda I)^*) \subset Z^\perp$ and $Z(T-\lambda I) \subset Y$, we have

$$\dim Z((T-\lambda I)^*) = \dim Z(A) = \dim Z(B^*) \ ,$$

$$\dim Z(T-\lambda I) = \dim Z(B) \ .$$

But since $\dim Y < \infty$, we have

$$\text{rank } B + \dim Z(B) = \dim Y = \dim Y^* = \text{rank } B^* + \dim Z(B^*) \ .$$

As rank B = rank B^* , we see that $\dim Z(B) = \dim Z(B^*)$. This shows that the geometric multiplicities of λ and $\bar{\lambda}$ are equal. //

Part (c) of the above corollary extends some well known linear algebra results to the discrete spectral values of an infinite dimensional operator T ; in particular, these results are applicable to the nonzero spectral values of a compact operator. Also, if $\lambda \in \sigma_d(T)$, then the nature of the solutions of the operator equation

$$Tx - \lambda x = y \ , \quad x,y \in X$$

can be described in terms of the solutions of the equation

$$T^*x^* - \bar{\lambda}x^* = y^* \ , \ x^*,y^* \in X^* \ .$$

See Problem 8.2, which gives an analogue of the Fredholm alternative.

If, however, an eigenvalue λ of T is not in the discrete spectrum of T , then $\bar{\lambda}$ need not be an eigenvalue of T^* . For example, let $X = \ell^2$, and for $[x(1),x(2),\ldots]^t \in \ell^2$, let

$$T[x(1),x(2),\ldots]^t = [x(2),x(3),\ldots]^t \ .$$

Then every λ with $|\lambda| < 1$ is an eigenvalue of T , but T^* has no eigenvalues at all ([L], 12.6(c) and Problem 12(vii)).

We now state a useful result which shows that if $\lambda \in \sigma_d(T)$, then the associated spectral projection has a simple representation that does not involve an integral.

THEOREM 8.3 Let $\lambda \in \sigma_d(T)$, m be its algebraic multiplicity, and ℓ be the order of the pole of $R(z)$ at λ . Let $x_1,\ldots,x_m$ form an ordered basis of the generalized eigenspace $Z((T-\lambda I)^{\ell})$ of T corresponding to λ .. There is a unique basis $\{x_1^*,\ldots,x_m^*\}$ of the generalized eigenspace $Z((T^*-\bar{\lambda}I)^{\ell})$ of T^* corresponding to $\bar{\lambda}$ such that

$$\langle x_j^*, x_i \rangle = \delta_{i,j} , \quad 1 \le i,j \le m .$$

Also, if P (resp., P^*) denotes the spectral projection associated with T and λ (resp., T^* and $\bar{\lambda}$) , then

$$(8.7) \qquad Px = \sum_{j=1}^{m} \langle x, x_j^* \rangle x_j , \; x \in X$$

$$(8.8) \qquad P^*x^* = \sum_{j=1}^{m} \langle x^*, x_j \rangle x_j^* , \; x^* \in X^* .$$

If, in particular, λ is semisimple, then $x_1,\ldots,x_m$ (resp., $x_1^*,\ldots,x_m^*$) form, in fact, an ordered basis of the eigenspace of T (resp., T^*) corresponding to λ (resp., $\bar{\lambda}$) .

Proof We have $R(P) = Z((T-\lambda I)^{\ell})$ by Lemma 7.1(b), and

$$Z((T^*-\bar{\lambda}I)^{\ell}) = R(P^*) = Z(P)^{\perp} ,$$

by Corollary 8.2(b) and (2.2). Letting $Y = R(P)$ and $\tilde{Z} = Z(P)$ in Theorem 3.2, we see that there are unique $x_1^*,\ldots,x_m^*$ in $\tilde{Z}^{\perp} = Z((T^*-\bar{\lambda}I)^{\ell})$ such that $\langle x_j^*, x_i \rangle = \delta_{i,j}$. The formulae (8.7) and (8.8) then follow from (3.3) and (3.4).

If λ is semisimple, i.e., $\ell = 1$, then $R(P) = Z(T-\lambda I)$ is the eigenspace of T corresponding to λ , and similarly for $R(P^*)$. The last statement of the theorem now follows. //

If X is a Hilbert space, $T \in BL(X)$ and $0 \neq x \in X$, then the complex number

$$q(x) = \langle Tx,x\rangle / \|x\|^2$$

is called the <u>Rayleigh quotient of</u> T <u>at</u> x, and the vector

$$r(x) = Tx - q(x)x$$

is called the <u>residual of</u> T <u>at</u> x. Clearly, $r(x)$ is orthogonal to x and hence for any complex number z, we have

$$\begin{aligned}\|Tx - zx\|^2 &= \|[Tx - q(x)x] + [q(x)x - zx]\|^2 \\ &= \|Tx - q(x)x\|^2 + |q(x) - z|^2\|x\|^2 .\end{aligned}$$

Thus,

(8.9) $\min_{z\in\mathbb{C}}\|Tx - zx\| = \|Tx - z_0x\|$ if and only if $z_0 = q(x)$.

This is known as the <u>minimum residual property of the Rayleigh quotient</u>. Note that x is an eigenvector of T if and only if $r(x) = 0$, and in that case $q(x)$ is the corresponding eigenvalue.

The set of Rayleigh quotients of T is sometimes called the <u>numerical range of</u> T. It is a bounded set since $|q(x)| \leq \|T\|$ for every $x \neq 0$. An interesting property of the numerical range is that it is a *convex* subset of $\mathbb{C}$. (See [K], 2. of p.571 for a simple proof.)

More generally, if X is a Banach space, $T \in BL(X)$, $x \in X$ and $x^* \in X^*$ with $\langle x,x^*\rangle \neq 0$, we define the <u>generalized Rayleigh quotient of</u> T <u>at</u> (x,x^*) by

$$q(x,x^*) = \langle Tx,x^*\rangle / \langle x,x^*\rangle .$$

Notice that in case X is a Hilbert space and we let $x^* = x \neq 0$, then $q(x,x^*) = q(x,x) = q(x)$, as defined earlier.

Let φ be an eigenvector of T corresponding to an eigenvalue λ. Assume that $\bar{\lambda}$ is an eigenvalue of T^* with φ^* as a corresponding eigenvector. We have seen in Corollary 8.2(c) that this assumption is satisfied if $\lambda \in \sigma_d(T)$. Now, let $\psi \in X$ and $\psi^* \in X^*$ be such that $\langle\psi,\psi^*\rangle \neq 0$. Then writing $\psi = \varphi + (\psi-\varphi)$ and $\psi^* = \varphi^* + (\psi^*-\varphi^*)$, we have

$$q(\psi,\psi^*) = \frac{\langle T\varphi,\varphi^*\rangle + \langle T\varphi,\psi^*-\varphi^*\rangle + \langle\psi-\varphi,T^*(\varphi^*)\rangle + \langle T(\psi-\varphi),\psi^*-\varphi^*\rangle}{\langle\varphi,\varphi^*\rangle + \langle\varphi,\psi^*-\varphi^*\rangle + \langle\psi-\varphi,\varphi^*\rangle + \langle\psi-\varphi,\psi^*-\varphi^*\rangle}$$

$$= \frac{\lambda[\langle\varphi,\varphi^*\rangle + \langle\varphi,\psi^*-\varphi^*\rangle + \langle\psi-\varphi,\varphi^*\rangle] + \langle T(\varphi-\psi),\varphi^*-\psi^*\rangle}{\langle\varphi,\varphi^*\rangle + \langle\varphi,\psi^*-\varphi^*\rangle + \langle\psi-\varphi,\varphi^*\rangle + \langle\varphi-\psi,\varphi^*-\psi^*\rangle}.$$

Hence

$$q(\psi,\psi^*) - \lambda = \frac{\langle(T-\lambda I)(\varphi-\psi),\varphi^*-\psi^*\rangle}{\langle\psi,\psi^*\rangle}, \tag{8.10}$$

$$|q(\psi,\psi^*) - \lambda| \leq \frac{\|(T-\lambda I)\|}{|\langle\psi,\psi^*\rangle|}\|\varphi-\psi\|\;\|\varphi^*-\psi^*\|. \tag{8.11}$$

The above relation is useful in estimating the eigenvalue λ by $q(\psi,\psi^*)$ if we know some approximations ψ and ψ^* of the eigenvectors φ and φ^*, respectively. In case X is a Hilbert space and $|\lambda| = \|T\|$, then $\bar{\lambda}$ is, in fact, an eigenvalue of T^* and $\varphi^* = \varphi$ is a corresponding eigenvector. (See Problem 8.4.) If T is normal, then this is the case for every eigenvalue λ of T since by (1.8) we have $\|(T^*-\bar{\lambda})\varphi\| = \|(T-\lambda)\varphi\|$. Thus, in these cases if we take $\psi^* = \psi$, we have

$$|q(\psi) - \lambda| \leq \frac{\|(T-\lambda I)\|}{\|\psi\|^2}\|\varphi-\psi\|^2. \tag{8.12}$$

If $\|\varphi-\psi\|$ is of order ϵ, then $|q(\psi) - \lambda|$ is of order ϵ^2. This phenomenon is called the <u>superconvergence of the Rayleigh quotient.</u>

We now prove some special results regarding the spectrum of a normal operator.

THEOREM 8.4 Let T be a normal operator on a Hilbert space X.

(a)
$$\|T\| = r_\sigma(T) ,$$

and for $z \in \rho(T)$, we have

$$\|R(z)\| = 1 / \operatorname{dist}(z,\sigma(T)) . \tag{8.13}$$

(b) Let λ be an isolated point of $\sigma(T)$. Then λ is a semisimple eigenvalue of T, P_λ is the orthogonal projection onto the eigenspace of T corresponding to λ, $D_\lambda = 0$ and

$$\|S_\lambda\| = 1 / \operatorname{dist}(\lambda,\sigma(T)\setminus\{\lambda\}) . \tag{8.14}$$

Proof (a) For $x \in X$, we have

$$\begin{aligned} \|T^2x\|^2 &= \langle T^2x,T^2x\rangle = \langle T^*T^2x,Tx\rangle \\ &= \langle TT^*Tx,Tx\rangle = \langle T^*Tx,T^*Tx\rangle \\ &= \|T^*Tx\|^2 . \end{aligned}$$

Hence $\|T^2\| = \|T^*T\| = \|T\|^2$. For $j = 2,3,\ldots$, we have $T^{2^j} = (T^{2^{j-1}})^2$, where $T^{2^{j-1}}$ is normal. Hence by induction on j,

$$\|T^{2^j}\| = \|T\|^{2^j} ,$$

for all $j = 1,2,\ldots$. The spectral radius formula (5.10) now gives

$$r_\sigma(T) = \lim_{j\to\infty} \|T^{2^j}\|^{1/2^j} = \|T\| .$$

Since T is normal, we see that $R(z)$ is normal for every $z \in \rho(T)$, and

$$\|R(z)\| = r_\sigma(R(z)) = 1 / \operatorname{dist}(z,\sigma(T)) ,$$

by (5.6).

(b) Let λ be an isolated point of $\sigma(T)$. Then since $[P_\lambda(T)]^* = P_{\bar\lambda}(T^*)$ by (8.3), and since $R(T,z)$ commutes $R(T^*,w)$ for z near λ and w near $\bar\lambda$, we see that P_λ is a normal operator. But since P_λ is a projection, it follows by Proposition 2.3 that $R(P_\lambda)^\perp = Z(P_\lambda)$, i.e., P_λ is an orthogonal projection.

Next, since $D_\lambda = (T-\lambda I)P_\lambda$ is normal, we have

$$\|D_\lambda\| = r_\sigma(D_\lambda) = 0 ,$$

by (7.4). As $P_\lambda \neq 0$ and $D_\lambda = 0$, we see from (7.7) that λ is a pole of order 1 of $R(z)$, i.e., λ is a semisimple eigenvalue of T. (cf. Proposition 7.3.) Thus, by Lemma 7.1(b), $P_\lambda(X)$ is the eigenspace of T corresponding to λ.

Lastly, since $S_\lambda = \frac{1}{2\pi i}\int_\Gamma \frac{R(z)}{z-\lambda}\,dz$ is likewise normal, we have by (7.3),

$$\|S_\lambda\| = r_\sigma(S_\lambda) = 1 / \operatorname{dist}(\lambda,\sigma(T)\setminus\{\lambda\}) . \qquad //$$

THEOREM 8.5 Let T be a normal operator on a Hilbert space X.

(a) Let $\lambda \in \sigma(T)$. Then there is a sequence (x_n) in X such that $\|x_n\| = 1$ and

(8.15) $$Tx_n - \lambda x_n \to 0 .$$

For this sequence, we have

(8.16) $$\langle Tx_n, x_n\rangle = q(x_n) \to \lambda .$$

(b) (Krylov-Weinstein) Given $x \in X$ with $\|x\| = 1$ and $z \in \mathbb{C}$, there is $\lambda \in \sigma(T)$ such that

$$(8.17) \qquad |\lambda - z| \leq \|Tx - zx\| .$$

Proof (a) Since $(T-\lambda I)$ is not invertible in $BL(X)$, either its range is not dense in X, or it is not bounded below. In the former case, by Proposition 1.3(c), we have

$$Z(T^{*}-\bar{\lambda}I) = R(T-\lambda I)^{\perp} \neq \{0\} .$$

Hence there is $x \in X$ with $\|x\| = 1$ such that $\|(T^{*}-\bar{\lambda}I)x\| = 0$. By (1.8), we have $\|(T-\lambda I)x\| = 0$ and (8.15) is satisfied. In the latter case, it is obvious that (8.15) holds. Next,

$$|q(x_n) - \lambda| = |\langle Tx_n-\lambda x_n,x_n\rangle|$$

Hence (8.16) holds.

(b) If $z \in \sigma(T)$, there is nothing to prove. Let $z \in \rho(T)$. Then $x = R(z)(Tx-zx)$, so that

$$1 = \|x\| \leq \|R(z)\| \ \|Tx - zx\| ,$$

i.e., $\text{dist}(z,\sigma(T)) \leq \|Tx - zx\|$ by (8.13). This shows that there is $\lambda \in \sigma(T)$ satisfying (8.17). //

We now prove the spectral theorem for a compact normal operator. We have seen in Section 7 that if T is a compact operator on a Banach space X, then $\sigma(T)$ consists of a countable number of points, and each such point, except possibly the point 0, is in the discrete spectrum of T. If, in addition, T is a normal operator on a Hilbert space X, then we get a complete description of T in terms of its nonzero eigenvalues and corresponding eigenvectors.

THEOREM 8.6 Let T be a nonzero compact normal operator on a Hilbert space X. Let $\lambda_1, \lambda_2, \ldots$ be the distinct nonzero eigenvalues of T, arranged so that

$$|\lambda_1| \geq |\lambda_2| \geq \ldots .$$

Let P_j denote the orthogonal projection onto the eigenspace of T corresponding to λ_j. Then each P_j has finite rank and

$$P_j P_k = 0 , \quad j \neq k .$$

For $n = 1,2,\ldots,$ we have

$$\|T - \sum_{j=1}^{n} \lambda_j P_j\| = |\lambda_{n+1}| , \tag{8.18}$$

which tends to zero whenever the sequence (λ_j) is infinite, so that

$$T = \sum_{j=1}^{\infty} \lambda_j P_j . \tag{8.19}$$

Let (u_k), $k = n_{j-1} + 1, \ldots, n_j$, denote an ordered orthonormal basis of the eigenspace $Z(T-\lambda_j I)$, $j = 1,2,\ldots,(n_0 = 0)$, and let $\mu_k = \lambda_j$ for $n_{j-1} + 1 \leq k \leq n_j$. Then

$$Tx = \sum_{k=1}^{\infty} \mu_k \langle x, u_k \rangle u_k , \quad x \in X . \tag{8.20}$$

Also, if P_0 denotes the orthogonal projection onto $Z(T)$, then

$$P_0 P_j = 0 , \quad j = 1,2,\ldots ,$$

$$x = P_0 x + \sum_{j=1}^{\infty} P_j x , \quad x \in X . \tag{8.21}$$

Proof Since T is compact, we know that

$$\sigma(T) \setminus \{0\} = \sigma_d(T) \setminus \{0\} = \{\lambda_1, \lambda_2, \ldots\} ,$$

where $|\lambda_1| \geq |\lambda_2| \geq \ldots$. Since T is normal, each λ_j is a semisimple eigenvalue of T, and the associated spectral projection is the orthongonal projection P_j onto $Z(T-\lambda_j I)$ (Theorem 8.4(b)). Since $\lambda_j \in \sigma_d(T)$, each P_j is of finite rank, and since $\lambda_j \neq \lambda_k$, we see from Lemma 7.8 that $P_j P_k = 0$ if $j \neq k$.

For $n = 1,2,\ldots,$ let

$$Q_n = P_1 + \ldots + P_n .$$

Since $D_j = (T-\lambda_j I)P_j = 0$, we have

$$TQ_n = TP_1 + \ldots + TP_n = \lambda_1 P_1 + \ldots + \lambda_n P_n .$$

Now, by the spectral decomposition theorem (cf. (6.10)), the spectrum of $T(I-Q_n)$ can differ from $\{\lambda_{n+1},\lambda_{n+2},\ldots\}$ only by 0. Hence

$$r_\sigma(T(I-Q_n)) = |\lambda_{n+1}| .$$

But since $TQ_n = Q_n T$ and $Q_n^* = Q_n$, we conclude that $T(I-Q_n)$ is normal. Hence

$$\|T - \sum_{j=1}^{n} \lambda_j P_j\| = \|T(I-Q_n)\| = r_\sigma(T(I-Q_n)) ,$$

by Theorem 8.4(a). This proves (8.18). Now, whenever (λ_j) is infinite, it must tend to 0, since 0 is the only limit point of $\sigma(T)$. Thus, T is the limit in $BL(X)$ of $\sum_{j=1}^{n} \lambda_j P_j$. In other words, (8.19) holds. The representation (8.20) is immediate from (8.19) since $P_j x = \sum_{k=n_{j-1}+1}^{n_j} \langle x,u_k\rangle u_k$.

Now consider the orthogonal projection P_0 onto $Z(T)$. Let $x \in R(P_0)$, and $y \in R(P_j)$ for some $j = 1,2,\ldots$. Then by (1.8), $\|T^* x\| = \|Tx\| = 0$, while $Ty = \lambda_j y$. Hence

$$\bar{\lambda}_j \langle x,y\rangle = \langle x,Ty\rangle = \langle T^* x,y\rangle = 0 .$$

But $\lambda_j \neq 0$, so that $\langle x,y\rangle = 0$. This shows that $P_0 P_j = 0$ for $j = 1,2,\ldots$.

It is clear that $\{u_1,u_2,\ldots\}$ is an orthonormal set in $R(P_0)^{\perp}$. Let $x \in R(P_0)^{\perp}$ and $\langle x,u_k\rangle = 0$ for each $k = 1,2,\ldots$. Then by (8.20), we see that $Tx = 0$, i.e., $x \in Z(T) = R(P_0)$. But since $x \in R(P_0)^{\perp}$, we have $x = 0$. The Fourier expansion theorem ([L], 22.10) now shows that $\{u_1,u_2,\ldots\}$ is, in fact, an orthonormal basis of $R(P_0)^{\perp}$. Since $x - P_0x \in R(P_0)^{\perp}$ for every $x \in X$, we have

$$x - P_0x = \sum_{k=1}^{\infty} \langle x,u_k\rangle u_k = \sum_{j=1}^{\infty} P_jx .$$

This proves (8.21). //

A self-adjoint operator T on a Hilbert space is normal, and hence the results of Theorem 8.5, and of Theorem 8.6 (in case T is also compact) hold for T. There are some interesting results regarding the spectrum of a self-adjoint operator. By (1.9), the Rayleigh quotient $q(x)$ of T at $0 \neq x \in X$ is a real number. Let

$$m_T = \min\{q(x) : x \in X , \|x\| = 1\}$$
$$M_T = \max\{q(x) : x \in X , \|x\| = 1\} .$$

THEOREM 8.7 Let T be a self-adjoint operator on a Hilbert space X.

(a) The spectrum $\sigma(T)$ of T is contained in the closed interval $[m_T,M_T]$ of the real line, and m_T as well as M_T belong to $\sigma(T)$.

(b) (Kato-Temple) Let $x \in X$ with $\|x\| = 1$. Then

(8.22) $$\operatorname{dist}(q(x),\sigma(T)) \leq \|r(x)\| .$$

Consider $\lambda \in \sigma(T)$ such that $|q(x) - \lambda| = \operatorname{dist}(q(x),\sigma(T))$. Then

(8.23) $$|q(x) - \lambda| \leq \|r(x)\|^2 / \operatorname{dist}(q(x), \sigma(T)\backslash\{\lambda\}) .$$

Proof (a) By part (a) of Theorem 8.5, we see that every $\lambda \in \sigma(T)$ is the limit of a sequence of Rayleigh quotients. Since each Rayleigh quotient belongs to $[m_T, M_T]$, it follows that $\sigma(T) \subset [m_T, M_T]$.

We show that $m_T \in \sigma(T)$. Let $x_n \in X$ be such that $\|x_n\| = 1$ and $q(x_n) \to m_T$. Then $\langle (T-m_T)x_n, x_n \rangle \to 0$. It can be verified by using the *generalized Schwarz inequality* for $\langle (T-m_TI)x, y \rangle$ that

$$\|Tx_n - m_Tx_n\|^4 \leq \|T-m_TI\|^3 \langle (T-m_TI)x_n, x_n \rangle .$$

(cf. [L], p.257.) Hence $\|Tx_n - m_Tx_n\| \to 0$. This implies that $(T-m_TI)$ is not bounded below, so that $m_T \in \sigma(T)$. The proof for $M_T \in \sigma(T)$ is very similar.

(b) Let $x \in X$ with $\|x\| = 1$, and $q = q(x)$. By part (b) of Theorem 8.5 with $z = q$, we immediately obtain (8.22). Let $\lambda \in \sigma(T)$ such that $|q-\lambda| = \operatorname{dist}(q, \sigma(T))$, and

$$d = \operatorname{dist}(q, \sigma(T)\backslash\{\lambda\}) .$$

For $t \in [m_T, M_T]$, consider the function

$$f(t) = (t-\lambda)[t-(q-d)] = t^2 - (\lambda+q-d)t + \lambda(q-d) .$$

Since no $t \in (q-d, \lambda)$ lies in $\sigma(T)$, we see that $f(t) \geq 0$ for all $t \in \sigma(T)$. Hence ([L], 31.4 and 32.6)

$$\int_{m_T}^{M_T} f(t)d\alpha(t) \geq 0 ,$$

where $\alpha(t) = \langle P_tx, x \rangle$, $\{P_t\}$ being the normalized resoltuion of the identity associated with the self-adjoint operator T. But

$$\int_{m_T}^{M_T} t^2 d\alpha(t) = \langle T^2x,x\rangle = \|Tx\|^2 , \int_{m_T}^{M_T} t\, d\alpha(t) = \langle Tx,x\rangle = q ,$$

$$\int_{m_T}^{M_T} d\alpha(t) = \langle x,x\rangle = 1 .$$

Thus,

$$\|Tx\|^2 - (\lambda+q-d)q + \lambda(q-d) \geq 0 , \quad \text{or} \quad \|Tx\|^2 - q^2 \geq d(\lambda-q) .$$

Since $\|r(x)\|^2 = \langle Tx-qx, Tx-qx\rangle = \|Tx\|^2 - q^2$, we have

$$\lambda - q \leq \|r(x)\|^2 / d .$$

Similarly, by considering the interval $(\lambda, q+d)$ and the function $g(t) = (t-\lambda)[t-(q+d)]$, we obtain

$$q - \lambda \leq \|r(x)\|^2 / d .$$

The above two inequalities imply (8.23). //

Problems

8.1 Let X be a Hilbert space, and $T \in BL(X)$. Then $\|T\| = [r_\sigma(T^*T)]^{1/2}$. If T is normal and $z \in \rho(T)$, then $\|TR(z)\| = \max\{|\lambda|/|\lambda-z| : \lambda \in \sigma(T)\}$.

8.2 Let $\lambda \in \sigma_d(T)$. Then the dimension of the solution space $\{x \in X : Tx - \lambda x = 0\}$ is the same as the dimension of the solution space $\{x^* \in X^* : T^*x^* - \bar{\lambda}x^* = 0\}$. Let $\{x_1,\dots,x_g\}$ and $\{x_1^*,\dots,x_g^*\}$ be bases of these two spaces, respectively. Given $y \in X$ (resp., $y^* \in X^*$), the nonhomogeneous equation

$$Tx - \lambda x = y \quad (\text{resp., } T^*x^* - \bar{\lambda}x^* = y^*)$$

possesses a solution if and only if

$$\langle x_j^*, y\rangle = 0 \quad (\text{resp.}, \ \langle x_j, y^*\rangle = 0) \ , \ j = 1,\dots,g \ .$$

If x_0 (resp., x_0^*) is a solution of this equation, then its most general solution is

$$x_0 + c_1x_1 + \dots + c_gx_g \ (\text{resp.}, \ x_0^* + c_1x_1^* + \dots + c_gx_g^*)$$

where $c_1,\dots,c_n$ are complex numbers.

8.3 Let $X = \mathbb{C}^5$, and the operators T and T^* be given by the matrices

$$\begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar\lambda & 0 & 0 & 0 & 0 \\ 1 & \bar\lambda & 0 & 0 & 0 \\ 0 & 1 & \bar\lambda & 0 & 0 \\ 0 & 0 & 0 & \bar\lambda & 0 \\ 0 & 0 & 0 & 1 & \bar\lambda \end{bmatrix} ,$$

respectively. Then e_1 and e_4 are eigenvectors of T , while e_2 , e_3 and e_5 are generalized eigenvectors. But e_3 and e_5 are eigenvectors of T^* , while e_1 , e_2 and e_4 are generalized eigenvectors. (Cf. Theorem 8.3 for a nonsemisimple eigenvalue λ .)

8.4 Let X be a Hilbert space, $T \in BL(X)$ and $|\lambda| = \|T\|$. If $Tx - \lambda x = 0$, then $T^*x - \bar\lambda x = 0$. If $\|x_n\| = 1$ and $\|Tx_n - \lambda x_n\| \to 0$, then $\|T^*x_n - \bar\lambda x_n\| \to 0$.

8.5 Let T be a normal operator on a Hilbert space X , and λ be an isolated point of $\sigma(T)$. Then by (4.7) and (8.13),

$$D_\lambda = \frac{1}{2\pi i}\int_\Gamma (z-\lambda)R(z)dz = 0 \ ,$$

where Γ is a small circle with centre λ , proving that every isolated point of $\sigma(T)$ is a semisimple eigenvalue of T. (This proof does not use the spectral decomposition theorem.)

8.6 Let $x \in X$ with $\|x\| = 1$ and $T \in BL(X)$ be self-adjoint. Let $\lambda \in \sigma_d(T)$ be such that $|q(x)-\lambda| = d(q(x),\sigma(T)) = d_0$, say. Let P denote the orthogonal projection onto $Z(T-\lambda I)$. Assume that $Px \neq 0$ and let θ be the acute angle between x and Px. Then

$$\sin\theta \leq \left[\frac{\|r(x)\|^2-d_0^2}{d^2-d_0^2}\right]^{1/2}, \tag{8.24}$$

where $d = \mathrm{dist}(q(x),\sigma(T)\setminus\{\lambda\})$. In particular,

$$\sin\theta \leq \|r(x)\| / d. \tag{8.25}$$

8.7 Let X be a Hilbert space and λ be an isolated point of $\sigma(T)$, $T \in BL(X)$. Assume that P_λ is orthogonal. Then

$$S_\lambda S_\lambda^*|_{Z(P_\lambda)} = \left[(T^*-\bar{\lambda}I)(T-\lambda I)|_{Z(P_\lambda)}\right]^{-1},$$

$$\sigma(S_\lambda S_\lambda^*) = \{0\} \cup \left\{\frac{1}{\mu} : 0 \neq \mu \in \sigma((T^*-\bar{\lambda}I)(T-\lambda I))\right\} \tag{8.26}$$

$$\|S_\lambda\| = 1 / \inf\left\{\sqrt{\mu} : 0 \neq \mu \in \sigma((T^*-\bar{\lambda}I)(T-\lambda I))\right\}.$$