SOME BASIC SEQUENCES AND THEIR MOMENT OPERATORS

Rodney Nillsen*

1. INTRODUCTION

A well known result in Fourier analysis (see [3, p.107], for example) says that if the Fourier series of a continuous function on the circle group is lacunary, then the series converges uniformly to the function. Equivalently, if \( (\alpha(n)) \) is a lacunary sequence of positive integers (that is, \( \alpha(n+1) \alpha(n)^{-1} \geq \gamma > 1 \), for all \( n \) and some \( \gamma \)), then the sequence \( 1, e^{i\alpha(1)t}, e^{-i\alpha(1)t}, e^{i\alpha(2)t}, e^{-i\alpha(2)t}, \ldots \) is basic in \( C(0,2\pi) \).

On the other hand, Gurarii and Macaev ([5]) proved some analogues of this result for power sequences in \( C([0,1]) \) and \( L^p(0,1) \). Letting \( 1 \leq p < \infty \) and letting \( (\alpha(n)) \) be a given increasing sequence of positive numbers, they proved that \( (\alpha(n)) \) is lacunary if and only if \( (\alpha(n)^{1/p} \alpha(n)^{-1/p}) \) is basic in \( L^p(0,1) \), in which case this basic sequence is equivalent to the standard basis in \( \theta^p \). They also proved that \( (\alpha(n)) \) is lacunary if and only if \( (t^{\alpha(n)}) \) is basic in \( C([0,1]) \).

In [4], Edwards has considered, in a dual form, a related problem concerning sequences of measures on a compact Hausdorff space \( K \). If \( (\mu_n) \) is a weak* convergent sequence of measures on \( K \) which satisfies a one term recurrence relation, he gives conditions which ensure that \( \{(\int_K f d\mu_n) : f \in C(K)\} = c \). This result is closely related to the problem of finding conditions for \( (\mu_n) \) to be a basic sequence of measures on \( K \).

The present paper presents some analogues of the preceding results which are derived by considering a general problem in Banach spaces. Throughout, \( X \) will denote a given Banach space with dual \( X^* \), \( (b_n) \) will denote a given sequence of scalars, \( \sigma = (\nu_n) \) will denote a given sequence of vectors in \( X \) and \( \tau = (x_n) \) will denote the sequence in \( X \)

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given by the recurrence relation

\[(1.1) \quad x_n - b_n x_{n-1} = v_n, \quad \text{for } n \geq 1, \quad \text{where } x_0 = 0.\]

The general problem considered is to find conditions which ensure that if \( \sigma \) is basic then \( \tau \) is basic, and also to find when \( \sigma \) and \( \tau \) are equivalent bases. If \( (z_n) \) is a sequence in \( X \), the moment operator \( A \) of \( (z_n) \) is defined on \( X^* \) by \( (Az^*)(n) = z^*(z_n) \), for \( n \in \mathbb{N} \) and \( z^* \in X^* \). Whether \( (z_n) \) is basic can often be expressed in terms of the range of \( A \) ([2,7]). These results are discussed in section 2.

In section 3, basic sequences in a space \( L^p(S, \mathcal{S}, \mu) \) are constructed which are of the form \( (f|K_n) \), where \( (K_n) \) is an increasing sequence of sets in \( S \), \( f \) is a given \( \mathcal{S} \)-measurable function, and \( f|K_n \) is the function equal to \( f \) on \( K_n \) and 0 elsewhere. In section 4, some bases are constructed for some subspaces of \( L^p(\mathbb{R}) \) consisting of piecewise linear functions. By taking Fourier transforms in some of these results with \( p = 2 \), conditions are found for weighted sequences of Dirichlet and Fejér kernels in \( L^2(\mathbb{R}) \) to be basic. The dual versions of these results give statements about the ranges of the various moment operators. For example, the following conditions are equivalent, where \( 1 \leq p < \infty \), \( p^{-1} + q^{-1} = 1 \), and \( (a(n)) \) is an increasing sequence of positive numbers:

\[
(\alpha(n)) \quad \text{is lacunary,}
\]

\[
\left\{ \left( \alpha(n)^{-\frac{1}{2}+1/p} \int_{-\infty}^{\alpha(n)} (\alpha(n) - |t|) f(t) dt \right) : f \in L^p(\mathbb{R}) \right\} = \ell^q,
\]

\[
\left\{ \left( \alpha(n)^{-3/2} \int_{-\infty}^{\infty} \left( \frac{\sin \alpha(n)t}{t} \right)^2 f(t) dt \right) : f \in L^2(\mathbb{R}) \right\} = \ell^2.
\]

Some definitions and notation used throughout the paper now follow. All sequences \( (z_n) \) in \( X \) or elsewhere are understood to be of the form \( (z_n)_{n=1}^{\infty} \), unless indicated otherwise. If \( (z_n) \) is a sequence in \( X \), \( [z_n : n \in \mathbb{N}] \) denotes the Banach subspace of \( X \) generated by \( \{z_n : n \in \mathbb{N}\} \). If \( \lambda = (x_n) \) is a sequence in \( X \) we define

\[
A_{\lambda} = \left\{ d : d \text{ is a scalar sequence and } \sum_{n=1}^{\infty} d_n z_n \text{ converges in } X \right\}.
\]
Let $S_\lambda : A_\lambda \rightarrow X$ be given by $S_\lambda (d) = \sum_{n=1}^{\infty} d_n z_n$. If $S_\lambda$ is a bijection from $A_\lambda$ onto $[z_n : n \in \mathbb{N}]$, $\lambda$ is said to be basic in $X$ and to be a basis for $[z_n : n \in \mathbb{N}]$. If $\sigma$ and $\tau$ are two basic sequences in $X$, they are said to be equivalent if $A_\sigma = A_\tau$.

A sequence $\lambda = (z_n)$ in $X$ is basic in $X$ and $A_\lambda = \ell^p$ (for some $1 \leq p < \infty$) if and only if there are $A, B > 0$ such that

\begin{equation}
A\|d\|_p \leq \left\| \sum_{n=1}^{\infty} d_n z_n \right\| \leq B\|d\|_p, \quad \text{for all } d \in A_\lambda.
\end{equation}

Also, $\lambda$ is basic and $A_\lambda = c_0$ if and only if an equality of type (1.2) holds with $p = \infty$ (see [11, p.354-355] or [12, p.30] for these facts). In the case where $\lambda$ is basic in a Hilbert space, $\lambda$ is said to be Riesz basic if $A_\lambda = \ell^2$. Standard results on bases may be found in [11] and [12] and used without explicit reference. For convenience rather than necessity, spaces such as $L^p(\mathbb{R})$, $\ell^p$ will be taken to consist of real valued functions and sequences. The bounded continuous real valued functions on $\mathbb{R}$ are denoted by $C(\mathbb{R})$, and $C_0(\mathbb{R})$ denotes those functions in $C(\mathbb{R})$ vanishing at infinity. The characteristic function of a set $A$ is denoted by $\chi(A)$.

2. GENERAL RESULTS

If the given sequence $\sigma = (v_n)$ in $X$ is basic, there is a sequence $(f_n)$ in $X^*$ which is biorthogonal to $\sigma$. That is, $f_n(v_m) = 0$ if $m \neq n$ and $f_n(v_n) = 1$, for all $m, n$. If $(b_n)$ is a given sequence of scalars we let $x_n - b_n x_{n-1} = v_n$, as in (1.1), and let $h_n = f_n - b_{n+1} f_{n+1}$, for all $n$.

**Lemma 2.1.** If $\sigma = (v_n)$ is basic in $X$, then $(h_n)$ is a sequence in $X^*$ which is biorthogonal to $(z_n)$. Also,

\[ \sum_{i=1}^{n} h_i(x) x_i = \sum_{i=1}^{n+1} f_i(x) u_i - f_{n+1}(x) x_{n+1}, \quad \text{for } x \in X, \ n \in \mathbb{N}. \]

**Proof.** It is straightforward to prove this from (1.1) and the definition of $h_n$ (see also [4,p.11] and [11,p.29]).
THEOREM 2.2. Let \( \sigma = (v_n) \) be a basis for \( X \), let \( (b_n) \) be a sequence of scalars with \( b_1 = 0 \), let \( \tau = (x_n) \) be given by (1.1), and let \( 1 \leq p < \infty \). Then the following hold.

(2.1) If \( \sigma \) is bounded away from 0 and \( \tau \) is bounded, then \( \tau \) is a basis for \( X \). If \( \sigma \) is bounded and \( \tau \) is a basis for \( X \), then \( \tau \) is bounded.

(2.2) If \( \tau \) is a basis for \( X \) which is bounded away from 0, then \( \sigma \) is bounded away from 0.

(2.3) If \( A_\sigma \) is \( \ell^p \) or \( c_0 \), \( \tau \) is bounded if and only if \( \tau \) is a basis for \( X \).

(2.4) If \( \|b\|_\infty < 1 \) and \( \sigma \) is bounded, then \( A_\sigma = \ell^p \) (respectively \( c_0 \)) if and only if \( \tau \) is a basis for \( X \) and \( A_\tau = \ell^p \) (respectively \( c_0 \)).

Proof. As \( \sigma \) is a basis for \( X \), \( x = \sum_{n=1}^\infty f_n(x)v_n \), for all \( x \in X \). Assume that \( \sigma \) is bounded away from 0. Then \( \lim_{n \to \infty} f_n(x) = 0 \), for \( x \in X \). Hence, if \( \tau \) is bounded, we deduce from Lemma 2.1 that \( x = \sum_{n=1}^\infty h_n(x)x_n \), for all \( x \in X \), and it follows that \( \tau \) is a basis for \( X \). This proves half of (2.1).

Now let \( \sigma \) be bounded and \( \tau \) be a basis for \( X \). Because \( (h_n) \) is biorthogonal to \( \tau \), there is \( K > 0 \) so that \( \|x_n\| \|h_n\| \leq K \) for all \( n \). Thus,

\[
\|x_n\| \leq K \|h_n\|^{-1} \leq K \|v_n\| \|h_n(v_n)\|^{-1} \leq K \|v_n\|,
\]

so that \( \tau \) is bounded. This proves the rest of (2.1).

If \( \tau \) is a basis for \( X \) bounded away from 0, choose \( K \) as above and observe that, using Lemma 2.1,

\[
\|v_n\| = \|x_n - b_n x_{n-1}\| \geq \|h_n(x_n - b_n x_{n-1})\| \|h_n\|^{-1} \geq \|h_n\|^{-1} \geq K^{-1} \|x_n\|.
\]

Hence \( \sigma \) is bounded away from 0. This proves (2.2).
If $A_\sigma$ is $\ell_p$ or $c_0$, an inequality of type (1.2) holds, so $\sigma$ is bounded and also bounded away from 0. Hence (2.3) is a consequence of (2.1).

If $\|b\|_\infty < 1$ and $\sigma$ is bounded, use (1.1) to obtain
\[
\|x_n\| \leq \|v_n\| + \sum_{j=1}^{n-1} \|b\|_{\infty}^{n-j} \|v_j\|,
\]
\[
\leq (1 - \|b\|_\infty)^{-1} \sup\{|\|v_n\| : n \in \mathbb{N}\}.
\]
Hence $\tau$ is bounded. Now let $A_\sigma$ be $\ell_p$ (respectively, $c_0$). It follows from (2.3) that $\tau$ is a basis for $X$ so that
\[
\tag{2.6} x = \sum_{n=1}^{\infty} (f_n(x) - b_{n+1} f_{n+1}(x)) x_n = \sum_{n=1}^{\infty} f_n(x) v_n, \quad \text{for } x \in X.
\]
Hence, $(S^{-1} \circ S_\sigma)(d) = (I - SM)(d)$, for $d \in A_\sigma$, where $S,M$ are the operators given by
\[Sd = (d_{n+1}), \quad Md = (b_n d_n)\] and $I$ is the identity. $SM$ maps $\ell_p$ into $\ell_p$ (respectively $c_0$ into $c_0$) and $\|SM\| \leq \|b\|_\infty < 1$. Hence $I - SM$ is a bounded invertible operator on $\ell_p$ (respectively $c_0$) and $A_\tau = (I - SM) A_\sigma = \ell_p$. This proves half of (2.4). For the other half, let $\tau$ be a basis with $A_\tau = \ell_p$ (respectively $c_0$). Then $\tau$ is bounded away from 0. By (2.3), $\sigma$ is bounded away from 0, so $A_\sigma \subseteq c_0$. It is easy to see that $I - SM$ is injective on $c_0$. Thus, as $\ell_p = A_\tau = (I - SM) A_\sigma$, we deduce that $A_\sigma = \ell_p$ (respectively, $c_0$). This proves (2.4).

To prove (2.5), observe that $\|I - SM\| \leq 1 + \|b\|_\infty$ and $\|(I - SM)^{-1}\| \leq (1 - \|b\|_\infty)^{-1}$. Then (1.2) and (2.6) give
\[
A \|d\|_p \leq \sum_{n=1}^{\infty} \|((I - SM)d)_n x_n\| \leq B \|d\|_p, \quad \text{for } d \in A_\sigma.
\]
Replacing $d$ by $(I - SM)^{-1}(d)$ now gives (2.5).

**Corollary 2.3.** Let $\sigma = (v_n)$ be a bounded basis for $X$ which is also bounded away from 0. Let $(d_n)$ be a sequence of non-zero scalars, let $y_n = \sum_{j=1}^{n} d_j v_j$ and let $\lambda = (d_n^{-1} y_n)$. Then the following conditions are equivalent: (i) $\lambda$ is basic in $X$, (ii) $\lambda$ is bounded, and (iii) $(d_n^{-1} y_n)$ is bounded. If there is $\theta < 1$ so that $|d_{j-1} d_j^{-1}| \leq \theta$ for all $j \geq 2$, then conditions (i) to (iii) do hold, and $A_\sigma = \ell_p$ (respectively $c_0$) if and only if $A_\lambda = \ell_p$ (respectively $c_0$).
Proof. If \( x_n = d_n^{-1} y_n, \) \( b_n = d_{n-1} d_n^{-1}, \) \( b_1 = 0 \) then \( x_n - b_n x_{n-1} = v_n, \) all \( n. \) The equivalence of (i), (ii) now follows from (2.1). As \( d_n^{-1} y_n - d_{n-1}^{-1} y_{n-1} = v_n \) and \( \sigma \) is bounded, (ii) and (iii) are equivalent. If \( \|b\|_{\infty} < 1, \) \( (x_n) \) is bounded and the remaining statements follow from (2.3) and (2.4).

REMARK. The equivalence of (i), (ii) and (iii) is known ([11, p.29]) and may be regarded as the special case of (2.1) which arises when it is assumed that in the recurrence relation (1.1), \( b_n \neq 0 \) for all \( n. \)

THEOREM 2.4. Let \( X \) be reflexive, let \( \sigma = (v_n) \) be a basis for \( X \) with \( \|v_1\| = 1 \) and \( \|v_n\| \leq 1 \) for \( n \geq 2. \) Let \( \sigma' = (\|v_n\|^{-1} v_n) \) and assume that \( \lambda' \leq \Theta, \) for some \( 1 < p < \infty. \) For \( n \geq 1 \) let \( b_n = (1 - \|v_n\|^p)^{1/p} \) and let \( \tau = (x_n) \) be the sequence in \( X \) given by (1.1). Let \( (f_n), (h_n) \) be the sequences in \( X^* \) which are biorthogonal to \( \sigma, \tau \) respectively, as described in Lemma 2.1. Then the following conditions are equivalent.

\[
\begin{align*}
(2.7) & \quad [h_n : n \in \mathbb{N}] = X^*, \\
(2.8) & \quad \prod_{j=r}^{\infty} b_j = 0, \quad \text{for all } r \in \mathbb{N}, \quad \text{and} \\
(2.9) & \quad \sum_{j=1}^{\infty} \|v_j\|^p = \infty.
\end{align*}
\]

Proof. By reflexivity, (2.7) holds if and only if \( x \in X \) and \( h_n(x) = 0 \) for all \( n \) implies \( x = 0. \) Let \( x = \sum_{n=1}^{\infty} d_n\|v_n\|^{-1} v_n, \) where \( d \in \ell^p, \) be such that \( h_n(x) = 0 \) for all \( n. \) Then \( \|v_n\|^{-1} d_n = b_{n+1}\|v_{n+1}\|^{-1} d_{n+1} \) for all \( n. \)

If \( b_n = 0 \) for an infinite number of \( n, \) we deduce that \( d = 0. \) In this case (2.7) to (2.9) hold.

On the other hand suppose that there is \( q \) so that \( b_q = 0 \) and \( b_n \neq 0 \) for \( n > q. \) Then \( d_j = 0 \) for \( 1 \leq j \leq q - 1 \) and \( d_n = \|v_n\|\|v_q\|^{-1}(b_n b_{n-1} \cdots b_{q+1})^{-1} b_q \) for \( n > q. \) Hence

\[
\sum_{n=q+1}^{\infty} |d_n|^p = |d_q|^p \|v_q\|^{-p} \lim_{n \to \infty} (b_q b_{q-1} \cdots b_{p+1})^{-p}.
\]
As $d \in \ell^p$, either $d = 0$ or $\prod_{n=p+1}^{\infty} b_n \neq 0$. Hence (2.8) implies (2.7). The converse argument may be used to show that if (2.8) fails, there is $x \in X$, $x \neq 0$ so that $h_n(x) = 0$ for all $n$. Hence (2.7) implies (2.8).

Thus, (2.7) and (2.8) are equivalent, and the latter is equivalent to (2.9) by a standard result on infinite products ([9, p.292]).

**COROLLARY 2.5.** Let $H$ be a Hilbert space, let $(x_n)$ be a normalized sequence in $H$, and let $(b_n)$ be a scalar sequence such that $b_1 = 0$ and the projection of $x_n$ into $[x_j : 1 \leq j \leq n - 1]$ is equal to $b_n x_{n-1}$ for all $n \geq 2$. Let $v_1 = x_1$ and $v_n = x_n - b_n x_{n-1}$ for $n \geq 2$. Then $(x_n)$ is basic in $H$ if and only if $(v_n)$ is bounded away from $0$, in which case $(x_n)$ is Riesz basic. The subspaces $[x_n : n \in \mathbb{N}]$ and $[[\|v_n\|^2 - b_{n+1}\|v_{n+1}\|^2 : n \in \mathbb{N}]$ of $H$ are equal if and only if $\sum_{n=1}^{\infty} \|v_n\|^2 = \infty$.

**Proof.** Let $X = [x_n : n \in \mathbb{N}]$. Then $(v_n)$ is an orthogonal basis for $H$, and

$1 = \|x_n\|^2 = |b_n|^2 + \|v_n\|^2$. Hence $(v_n)$ is bounded away from $0$ if and only if $\|b\|_\infty < 1$. The first statement now follows from Theorem 2.2. The rest follows from Theorem 2.4 with $p = 2$.

The following result concerns the relationship between a sequence in $X$ and its associated moment operator. The result is essentially known (see [2], [7, Theorem 1] and [12, p.169], for similar results) and is included for completeness.

**THEOREM 2.6.** Let $\sigma = (z_n)$ be a sequence in $X$, let $M = [z_n : n \in \mathbb{N}]$ and for $x^* \in X^*$ let $S x^* = (x^*(z_n))$. Then the following hold.

(2.10) If $S(X^*)$ is equal to $\ell^r$ for some $1 \leq r \leq \infty$ (respectively $c_0$), then $S$ is bounded from $X^*$ onto $\ell^r$ (respectively, $c_0$).

(2.11) If $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, then $S(X^*) = \ell^q$ (respectively $\ell^r$) if and only if $\sigma$ is a basic sequence in $X$ which is equivalent to the standard basis in $\ell^q$ (respectively $c_0$).

(2.12) If $\sigma$ is a basic sequence in $X$ which is equivalent to the standard basis in $\ell^r$, ...
then $S(X^*) = \ell^\infty$.

(2.13) If $M$ is complemented in $X$ and $\pi$ is a projection from $X$ onto $M$, the restriction of $S$ to $\pi^*(M^*)$ is a bijection onto $S(X^*)$.

Proof. (2.10) follows from the closed graph theorem.

Now let $1 < p < \infty$ and $S(X^*) = \ell^p$. Define $T$ on $M^*$ by $T\mu = (\mu(z_n))$. Then $T(M^*) = \ell^p$ and $T$ is a bounded bijection from $M^*$ to $\ell^p$. Hence $T^*$ is a bounded bijection from $\ell^p$ to $M^{**}$. If $d \in \ell^p$ and $\mu \in M^*$ we have $(T^*d)(\mu) = \sum_{n=1}^\infty d_n \mu(z_n)$, and it is easy to see that this series converges uniformly on the unit ball in $M^*$. It follows that $\sum_{n=1}^\infty d_n z_n$ converges in $M$ and that $T^*d = \sum_{n=1}^\infty d_n z_n$, for $d \in \ell^p$. As $T^*$ is a bounded bijection onto $M^{**}$, it follows that $M = M^{**}$ and that $\sigma$ is a basic sequence with $A_\sigma = \ell^p$. When $p = \infty$ and $q = 1$, apply a similar argument to prove that $T^*$ is a bounded bijection from $c_0$ onto $M$ and that $T^*d = \sum_{n=1}^\infty d_n z_n$ for $d \in c_0$ -- then $\sigma$ is basic with $A_\sigma = c_0$. This proves part of (2.11).

Conversely, if $\sigma$ is basic and $A_\sigma = \ell^p$ for some $1 \leq p < \infty$, let $S_\sigma d = \sum_{n=1}^\infty d_n z_n$, for $d \in \ell^p$. Then $S_\sigma^* (X^*) = \ell^p$. As $S = S_\sigma^*$, this proves (2.12) and the rest of (2.11). The proof of (2.13) is straightforward.

3. BASES AND RESTRICTIONS

In this section, $(S, S, \mu)$ will denote a given measure space, $K = (K_n)$ will denote an increasing sequence of sets in $S$ such that $\mu(K_{n+1} - K_n) > 0$ for all $n$, and $f$ will denote a given $S$-measurable scalar valued function on $S$. It will be assumed that $1 \leq p \leq \infty$ is given and that, for all $n$, $f_X(K_n - K_{n-1})$ is a non-zero element of $L^p(S, S, \mu)$, where $K_0 = \emptyset$ when $n = 1$. We let $R(f, p, K)$ denote all functions $g$ in $L^p(S, S, \mu)$ such that $g = 0$ on $S - \bigcup_{n=1}^\infty K_n$ and on each set $K_n - K_{n-1}$, the restriction of $g$ is a multiple of the restriction of $f$. Then $R(f, p, K)$ is a Banach subspace of $L^p(S, S, \mu)$ and it is clear that $(f_X(K_n - K_{n-1}))$ is a basis for $R(f, p, K)$. This section is concerned with when $(f_X(K_n))$ is also a basis for
\( R(f, p, K) \). Let, for \( n \in \mathbb{N} \),

\[
(3.1) \quad f_n = \frac{1}{\|f(K_n)\|_p} f(K_n), \quad f_0 = 0,
\]
\[
b_n = \frac{1}{\|f(K_n)\|_p} \|f(K_{n-1})\|_p, \quad \text{and}
\]
\[
v_n = \frac{1}{\|f(K_n)\|_p} f(K_{n-1}).
\]

It is immediate from (3.1) that

\[
(3.2) \quad f_n - b_n f_{n-1} = v_n, \quad \text{for } n \in \mathbb{N}.
\]

**THEOREM 3.1.** Let \( \tau = (f_n) \) and consider the following conditions.

There is \( \delta > 0 \) such that for all \( n \in \mathbb{N} \),

\[
(3.3) \quad \|f(K_n - K_{n-1})\|_p \geq \delta \|f(K_n)\|_p,
\]
\[
(3.4) \quad \tau \text{ is a basis for } R(f, p, K), \text{ and}
\]
\[
(3.5) \quad \text{there is } \gamma > 1 \text{ such that for all } n \in \mathbb{N}, \|f(K_n)\|_p \geq \gamma \|f(K_{n-1})\|_p.
\]

Then if \( 1 \leq p \leq \infty \), (3.3) and (3.4) are equivalent. If \( 1 \leq p < \infty \), (3.3), (3.4) and (3.5) are equivalent and imply that \( A_r = \ell^p \). If \( p = \infty \) and (3.5) holds, (3.3) and (3.4) also hold and \( A_r = c_0 \).

Proof. (3.1) shows that \( b_1 = 0 \) and it follows from (3.2) that Theorem 2.2 applies. Also \( \tau \) is bounded, by (3.1). Now if (3.3) holds, \((v_n)\) is bounded away from 0 and (3.4) follows from (2.1). Conversely, if (3.4) holds, (3.3) is a consequence of (2.2).

When \( 1 \leq p < \infty \), it is easy to prove that (3.3) and (3.5) are equivalent. As (3.5) means that \( \|b\|_\infty < 1 \), it follows from (2.4) that \( A_r = \ell^p \).

When \( p = \infty \), (3.5) implies that \( \|f(K_n)\|_\infty = \|f(K_n - K_{n-1})\|_\infty \) so that (3.3) holds. (3.5) also implies that \( \|v_n\|_\infty = 1 \) and that \( \|b\|_\infty < 1 \), so that \( A_r = c_0 \) (where \( \sigma = (v_n) \)) and \( A_r = c_0 \) by (2.4). This completes the proof.

If \((\alpha(n))\) is a strictly increasing sequence of positive numbers let

\[
(3.6) \quad \gamma(\alpha) = \inf \{ \alpha(n+1)\alpha(n)^{-1} : n \in \mathbb{N} \} \quad \text{and} \quad \psi(\alpha) = \sup \{ \alpha(n+1)\alpha(n)^{-1} : n \in \mathbb{N} \}.
\]
We allow the possibility that $\psi(\alpha) = \infty$, in which case $\psi(\alpha)^{-1} = 0$. Clearly, $\gamma(\alpha) \geq 1$.

**COROLLARY 3.2.** Let $(\alpha(n))$ be a strictly increasing sequence of positive real numbers and let $1 \leq p < \infty$. Then $\gamma(\alpha) > 1$ if and only if there are $C, D > 0$ such that

$$C \left( \sum_{n=1}^{r} |d_n|^p \right)^{1/p} \leq \left( \sum_{j=1}^{r} (\alpha(j) - \alpha(j-1)) \left| \sum_{n=j}^{r} \frac{d_n}{\alpha(n)^{1/p}} \right|^p \right)^{1/p} \leq D \left( \sum_{n=1}^{r} |d_n|^p \right)^{1/p},$$

for all scalars $d_1, d_2, \ldots, d_r$ and $r \in \mathbb{N}$. In this case we may take

$$C = \frac{(\gamma(\alpha) - 1)^{1/p}}{\gamma(\alpha)^{1/p} + 1} \quad \text{and} \quad D = \frac{\gamma(\alpha)^{1/p}}{\gamma(\alpha)^{1/p} - 1} (1 - \psi(\alpha)^{-1})^{1/p}.$$

**Proof.** Apply Theorem 3.1 to $L^p(\mathbb{R})$ with $f = 1$ and $K_n = (0, \alpha(n))$. Then $f_n = \alpha(n)^{-1/p} \chi(0, \alpha(n))$ and (3.5) holds if and only if $\gamma(\alpha) > 1$. Now observe that

$$\left\| \sum_{n=1}^{r} d_n f_n \right\|_p = \left( \sum_{j=1}^{r} (\alpha(j) - \alpha(j-1)) \left| \sum_{n=j}^{r} \frac{d_n}{\alpha(n)^{1/p}} \right|^p \right)^{1/p}.$$

Thus, an inequality of the above type is equivalent to saying that $\tau = (f_n)$ is basic in $L^p(\mathbb{R})$ with $A_\tau = \mathbb{E}$ (see (1.2)). The estimates for $C, D$ are consequences of applying (2.5) with $\sigma = (\alpha(n)^{-1/p} \chi((\alpha(n-1), \alpha(n))))$, $\tau$ as above and $b_n = \alpha(n-1)^{1/p} \alpha(n)^{-1/p}$. This completes the proof.

**PROPOSITION 3.3.** Let $(H, <, >)$ be a Hilbert space, let $(e_n)$ be a Riesz basis for $H$, let $(c_n)$ be a sequence of scalars and let $(\alpha(n))$ be a strictly increasing sequence of positive integers. Then the following conditions are equivalent.

(3.7) There is $\eta > 0$ such that for all $n \in \mathbb{N}$,

$$\left( \sum_{j=1}^{\alpha(n)} |c_j|^2 \right)^{-1} \left( \sum_{j=\alpha(n)+1}^{\alpha(n)} |c_j|^2 \right) \geq \eta.$$

(3.8) The sequence $\left( \sum_{j=1}^{\alpha(n)} c_j e_j \right)$ is basic in $H$.

(3.9) If we let

$$a_{j,k} = \frac{\sum_{r=1}^{\alpha(j)} \sum_{s=1}^{\alpha(k)} c_r c_s < e_r, e_s >}{\left( \sum_{r=1}^{\alpha(j)} |c_r|^2 \right)^{1/2} \left( \sum_{s=1}^{\alpha(k)} |c_s|^2 \right)^{1/2}},$$

...
then there are \( A, B > 0 \) such that for all scalar sequences \((d_n)\) of finite support,
\[
A \|d\|_2^2 \leq \left| \sum_{j,k=1}^{\infty} a_{j,k} d_j d_k \right| \leq B \|d\|_2^2.
\]

When the above conditions hold,
\[
\left( \sum_{j=1}^{\alpha(n)} |c_j|^2 \right)^{-1/2} \sum_{j=1}^{\alpha(n)} c_j e_j
\]
is Riesz basic in \( H \).

Proof. Apply Theorem 3.1 to \( \ell^2(\mathbb{N}) \), with \( f = (c_n) \) and \( K_n = \{1, 2, \ldots, \alpha(n)\} \). Then (3.7) is equivalent to (3.3) with \( p = 2 \). Let \( Jd = \left( \sum_{j=1}^{\infty} |d_j|^2 \right)^{-1/2} \left( \sum_{j=1}^{\infty} d_j e_j \right) \), for \( d \in \ell^2 \). Then \( J \) is an isomorphism from \( \ell^2(\mathbb{N}) \) onto \( H \) such that \( J(f \chi(K_n)) = \left( \sum_{j=1}^{\alpha(n)} |c_j|^2 \right)^{-1/2} \left( \sum_{j=1}^{\alpha(n)} c_j e_j \right) \).

Hence the equivalence of (3.7) and (3.8) is a consequence of the equivalence of (3.3) and (3.4). Condition (3.9) is equivalent to saying that \((J(f \chi(K_n)))\) is Riesz basic in \( H \). This observation and Theorem 3.1 give the remaining conclusions.

REMARKS. 1. An alternative proof of Proposition 3.3 may be based upon Corollary 2.5.

2. If \((c_n)\) is an orthonormal basis for \( H \) and \( c_n = 1 \) for all \( n \), then
\[
a_{j,k} = \text{minimum} \left( \alpha(j)^{1/2} \alpha(k)^{-1/2}, \alpha(k)^{1/2} \alpha(j)^{-1/2} \right).
\]

In this case the inequality (3.9) is the same as the one in Corollary 3.2 with \( p = 2 \).

COROLLARY 3.4. Let \((\alpha(n))\) be an increasing sequence of positive integers. For \( n \in \mathbb{N} \), let \( D_n(t) = \sin(n + \frac{1}{2})t/\sin \frac{1}{2}t \), for \( t \in (0,2\pi) \). Then \( \gamma(\alpha) > 1 \) if and only if \((D_{\alpha(n)})\) is basic in \( L^2(0,2\pi) \), in which case \((\alpha(n))^{-1/2} D_{\alpha(n)}\) is Riesz basic. If \( f \in L^2(0,2\pi) \), then \((D_{\alpha(n)} \ast f)\) is not basic in \( L^2(0,2\pi) \). If \( \gamma(\alpha) > 1 \), a function \( f \in L^2(0,2\pi) \) has a unique expression in \( L^2(0,2\pi) \) of the form
\[
\sum_{n=1}^{\infty} d_n \alpha(n)^{-1/2} D_{\alpha(n)}, \quad d \in \ell^2,
\]
if and only if the Fourier transform of \( f \) is constant on the set \( \{-\alpha(1), \ldots, \alpha(1)\} \) and also upon each set of the form \( \{-\alpha(n), \ldots, -\alpha(n-1) - 1\} \cup \{\alpha(n-1) + 1, \ldots, \alpha(n)\} \), for \( n \geq 2 \).
Proof. Apply Proposition 3.3 with $H = L^2(0,2\pi)$, $c_n = 1$ for all $n$, $e_1 = 1$ and $e_n(t) = e^{i(n-1)t} + e^{-i(n-1)t}$, for $n \geq 2$. Then (3.7), (3.8) imply that $\gamma(\alpha) > 1$ if and only if $(D_{\alpha(n)})$ is basic in $L^2(\mathbb{R})$. $D_{\alpha(n)} * f$ is the $n$th partial sum of the Fourier series of $f$, and $(D_{\alpha(n)} * f)$ is thus not basic by Corollary 2.3. Finally, observe that the Fourier transform of $f$ is constant on $\{-\alpha(1), \ldots, \alpha(1)\}$ and upon each set

$$\{-\alpha(n), \ldots, -\alpha(n-1) - 1\} \cup \{\alpha(n-1) + 1, \ldots, \alpha(n)\}$$

if and only if $f \in [D_{\alpha(n)} : n \in \mathbb{N}]$. This completes the proof.

REMARKS. A consequence of Corollary 3.6 is that there exist basic sequences $(D_{\alpha(n)})$ in $L^2(0,2\pi)$ such that for no $f \in L^2(0,2\pi)$ is $(D_{\alpha(n)} * f)$ basic in $L^2(0,2\pi)$.

COROLLARY 3.5. Let $(\alpha(n))$ be an increasing sequence of positive real numbers. For $\beta \in \mathbb{R}$, let $D_{\beta}^{\mathbb{R}}(t) = \sin \beta t$, for $t \in \mathbb{R}$. Then $\gamma(\alpha) > 1$ if and only if $(D_{\alpha(n)}^{\mathbb{R}})$ is basic in $L^2(\mathbb{R})$, in which case $(\alpha(n)^{-1/2} D_{\alpha(n)}^{\mathbb{R}})$ is Riesz basic. If $\gamma(\alpha) > 1$, a function $f \in L^2(\mathbb{R})$ has a unique expansion in $L^2(\mathbb{R})$ of the form $\sum_{n=1}^{\infty} d_n \alpha(n)^{-1/2} D_{\alpha(n)}^{\mathbb{R}}$, $d \in l^2$, if and only if the Fourier transform of $f$ is constant on each subset of $\mathbb{R}$ of the form $(-\alpha(n), -\alpha(n-1)] \cup [\alpha(n-1), \alpha(n))$.

Proof. This is similar to Corollary 3.4.

PROPOSITION 3.6. Let $1 \leq p < \infty$, let $(\alpha(n))$ be a strictly increasing sequence of positive integers, let $a_{ij} = \alpha(i)^{-1/p}$ for $1 \leq j \leq \alpha(i)$, and let $a_{ij} = 0$ if $j > \alpha(i)$. Let $A$ denote the operator obtained by multiplying by $(a_{ij})$. Then $A$ is a bounded operator from $\ell^p$ onto $\ell^q$ (where $p^{-1} + q^{-1} = 1$) if and only if $\gamma(\alpha) > 1$. In this case, the restriction of $A$ to the subspace of $\ell^q$ consisting of those sequences which are constant on each interval $\{\alpha(n-1) + 1, \ldots, \alpha(n)\}$ in $\mathbb{N}$ is a bounded invertible operator on $\ell^q$.

Proof. Let $a_n$ denote the $n$th row of $A$. Then by Theorem 3.1, $\sigma = (a_n)$ is basic in $\ell^p$ if and only if $\gamma(\alpha) > 1$, in which case $A_\sigma = \ell^q$. By (2.10), (2.11) and (2.12), $A$ is bounded.
from $\ell^p$ onto $\ell^q$. If $1 < p < \infty$ and $A(\ell^q) = \ell^q$, then (2.11) implies $\sigma$ is basic and thus $\gamma(\alpha) > 1$. If $p = 1$ and $A(\ell^\infty) = \ell^\infty$, we have for $d \in \ell^\infty$,

$$(Ad)(n) - (Ad)(n + 1) = \alpha(n + 1)^{-1} (\alpha(n + 1)\alpha(n)^{-1} - 1) \left( \sum_{i=1}^{\alpha(n)} d_i \right) - \alpha(n + 1)^{-1} \left( \sum_{i=\alpha(n)+1}^{\alpha(n+1)} d_i \right),$$

so that

$$|(Ad)(n) - (Ad)(n + 1)| \leq ||d||_{\infty} 2(1 - \alpha(n)\alpha(n + 1)^{-1}).$$

Hence, if $A(\ell^\infty) = \ell^\infty$, $\gamma(\alpha) > 1$.

Now let $M_p$ denote the subspace of $\ell^p$ consisting of those sequences which are constant on each interval $[\alpha(n - 1) + 1, \alpha(n)]$. Then if

$$(\pi d)_n = (\alpha(k) - \alpha(k - 1))^{-1} \left( \alpha(k) \sum_{i=\alpha(k-1)+1}^{\alpha(k)} d_i \right),$$

for $d \in \ell^p$ and $n \in [\alpha(k-1)+1, \alpha(k)]$, then $\pi$ is a projection from $\ell^p$ onto $M_p$ and $\pi^*(M_p^*) = M_q$. By (2.13) the restriction of $A$ to $M_q$ is a bounded invertible operator onto $\ell^q$, as required.

REMARK. Proposition 3.6 should perhaps be compared with the result ([1] and [6, p.239]) that if $p > 1$, the Cesàro operator is bounded on $\ell^p$, and with a recent result ([8]) on the partial invertibility of the Cesàro operator.

PROPOSITION 3.7. Let $1 \leq p < \infty$, let $(\alpha(n))$ be a strictly increasing sequence of positive integers and let

$$(Af)(n) = \alpha(n)^{-1/p} \int_{-\alpha(n)}^{\alpha(n)} f(t)dt,$$

for $n \in \mathbb{N}$ and $f \in L^q(\mathbb{R})$, where $p^{-1} + q^{-1} = 1$. Then $\gamma(\alpha) > 1$ if and only if $A$ is a bounded operator from $L^q(\mathbb{R})$ onto $\ell^q$. In this case the restriction of $A$ to the subspace of $L^q(\mathbb{R})$ consisting of those functions which are constant on each set $[-\alpha(n), -\alpha(n - 1)] \cup [\alpha(n - 1), \alpha(n)]$ is a bounded invertible operator onto $\ell^q$.

Proof. This is similar to Proposition 3.6.
THEOREM 3.8. Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and for $n \in \mathbb{N}$ let

$$w_n = \left( \int_{K_n - K_{n-1}} |f|^p d\mu \right)^{-1/q} \chi(K_n - K_{n-1})(\text{sign } f)|f|^{p-1},$$

and

$$h_n = \left( \int_{K_n - K_{n-1}} |f|^p d\mu \right)^{-1} \chi(K_n - K_{n-1})(\text{sign } f)|f|^{p-1} - \left( \int_{K_{n+1} - K_n} |f|^p d\mu \right)^{-1} \chi(K_{n+1} - K_n)(\text{sign } f)|f|^{p-1}.$$

Then $[w_n : n \in \mathbb{N}] = [h_n : n \in \mathbb{N}]$ in $L^q(S, \mathcal{S}, \mu)$ if and only if $\lim_{n \to \infty} \|f\chi(K_n)\|_p = \infty$.

Proof. Let $X = [v_n : n \in \mathbb{N}]$ in $L^p(S, \mathcal{S}, \mu)$. As the $v_n$ have disjoint supports, $\sigma' = (||v_n||^{-1} v_n)$ is a basis for $X$ and $A_{\sigma'} = \varnothing$. It is easy to check that $w_n \in L^q(S, \mathcal{S}, \mu)$, that $||w_n||_q = 1$ and that $\int_S v_n w_n d\mu = ||v_n||_p$. It follows that (||v_n||^{-1} w_n) is a sequence in $L^q(S, \mathcal{S}, \mu)$ which is biorthogonal to $(v_n)$. Also, $X^*$ is isometrically isomorphic to $[w_n : n \in \mathbb{N}]$ in $L^q(S, \mathcal{S}, \mu)$ under $T$, where $T \lambda = \sum_{n=1}^{\infty} \lambda(v_n)||v_n||^{-1} w_n$, for $\lambda \in X^*$. From (3.1) it follows that $b_n = (1 - ||v_n||^p)^{1/p}$, and, as $X$ is reflexive and (3.2) holds, we may apply Theorem 2.4. The result now follows from the equivalence of (2.7) and (2.8) by observing that, in the present context, (2.7) means that $[w_n : n \in \mathbb{N}]$ equals $[h_n : n \in \mathbb{N}]$ and (2.8) means that $\lim_{n \to \infty} \|f\chi(K_n)\|_p = \infty$. This completes the proof.

4. BASES IN SPACES OF PIECEWISE LINEAR FUNCTIONS

Let $\alpha = (\alpha(n))$ denote a given strictly increasing sequence of positive numbers and let $\gamma(\alpha)$ be defined as in (3.6). If $1 \leq p < \infty$, $PLC(p, \alpha)$ will denote the piecewise linear, even functions in $L^p(\mathbb{R})$ which are linear on each interval $[\alpha(n-1), \alpha(n))$, continuous on $\bigcup_{n=1}^{\infty} (-\alpha(n), \alpha(n))$, and zero off this union. Let $PLC_0(\infty, \alpha) = PLC(\infty, \alpha) \cap C_0(\mathbb{R})$. Then for $1 \leq p < \infty$, $PLC(p, \alpha)$ is a Banach subspace of $L^p(\mathbb{R})$. Also, $PLC_0(\infty, \alpha)$ is a Banach subspace of $C_0(\mathbb{R})$. Let

$$f(t) = \text{maximum}(0, 1 - |t|), \quad \text{for } t \in \mathbb{R},$$

and for $n \in \mathbb{N}$ and $t \in \mathbb{R}$ let

$$g_n(t) = 2^{-1/p}(p + 1)^{1/p} \alpha(n)^{-1/p} f(\alpha(n)^{-1} t), \quad \text{if } 1 \leq p < \infty,$$

and
\[ g_n(t) = f(\alpha(n)^{-1}t), \quad \text{if} \quad p = \infty. \]

Then, for \( 1 \leq p \leq \infty \), \( g_n \in PLC(p, \alpha) \) and \( \|g_n\|_p = 1 \). We also let, for \( n \in \mathbb{N} \) and \( t \in \mathbb{R} \),

\[
\begin{align*}
\phi_n(t) &= \frac{\alpha(n) - |t|}{\alpha(n) - \alpha(n - 1)}, \quad \text{for} \quad \alpha(n - 1) \leq |t| \leq \alpha(n), \\
\phi_n(t) &= 0, \quad \text{if} \quad |t| < \alpha(n - 1) \quad \text{or} \quad |t| > \alpha(n), \\
\phi'_n(t) &= \frac{|t| - \alpha(n - 1)}{\alpha(n) - \alpha(n - 1)}, \quad \text{for} \quad \alpha(n - 1) \leq |t| \leq \alpha(n), \\
\phi'_n(t) &= 0, \quad \text{if} \quad |t| < \alpha(n - 1) \quad \text{or} \quad |t| > \alpha(n), \quad \text{and} \\
z_n &= 2^{-1/p}(p + 1)^{1/p} \alpha(n)^{-1/p} (\phi'_{n-1} + \phi_n).
\end{align*}
\]

Let \( \phi_0 = 0 \) and \( \alpha(0) = \alpha(-1) = 0 \). Note that \( g_n, z_n \) depend upon \( p \). The function \( z_n \) is a type of Schauder hat function used in discussing bases of \( C([0,1]) \) (see [10, section 2.3]).

Expressions of the from \( a^{1/p}, (p + 1)^{1/p} \), etc., will be taken to be 1 when \( p = \infty \). The main result in this section is the following.

**THEOREM 4.1.** Let \( 1 \leq p < \infty \), let \( (g_n) \) be given by (4.1) and \( \nu = (z_n) \) be given by (4.2). Then \( \nu \) is a basis for \( PLC(p, \alpha) \) and \( \nu \subseteq A_\nu \). Also, if we consider the conditions

\[
\begin{align*}
(4.3) \quad A_\nu &= \mathcal{E}_\nu, \\
(4.4) \quad \gamma(\alpha) > 1, \\
(4.5) \quad (g_n) \text{ is a basis for } PLC(p, \alpha), \text{ and} \\
(4.6) \quad (\alpha(n)^{-3/2} t^{-2} \sin^2 2^{-1} \alpha(n) t) \text{ is basic in } L^2(\mathbb{R}),
\end{align*}
\]

then (4.4), (4.5) and (4.6) are equivalent, (4.4) implies (4.3), and if \( (\alpha(n) - \alpha(n - 1)) \) is increasing then (4.3) and (4.4) are equivalent. When conditions (4.4) to (4.6) hold, \( (g_n) \) is equivalent to the standard basis in \( \mathcal{E}_\nu \), and the sequence in (4.6) (which is a sequence of weighted Fejér kernels in \( L^2(\mathbb{R}) \)) is Riesz basic in \( L^2(\mathbb{R}) \).

The case \( p = \infty \) is covered by

**THEOREM 4.2.** Let \( p = \infty \), let \( (g_n) \) be given by (4.1) and let \( \nu = (z_n) \) be given by (4.2). Then \( \nu \) is a basis for \( PLC_0(\infty, \alpha) \) and \( A_\nu = c_0 \). Also, \( \gamma(\alpha) > 1 \) if and only if \( (g_n) \) is a basis for \( PLC_0(\infty, \alpha) \).
A function \( f \in \text{PLC}(\infty, \alpha) \) if and only if there exists a (necessarily unique) \( d \in \ell^\infty \) so that the series \( \sum_{n=1}^{\infty} d_n z_n \) converges uniformly to \( f \) on each compact subset of \( \mathbb{R} \).

The proofs of Theorems 4.1 and 4.2 require some preliminary results and observations. Let \( 1 \leq p \leq \infty \) be given. We define

\[
    r_n(t) = \alpha(n) - \alpha(n-1), \quad \text{for} \quad |t| \leq \alpha(n-1),
\]

\[
    r_n(t) = \alpha(n) - |t|, \quad \text{for} \quad \alpha(n-1) \leq |t| \leq \alpha(n), \quad \text{and}
\]

\[
    r_n(t) = 0, \quad \text{for} \quad |t| > \alpha(n).
\]

Also, let

\[
    w_n = 2^{-1/p}(p+1)^{1/p} \frac{\alpha(n) - \alpha(n-1)}{p} r_n.
\]

From (4.2) and (4.7) we now have

\[
    \|\phi_n\|_p = \|\phi'_n\|_p = 2^{1/p}(p+1)^{1/p} \frac{\alpha(n) - \alpha(n-1)}{p},
\]

\[
    \|z_n\|_p = (1 - \alpha(n-2)\alpha(n^{-1})^{1/p}, \quad \text{and}
\]

\[
    \|w_n\|_p = (1 + p\alpha(n-1)\alpha(n^{-1})^{1/p}.
\]

The sequences \((g_n)\) and \((w_n)\) also satisfy the following recurrence relations.

\[
    g_n - \alpha(n-1)^{1+1/p} \alpha(n)^{-1(1+1/p)} g_{n-1} = (1 - \alpha(n-1)\alpha(n^{-1})) w_n, \quad \text{and}
\]

\[
    w_n - \alpha(n-1)^{1/p} \alpha(n)^{-1/p} w_{n-1} = z_n, \quad \text{for all} \quad n \in \mathbb{N}.
\]

**Lemma 4.3.** Let \( 1 \leq p < \infty \) and let \( C_p = \inf\{(1+t)^{-1}(1+t^{p+1}): 0 \leq t \leq 1\} \). Then for all \( a, b \in \mathbb{R} \),

\[
    C_p^{1/p} 2^{1/p}(p+1)^{-1/p} \max \{ |a|, |b| \}
\]

\[
    \leq \|a\phi'_n + b\phi_n\|_p \leq 2^{1+1/p}(p+1)^{-1/p} \max \{ |a|, |b| \}.
\]

Proof. The right hand inequality follows easily from (4.8). For the left hand inequality, note that \( \|a\phi'_n + b\phi_n\|_p \) is symmetric in \( a, b \). If \( ab \geq 0 \) and \( |a| > |b| \),

\[
    \|a\phi'_n + b\phi_n\|_p = 2^{1/p}(p+1)^{-1/p} \alpha(n) - \alpha(n-1))^{1/p} |a| \left( \frac{1 - |b/a|^{p+1}}{1 - |b/a|} \right)^{1/p},
\]

\[
    \geq 2^{1/p}(p+1)^{-1/p} \max \{ |a|, |b| \}.
\]
If $a = b$, then
\[ \|a \phi'_n + b \phi_n\|_p = 2^{1/p} (\alpha(n) - \alpha(n-1))^{1/p}|a|. \]

If $ab < 0$ and $|a| \geq |b|$, then
\[ \|a \phi'_n + b \phi_n\|_p = 2^{1/p} (p+1)^{1-1/p} |a(n) - a(n-1))^{1/p}|a| \left( \frac{1 + |b/a|^{p+1}}{1 + |b/a|} \right)^{1/p}, \]
\[ \geq C_p^{1/p} (p+1)^{-1/p} |a(n) - a(n-1))^{1/p}|a|. \]

Lemma 4.3 now follows from these observations.

**LEMMA 4.4.** Let $1 \leq p < \infty$ and let $(d_n)$ be a sequence of scalars. Then the following conditions are equivalent.

(4.13) \[ \sum_{n=1}^{\infty} d_n z_n \text{ converges in } PLC(p, \alpha), \]

(4.14) \[ 2^{-1/p} (p+1)^{1/p} \left( \sum_{n=1}^{\infty} \left( d_{n+1} \alpha(n+1)^{-1/p} \phi'_n + d_n \alpha(n)^{-1/p} \phi_n \right) \right) \text{ converges in } L^p(\mathbb{R}), \]

(4.15) \[ \sum_{n=1}^{\infty} (\alpha(n) - \alpha(n-1)) \max \left( \frac{|d_n|^p}{\alpha(n)}, \frac{|d_{n+1}|^p}{\alpha(n+1)} \right) < \infty. \]

When these conditions hold, the sums of the series in (4.13), (4.14) are equal. If $d \in \mathcal{V}$, then \[ \sum_{n=1}^{\infty} d_n z_n \text{ converges in } PLC(p, \alpha). \] If $\gamma(\alpha) > 1$, \[ \sum_{n=1}^{\infty} d_n z_n \text{ converges in } PLC(p, \alpha) \text{ if and only if } d \in \mathcal{V}, \text{ and in this case,} \]

(4.16) \[ C_p^{1/p} (1 - \gamma(\alpha)^{-1})^{1/p} \|d\|_p \leq \left\| \sum_{n=1}^{\infty} d_n z_n \right\|_p \leq 2\|d\|_p. \]

**Proof.** First observe that
\[ \sum_{j=1}^{n} d_j z_j = 2^{-1/p} (p+1)^{1/p} \left\{ \left( \sum_{j=1}^{n} \left( d_{j+1} \alpha(j+1)^{-1/p} \phi'_j + d_j \alpha(j)^{-1/p} \phi_j \right) \right) + d_n \alpha(n)^{-1/p} \phi_n \right\}. \]

Now let (4.15) hold. Then by (4.8), \[ \lim_{n \to \infty} d_n \alpha(n)^{-1/p} \phi_n = 0. \] Also, by Lemma 4.3,
\[ \sum_{j=1}^{\infty} \left( d_{j+1} \alpha(j+1)^{-1/p} \phi'_j + d_j \alpha(j)^{-1/p} \phi_j \right)^p \]
\[ \leq 2^{p+1} (p+1)^{-1} \left( \sum_{j=1}^{\infty} (\alpha(n) - \alpha(n-1)) \max \left( \frac{|d_n|^p}{\alpha(n)}, \frac{|d_{n+1}|^p}{\alpha(n+1)} \right) \right), \]

\[ (4.16) \]

\[ \sum_{n=1}^{\infty} \left( d_{n+1} \alpha(n+1)^{-1/p} \phi'_n + d_n \alpha(n)^{-1/p} \phi_n \right)^p \]
\[ \leq 2^{p+1} (p+1)^{-1} \left( \sum_{j=1}^{\infty} (\alpha(n) - \alpha(n-1)) \max \left( \frac{|d_n|^p}{\alpha(n)}, \frac{|d_{n+1}|^p}{\alpha(n+1)} \right) \right), \]
and we deduce that the series in (4.13) and (4.14) converge and have equal sums.

Now let (4.13) hold. Then \( \lim_{n \to \infty} d_n z_n = 0 \) and it follows from (4.2) that
\[ \lim_{n \to \infty} d_n \alpha(n)^{-1/p} \phi_n = 0. \]
From the initial observation in the proof, we now see that (4.14) holds and that the series in (4.13), (4.14) have equal sums.

Let (4.14) hold. Then
\[ \sum_{n=1}^{\infty} \left\| \left( d_{n+1} \alpha(n+1)^{-1/p} \phi_n' + d_n \alpha(n)^{-1/p} \phi_n \right) \right\|_p^p < \infty. \]
Applying Lemma 4.3 shows that (4.15) then holds. This proves the equivalence of (4.13) to (4.15).

If \( d \in \ell^p \), (4.15) holds and hence (4.13) holds.

Now let \( \gamma(\alpha) > 1 \). Then if \( \sum_{n=1}^{\infty} d_n z_n \) converges, we deduce from Lemma 4.3 and (4.14) that
\[
C_p^2 (p+1)^{-1} (1 - \gamma(\alpha)^{-1}) \|d\|_p^p \leq C_p^2 (p+1)^{-1} \left( \sum_{n=1}^{\infty} (1 - \alpha(n-1) \alpha(n)^{-1}) |d_n|^p \right),
\]
\[
\leq C_p^2 (p+1)^{-1} \left( \sum_{n=1}^{\infty} (\alpha(n) - \alpha(n-1)) \max \left( \frac{|d_n|^p}{\alpha(n)}, \frac{|d_{n+1}|^p}{\alpha(n+1)} \right) \right),
\]
\[
\leq \left\| \sum_{n=1}^{\infty} \left( d_{n+1} \alpha(n+1)^{-1/p} \phi_n' + d_n \alpha(n)^{-1/p} \phi_n \right) \right\|_p^p,
\]
\[
= 2(p+1)^{-1} \left\| \sum_{n=1}^{\infty} d_n z_n \right\|_p^p.
\]
Hence \( d \in \ell^p \) and the left hand side of (4.16) holds.

If \( d \in \ell^p \),
\[
\left\| \sum_{n=1}^{\infty} d_n z_n \right\|_p \leq \left\| \sum_{n=1}^{\infty} d_{2n} z_{2n} \right\|_p + \left\| \sum_{n=1}^{\infty} d_{2n-1} z_{2n-1} \right\|_p
\]
\[
\leq 2\|d\|_p,
\]
as \( \|z_n\|_p \leq 1 \) by (4.9), and the \( z_{2n} \) have disjoint supports, as do the \( z_{2n-1} \). This proves the right hand side of (4.16).

**Lemma 4.5.** Let \( 1 \leq p < \infty \). Then a function is in \( PLC(p, \alpha) \) if and only if it is the sum of a convergent series in \( L^p(\mathbb{R}) \) which is of the form \( \sum_{n=1}^{\infty} (a_n \phi_n + a_{n+1} \phi_n') \).
Proof. The condition is clearly sufficient. For necessity, observe that if \( f \in PLC(p, \alpha) \), there are \( a_n, b_n \) so that \( f = a_n \phi_n + b_n \phi'_n \) on \([\alpha(n-1), \alpha(n)]\). As \( f \) is continuous, we must have \( b_n = a_{n+1} \). This completes the proof.

Proof of Theorem 4.1. Let \( 1 \leq p < \infty \) and let \( f \in PLC(p, \alpha) \). By Lemma 4.5 choose a sequence \( (a_n) \) so that \( f = \sum_{n=1}^{\infty} (a_n \phi_n + a_{n+1} \phi'_n) \) in \( L^p(\mathbb{R}) \). Let \( d_n = 2^{1/p} (p+1)^{-1/p} \alpha(n)^{1/p} a_n \).

Then by Lemma 4.4, \( f = \sum_{n=1}^{\infty} d_n z_n \), where this series converges in \( L^p(\mathbb{R}) \). Also, if \( \sum_{n=1}^{\infty} d_n z_n = 0 \), then \( d_n z_n + d_{n+1} z_{n+1} = 0 \) on \([\alpha(n-1), \alpha(n)]\). As \( z_n, z_{n+1} \) are independent on \([\alpha(n-1), \alpha(n)]\) we deduce that \( d_n = d_{n+1} = 0 \), hence \( d = 0 \). This proves that \( \nu = (z_n) \) is a basis for \( PLC(p, \alpha) \). Lemma 4.4 implies that \( \nu' \subseteq A_\nu \).

If \( \gamma(\alpha) > 1 \), Lemma 4.4 shows that \( A_\nu = \nu' \). If \( (\alpha(n) - \alpha(n-1)) \) is increasing, then (4.15) is equivalent to having \( \sum_{n=1}^{\infty} (1 - \alpha(n-1)\alpha(n))^{-1} |d_n|^p < \infty \). Together with Lemma 4.4, this implies that if \( (\alpha(n) - \alpha(n-1)) \) is increasing, then \( \gamma(\alpha) > 1 \) if and only if \( A_\nu = \nu' \).

Let \( \gamma(\alpha) > 1 \). The recurrence relation (4.12) shows that we may apply Theorem 2.2 with \( \sigma = (z_n), \tau = (w_n) \) and \( b_n = (\alpha(n-1)\alpha(n))^{-1/p} \). We see from (4.9) that \( \sigma \) is bounded away from 0, and from (4.10) that \( \tau \) is bounded, so we deduce from (2.1) that \( \tau = (w_n) \) is a basis for \( PLC(p, \alpha) \). As \( \|b\|_\infty = \gamma(\alpha)^{-1/p} < 1 \), (2.4) implies that \( A_\tau = \nu' \).

Now as \( \gamma(\alpha) > 1 \), \( ((1 - \alpha(n-1)\alpha(n))^{-1} w_n) \) is also a basis for \( PLC(p, \alpha) \) which is equivalent to the standard basis for \( \nu' \). The recurrence relation (4.11) shows that Theorem 2.2 may be applied again, with \( \sigma = ((1 - \alpha(n-1)\alpha(n))^{-1} w_n), \tau = (g_n) \) and \( b_n = (\alpha(n-1)\alpha(n))^{-1+p} \). Then \( \tau \) is bounded, \( \sigma \) is bounded away from 0 and

\[ \|b\|_\infty = \gamma(\alpha)^{-1+p} < 1 \]. It follows from (2.1) and (2.4) that \( (g_n) \) is a basis for \( PLC(p, \alpha) \) which is equivalent to the standard basis in \( \nu' \). This proves that (4.4) implies (4.5).

Conversely, let \( (g_n) \) be a basis for \( PLC(p, \alpha) \). As \( \|g_n\|_p = 1 \), (2.2) and (4.11) imply that \( ((1 - \alpha(n-1)\alpha(n))^{-1} w_n) \) is bounded away from 0. As (4.10) shows that \( (w_n) \) is bounded, we deduce that \( \gamma(\alpha) > 1 \). Thus, (4.5) implies (4.4).

If \( p = 2 \), observe that the Fourier transform of \( g_n \) in \( L^2(\mathbb{R}) \) is a multiple, independent
of \( n \), of \( \alpha(n)^{-3/2} ((\sin \alpha(n)t/2)/t)^2 \). The equivalence of (4.5) and (4.6) is thus a consequence of Plancherel's theorem. If \((g_n)\) is basic in \( L^2(\mathbb{R}) \), we have seen that it is Riesz basic, so in this case Plancherel's theorem also implies that the sequence in (4.6) is Riesz basic in \( L^2(\mathbb{R}) \). This completes the proof of Theorem 4.1.

REMARK. If we let \( \alpha(2n) = 2^n \) and \( \alpha(2n + 1) = 2^n + 1 \), it can be shown that (4.15) holds if and only if \( d \in \ell^p \). By Lemma 4.4, \( A_\nu = \ell^p \). Thus \( \gamma(\alpha) = 1 \) but \( A_\nu = \ell^p \), so (4.3) does not, in general, imply (4.4).  

COROLLARY 4.6. If \( m, n \in \mathbb{N} \) let

\[
a_{m,n}(\alpha) = \left( \frac{\alpha(m)}{\alpha(n)} \right)^{1/2} \left( 3 - \frac{\alpha(m)}{\alpha(n)} \right), \quad \text{if } m \leq n, \quad \text{and}
\]

\[
a_{m,n}(\alpha) = \left( \frac{\alpha(n)}{\alpha(m)} \right)^{1/2} \left( 3 - \frac{\alpha(n)}{\alpha(m)} \right), \quad \text{if } n \leq m.
\]

Then \( \gamma(\alpha) > 1 \) if and only if there are \( A, B > 0 \) such that for all scalar sequences \((d_n)\) of finite support,

\[
A \|d\|_2^2 \leq \sum_{m,n=1}^{\infty} d_m d_n a_{m,n}(\alpha) \leq B \|d\|_2^2.
\]

Proof. Let \( p = 2 \). Then \((a_{m,n}(\alpha))_{m,n=1}^{\infty}\) is the Gram matrix of \((g_n)\), except for a constant factor. The inequality is thus equivalent to saying that \((g_n)\) is Riesz basic in \( L^2(\mathbb{R}) \) (see [12, p.32]). The result now follows from Theorem 4.1.

COROLLARY 4.7. Let \( \gamma(\alpha) > 1 \). Then a function \( h \in L^2(\mathbb{R}) \) has an expansion as a convergent series in \( L^2(\mathbb{R}) \) of the form \( \sum_{n=1}^{\infty} \alpha(n)^{-3/2} d_n t^{-2} \sin^2 2^{-1} \alpha(n) t, \) for \( d \in \ell^2 \), if and only if the Fourier transform of \( h \) is in \( PLC(2, \alpha) \).

Proof. Observe that the Fourier transform \( \hat{h} \) of \( h \) is in \( PLC(2, \alpha) \) if and only if \( h \in [g_n : n \in \mathbb{N}] \), where \( g_n \) is given by (4.1) with \( p = 2 \). Now apply Theorem 4.1.

Proof of Theorem 4.2. Let \( p = \infty \). Note that \( \|\tau_n\|_\infty = 1 \) and that \( \tau_n \) is supported by \([\alpha(n-2), \alpha(n)]\). Hence \( \sum_{n=1}^{\infty} d_n \tau_n \) converges in \( PLC_0(\infty, \alpha) \) if and only if \( d \in c_0 \). It also
follows that \( f \in PLC(\infty, \alpha) \) if and only if there is \( d \in \ell^\infty \) so that \( \sum_{n=1}^{\infty} d_n z_n \) converges uniformly to \( f \) on compact subsets of \( \mathbb{R} \). It is easy to prove that \( \nu = (z_n) \) is a basis for \( PLC_0(\infty, \alpha) \) by analogy with the case \( 1 \leq p < \infty \) in Theorem 4.1.

If \( \alpha(\alpha) > 1 \), we apply (2.1) of Theorem 2.2 twice, using the recurrence relations (4.11) and (4.12) with \( p = \infty \). This is similar to the case \( 1 \leq p < \infty \) in Theorem 4.1, and we deduce in a similar way that \( (g_n) \) is a basis for \( PLC_0(\infty, \alpha) \).

Conversely, if \( (g_n) \) is a basis for \( PLC_0(\infty, \alpha) \), then \( ||g_{n+1} - g_n||_\infty \) is bounded away from 0. As

\[
||g_{n+1} - g_n||_\infty = ||g_{n+1}(\alpha(n))|| = (1 - \alpha(n)\alpha(n + 1)^{-1}),
\]

we deduce that \( \alpha(\alpha) > 1 \). This proves Theorem 4.2.

**Proposition 4.8.** If \( \alpha(\alpha) > 1 \), there is a projection \( \pi_1 \) from \( C_0(\mathbb{R}) \) onto \( PLC_0(\infty, \alpha) \) such that \( \pi_1^*(PLC_0(\infty, \alpha)^*) = PLC(1, \alpha) \).

If \( 1 \leq p < \infty \), \( p^{-1} + q^{-1} = 1 \) and \( \alpha(\alpha) > 1 \), there is a projection \( \pi_2 \) from \( L^p(\mathbb{R}) \) onto \( PLC(p, \alpha) \) such that \( \pi_2^*(PLC(p, \alpha)^*) = PLC(q, \alpha) \).

**Proof.** Let \( 1 < p < \infty \), \( p^{-1} + q^{-1} = 1 \) and \( \alpha(\alpha) > 1 \). We let

\[
z'_n = 2^{-1/q}(q + 1)^{1/q}\alpha(n)^{-1/q}(\phi'_{n-1} + \phi_n).
\]

By (4.9), \( ||z'_n||_q \leq 1 \). Also, \( z'_n \) is supported by \( F_n \cup -F_n \), where \( F_n = [\alpha(n-2), \alpha(n)] \). Hence, for \( f \in L^p(\mathbb{R}) \),

\[
(4.17) \quad \left| \int \mathbb{R} f(t)z'_n(t)dt \right| \leq \left\| f\chi(F_n \cup -F_n) \right\|_p, \quad \text{for} \ n \in \mathbb{N}.
\]

Now let \( A_1 = 0 \),

\[
A_n = \int \mathbb{R} z_n(t)z'_{n-1}(t)dt = \frac{(p + 1)^{1/p}(q + 1)^{1/q}}{6} \left( \frac{\alpha(n - 1)}{\alpha(n)} \right)^{1/p} \left( 1 - \frac{\alpha(n - 2)}{\alpha(n - 1)} \right), \quad \text{for} \ n \geq 2,
\]

\[
B_n = \int \mathbb{R} z_n(t)z'_n(t)dt = \frac{(p + 1)^{1/p}(q + 1)^{1/q}}{3} \left( 1 - \frac{\alpha(n - 2)}{\alpha(n)} \right), \quad \text{for} \ n \geq 1, \quad \text{and}
\]

\[
C_n = \int \mathbb{R} z_n(t)z'_{n+1}(t)dt = \frac{(p + 1)^{1/p}(q + 1)^{1/q}}{6} \left( \frac{\alpha(n)}{\alpha(n + 1)} \right)^{1/q} \left( 1 - \frac{\alpha(n - 1)}{\alpha(n)} \right), \quad \text{for} \ n \geq 1.
\]
As \( \gamma(\alpha) > 1 \), \((B_n^{-1})\) is bounded. If \( f \in L^p(\mathbb{R}) \), we now let

\[
\pi(f) = \sum_{n=1}^{\infty} B_n^{-1} \left( \int_{\mathbb{R}} f(t) z_n'(t) \, dt \right) z_n.
\]

From (4.16) and (4.17) we see that the series of \( \pi f \) converges in \( L^p(\mathbb{R}) \) and that

\[
\|\pi(f)\|_p \leq 2^{1+1/p} \|B_n^{-1}\|_\infty \|f\|_p.
\]

Hence \( \pi \) is bounded from \( L^p(\mathbb{R}) \) into \( PLC(p, \alpha) \). We will now show that \( \pi \) is invertible on \( PLC(p, \alpha) \). If \( f \in PLC(p, \alpha) \), as \((z_n)\) is a basis for \( PLC(p, \alpha) \) by Theorem 4.1, there is \( d \in \ell^p \) so that \( f = \sum_{n=1}^{\infty} d_n z_n \). Then

\[
\pi(f) = \sum_{n=1}^{\infty} (A_{n+1} B_n^{-1} d_{n+1} + d_n + B_n^{-1} C_{n-1} d_{n-1}) z_n, \quad \text{where,} \quad d_0 = 0,
\]

\[
= \sum_{n=1}^{\infty} ((I + S)d)_n z_n,
\]

where \( I \) is the identity operator on \( \ell^p \), and

\[
(Sd)_n = A_{n+1} B_n^{-1} d_{n+1} + B_n^{-1} C_{n-1} d_{n-1}, \quad \text{for} \quad d \in \ell^p.
\]

Now,

\[
A_{n+1} B_n^{-1} = 2^{-1} \alpha(n+1)^{1/p} \alpha(n+1)^{-1/p} (\alpha(n) - \alpha(n-1)) (\alpha(n) - \alpha(n-2))^{-1},
\]

\[
\leq 2^{-1} \gamma(\alpha)^{-1/p}, \quad \text{and}
\]

\[
B_n^{-1} C_{n-1} = 2^{-1} \alpha(n-1)^{1/4} \alpha(n)^{-1/4} (1 - \alpha(n-2) \alpha(n-1)) (1 - \alpha(n-2) \alpha(n)^{-1})^{-1},
\]

\[
\leq 2^{-1} \gamma(\alpha)^{-1/4}.
\]

Hence \( S \) is bounded on \( \ell^p \) and \( \|S\| \leq 2^{-1} (\gamma(\alpha)^{-1/p} + \gamma(\alpha)^{-1/4}) < 1 \), so \( I + S \) is invertible on \( \ell^p \). By (4.16), \( PLC(p, \alpha) \) is isomorphic to \( \ell^p \), and we deduce from (4.18) that \( \pi \) is invertible on \( PLC(p, \alpha) \). Denote this inverse by \( \lambda \) and let \( \pi_2 = \lambda \circ \pi \). Then \( \pi_2 \) is a projection from \( L^p(\mathbb{R}) \) onto \( PLC(p, \alpha) \).

Now by Theorem 4.1, \( \nu = (z_n) \) is a basis for \( PLC(p, \alpha) \) and \( A_\nu = \ell^p \). Then (2.11) shows that \( \{(\mu(z_n) : \mu \in PLC(p, \alpha)^*) = \ell^p \). Hence if \( \mu \in PLC(p, \alpha)^* \), \((B_n^{-1} \mu(z_n)) \in \ell^p \) and the
series \( \sum_{n=1}^{\infty} B_n^{-1} \mu(z_n) z'_n \) converges in \( PLC(q, \alpha) \). It is easy to prove that \( \pi^*(\mu) = \sum_{n=1}^{\infty} B_n^{-1} \mu(z_n) z'_n \), for \( \mu \in PLC(p, \alpha)^* \), and it follows that \( \pi^*(PLC(p, \alpha)^*) = PLC(q, \alpha) \) (here we have used the fact that \( (B_n) \) is bounded above and below and that \( (z'_n) \) is a basis for \( PLC(q, \alpha) \) equivalent to the standard basis in \( \mathbb{C} \)). Finally, as \( \lambda \) is invertible on \( PLC(p, \alpha) \),

\[
\pi^*_\lambda(PLC(p, \alpha)^*) = \pi^*(\lambda^*(PLC(p, \alpha)^*)) = PLC(q, \alpha),
\]

This proves the proposition for \( 1 < p < \infty \).

When \( p = 1 \) and \( q = \infty \), the proof proceeds on the lines above, except that when we have \( \pi^*(\mu) = \sum_{n=1}^{\infty} B_n^{-1} \mu(z_n) z'_n \), this series is taken as converging uniformly on compact sets, rather than in the \( L^\infty(\mathbb{R}) \) norm.

When \( p = \infty \) and \( q = 1 \), the proof is again similar to the preceding. Instead, \( \pi \) is defined on \( C_0(\mathbb{R}) \), \( \mathcal{E} \) is replaced by \( c_0 \), and Theorem 4.2 is used in place of Theorem 4.1.

REMARKS. 1. If one only wishes to show that \( PLC_0(\infty, \alpha) \) is complemented in \( C_0(\mathbb{R}) \) a simpler proof than the one above may be found in [10, p.27] – this proof does not require \( \gamma(\alpha) > 1 \), but it does not give the identity \( \pi^*_\lambda(PLC_0(\infty, \alpha)^*) = PLC(1, \alpha) \).

2. Let \( PL(p, \alpha) \) denote those (not necessarily continuous) functions in \( L^p(\mathbb{R}) \) which are even and linear on each interval \([\alpha(n-1), \alpha(n)]\). Then it can be proved that for \( 1 \leq p < \infty \), \( PL(p, \alpha) \) is complemented in \( L^p(\mathbb{R}) \) under a projection \( \pi \) such that \( \pi^*(PL(p, \alpha)^*) = PL(q, \alpha) \). This is true without restriction on \( \gamma(\alpha) \). Thus, it is not clear whether the role played in Proposition 4.8 by the condition \( \gamma(\alpha) > 1 \) is essential, although \( \gamma(\alpha) > 1 \) is essential for the next result.

PROPOSITION 4.9. Let \( 1 \leq p < \infty \) and \( p^{-1} + q^{-1} = 1 \). For \( g \in L^q(\mathbb{R}) \) and \( n \in \mathbb{N} \) let

\[
(Ag)(n) = \alpha(n)^{-1/p} \int_{-\alpha(n)}^{\alpha(n)} (\alpha(n) - |t|) g(t) dt.
\]

Then \( \gamma(\alpha) > 1 \) if and only if \( A(L^q(\mathbb{R})) = \mathcal{E} \). In this case, the restriction of \( A \) to the subspace \( PLC(q, \alpha) \) of \( L^q(\mathbb{R}) \) is a bounded invertible operator onto \( \mathcal{E} \).
Proof. By Theorem 4.1, \( \gamma(\alpha) > 1 \) is equivalent to saying that \((g_n)\) is a basis for \( PLC(p, \alpha) \) which is equivalent to the standard basis for \( \ell^p \). When \( 1 < p < \infty \), we deduce from (2.11) that this is equivalent to \( A(L^q(\mathbb{R})) = \ell^q \). When \( p = 1 \) and \( q = \infty \), \( \gamma(\alpha) > 1 \) implies that \( A(L^\infty(\mathbb{R})) = \ell^\infty \) is a consequence of (2.12). Conversely, if \( \gamma(\alpha) = 1 \) and \( g \in L^\infty(\mathbb{R}) \) let \( a_n = \alpha(n)^{-1} \int_{-\alpha(n)}^{\alpha(n)} (\alpha(n) - |t|) g(t) dt \). Then it can be shown that \( \liminf_{n \to \infty} |a_{n+1} - a_n| = 0 \) (compare with the corresponding part of the proof of Proposition 3.6). Hence, if \( \gamma(\alpha) = 1 \), \( A(L^\infty(\mathbb{R})) \subseteq \ell^\infty \) and \( A(L^\infty(\mathbb{R})) \neq \ell^\infty \). The final statement in the proposition comes from (2.13) and Proposition 4.8.

There are also discrete versions of the preceding results, some of which are presented.

**THEOREM 4.10.** Let \( 1 \leq p \leq \infty \) be given, and let \((\alpha(n))\) be an increasing sequence of positive integers. Let \( h_n \in \ell^p(\mathbb{Z}) \) be given by

\[ h_n(j) = \alpha(n)^{-1}(j+1/p)(\alpha(n) - |j|), \quad \text{for} \quad |j| \leq \alpha(n), \quad \text{and} \]

\[ h_n(j) = 0, \quad \text{for} \quad |j| > \alpha(n). \]

Then \( \gamma(\alpha) > 1 \) if and only if \((h_n)\) is basic in \( \ell^p(\mathbb{Z}) \). If \( \gamma(\alpha) > 1 \) and \( 1 \leq p < \infty \), \((h_n)\) is equivalent to the standard basis in \( \ell^p \). Also, \( \gamma(\alpha) > 1 \) if and only if the sequence \((\alpha(n)^{-3/2} \sin^2(\alpha(n)t/2) \sin^{-2} t/2)\) is basic in \( L^2([0,2\pi]) \), in which case it is Riesz basic.

Proof. Let \( PLC(p) \) denote the closed subspace of \( L^p(\mathbb{R}) \) consisting of the even, continuous functions which are linear on \([n-1, n]\) for \( n \in \mathbb{N} \). If \( f \in PLC(p) \), let \((Tf)(n) = f(n)\), for \( n \in \mathbb{Z} \). It follows from Lemma 4.3 that \( T \) is an isomorphism from \( PLC(p) \) into \( \ell^p(\mathbb{Z}) \). Also, \( T(g_n) = 2^{-1/p}(p+1)^{1/p} h_n \), for all \( n \). The statements concerning \((h_n)\) are thus a consequence of the equivalence of (4.4) and (4.5), and Theorems 4.1 and 4.2. When \( p = 2 \) the Fourier transform of \( \alpha(n)^{3/2} h_n \) is the Fejér kernel \( \sin^2(\alpha(n)t/2) \sin^{-2} t/2 \). The remainder of Theorem 4.10 now follows from Plancherel's theorem.
COROLLARY 4.11. Let $\gamma(\alpha) > 1$. Then a function $h \in L^2([0, 2\pi])$ has an expansion as a convergent series in $L^2([0, 2\pi])$ of the form
\[
\sum_{n=1}^{\infty} \alpha(n)^{-3/2} d_n \sin^2 (\alpha(n)t/2) \sin^{-3/2} t/2, \quad \text{for } d \in \ell^2,
\]
if and only if the Fourier transform of $h$ is the restriction to $Z$ of some function in $PCL(2, \alpha)$.

Proof. This is analogous to the proof of Corollary 4.7.

COROLLARY 4.12. Let $1 \leq p < \infty$ and $p^{-1} + q^{-1} = 1$. Let $a_{ij} = \alpha(i)^{-1+1/p} (\alpha(i) - j + 1)$, for $1 \leq j \leq \alpha(i)$, and $a_{ij} = 0$, for $j > \alpha(i)$. Let $A$ denote the operator obtained by multiplying by the matrix $(a_{ij})$. Then $A$ is a bounded operator from $\ell^q$ onto $\ell^p$ if and only if $\gamma(\alpha) > 1$.

Proof. This is similar to the proof of Proposition 4.9.

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REFERENCES


The University of Wollongong,
Wollongong,
New South Wales, 2500
Australia.