REGULARITY OF GENERALIZED SCALAR OPERATORS WITH SPECTRUM CONTAINED IN A LINE

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(Dedicated to Professor Igor Kluvánek)

Generalized scalar operators were introduced by C. Foiaş [4] and a detailed study of such operators can be found in the monograph [3]. An important subclass consists of regular generalized scalar operators which enjoy properties not shared by all generalized scalar operators. For example, the sum and product of commuting generalized scalar operators S and T need not be generalized scalar operators [2; §3]. However, if, in addition, S and T are both regular, then ST and S + T are again generalized scalar operators [3; p.106], although they need not be regular [2; §3]. In particular, there exist generalized scalar operators which are not regular [1,2].

A closed subset F of the complex plane $\mathbb{C}$ is called *thin* [3; p.100] if the function $\lambda \to \overline{\lambda}$ on F (the bar denotes complex conjugation) is the restriction of a function which is analytic in a neighbourhood of F. It is clear that any closed subset of a thin set is also a thin set and that segments of a line are thin sets. Accordingly, the following result is an immediate consequence of Theorem 4.1.11 in [3].

THEOREM. *A generalized scalar operator whose spectrum is contained in a line in the complex plane is necessarily regular.*

The proof of this result given in [3] is based on the theory of distributions and N. Dunford's analytic functional calculus. The purpose of this note is to present another proof of the above Theorem based on some recent work of A. McIntosh and A. Pryde [6] in which they develop a specific functional calculus for certain operators via the theory of integration.

It is time to be more precise. Let X be a complex Banach space and L(X) denote the Banach algebra of all continuous linear operators from X into itself equipped with the uniform operator topology. An element T of L(X) is a *generalized scalar operator* if there exists a continuous algebra homomorphism $U: C^\infty(\mathbb{C}) \to L(X)$ such that $U(p) = p(T)$ for all polynomials p, and the function $\xi \to U(fg_\xi)$ from $\mathbb{C}\setminus \mathrm{Supp}(f)$ into L(X) is holomorphic for each $f \in C^\infty(\mathbb{C})$, where $g_\xi(z) = (\xi-z)^{-1}$ and Supp(f) is the support of f. Such a U is called a *spectral distribution* for T. Here $C^\infty(\mathbb{C}) \simeq C^\infty(\mathbb{R}^2)$ is the algebra of all infinitely differentiable functions on $\mathbb{R}^2$ equipped with the topology of uniform convergence on compacta of functions and all their derivatives. We say that T is *regular* if it has a spectral distribution which takes its values in the bicommutant, {T}", of T. Such a distribution is called a *regular spectral distribution* for T.

The first assertion is that it suffices to establish the Theorem for the case when the line in $\mathbb{C}$ is the real axis $\mathbb{R}$. Indeed, suppose that T satisfies the hypotheses of the Theorem. Then there exist complex numbers α and β, with $|\alpha| = 1$, such that $S = \alpha(T-\beta I)$ satisfies $\sigma(S) \subset \mathbb{R}$. Let V be a spectral distribution for T and let $g(z) = \alpha(z-\beta)$, $z \in \mathbb{C}$. Since $S = V(g)$ it follows that S is also a generalized scalar operator [3; p.105, Lemma 3.2] and hence, there would exist a regular spectral distribution for S, say W. If $h(z) = \beta + \alpha^{-1}z$, $z \in \mathbb{C}$, then $U(f) = W(f\circ h)$ defines a spectral distribution for the operator

$$U(\lambda) = W(h) = \beta I + \alpha^{-1}W(\lambda) = \beta I + \alpha^{-1}S = T ,$$

where λ denotes the identity function on $\mathbb{C}$ [3; p.105, Lemma 3.2]. But,

$$\{U(f);\ f \in C^\infty(\mathbb{C})\} \subset \{W(\psi);\ \psi \in C^\infty(\mathbb{C})\} \subset \{S\}'' = \{T\}''$$

and hence, U would be a regular spectral distribution for T.

So, suppose that T is a generalized scalar operator with real spectrum. Then there exist constants $M \geq 1$ and $s \geq 0$ such that

$$\| e^{itT} \| \leq M(1 + |t|)^s, \quad t \in \mathbb{R},$$

[3; p.160]. Let $\overset{\vee}{L_1}(s)$ denote the space of inverse Fourier transforms $f = \overset{\vee}{g}$ of functions $g: \mathbb{R} \to \mathbb{C}$ for which $t \to (1 + |t|)^s g(t)$, $t \in \mathbb{R}$, belongs to $L^1(\mathbb{R})$. We shall write $\hat{f}$ for g. The Fourier inversion formula being used is

$$f(x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{ixw} g(w)\,dw, \quad x \in \mathbb{R}.$$

It follows [6; §8] that $\overset{\vee}{L_1}(s)$ is a Banach algebra with respect to pointwise addition and multiplication and with norm

$$\| f \| = (2\pi)^{-1} \|(1 + |\cdot|)^s \hat{f}(\cdot)\|_{L^1(\mathbb{R})}.$$

Define a linear map $\Phi: \overset{\vee}{L_1}(s) \to L(X)$ by

$$\Phi(f) = (2\pi)^{-1} \int_{\mathbb{R}} \hat{f}(t) e^{itT} dt, \quad f \in \overset{\vee}{L_1}(s),$$

where the integral is a Bochner integral. Indeed, the integrand is a strongly measurable $L(X)$-valued function of t (using continuity of $t \to e^{itT}$ and the separability of $\mathbb{R}$) and $\int_{\mathbb{R}} \| \hat{f}(t) e^{itT} \|\,dt \leq 2\pi M \| f \| < \infty$. This also establishes the continuity of Φ. Actually, it is shown in Section 8 of [6] that Φ is a multiplicative functional calculus for T based on the algebra $\overset{\vee}{L_1}(s)$, in the sense of Definition 6.1 of [6]. In particular, the support, $\mathrm{Supp}(\Phi)$, of Φ is precisely $\sigma(T)$ and $p(T) = \Phi(\theta p)$ for all polynomials $p: \mathbb{R} \to \mathbb{C}$. Here θ is any compactly supported C^∞-function on $\mathbb{R}$ which is equal to 1 in a neighbourhood of $\mathrm{Supp}(\Phi)$. It is then straightforward to verify that the mapping $U: C^\infty(\mathbb{C}) \to L(X)$ defined by

$$U(f) = \Phi(\theta f|_{\mathbb{R}}), \quad f \in C^\infty(\mathbb{C}),$$

where $f|_{\mathbb{R}}$ denotes the restriction of f to $\mathbb{R}$, is a spectral distribution for T.

So, it remains to verify that U assumes its values in $\{T\}''$ or, equivalently, that Φ assumes its values in $\{T\}''$. Let $f \in \check{L}_1(s)$. Then define the set $P(f) = \{t \in \mathbb{R};\ \|(\theta f)^\wedge(t)e^{itT}\| > 0\}$. If $\varepsilon > 0$ there exists a decomposition of $P(f)$ into disjoint measurable sets $\{E_k(\varepsilon)\}_{k=1}^\infty$ such that for arbitrary $t_k \in E_k(\varepsilon)$, the function f_ε given by $f_\varepsilon(t) = (\theta f)^\wedge(t_k)e^{it_kT}$ if $t \in E_k(\varepsilon)$, $k = 1,2,\ldots,$ and $f_\varepsilon(t) = 0$ otherwise, is Bochner integrable and satisfies

$$(1) \qquad \int_{\mathbb{R}} \|(\theta f)^\wedge(t)e^{itT} - f_\varepsilon(t)\|\,dt < \varepsilon\ ;$$

see [5; p.81, Corollary], for example. Since $e^{it_kT} = \Sigma_{n=0}^\infty (it_k)^n(n!)^{-1}T^n$, for each $k = 1,2,\ldots,$

$$\int_{\mathbb{R}} f_\varepsilon(t)dt = \Sigma_{k=1}^\infty (\theta f)^\wedge(t_k)\mu(E_k(\varepsilon))e^{it_kT}\ ,$$

where μ is Lebesgue measure in $\mathbb{R}$, and all series involved converge in $L(X)$, it is clear that $\int_{\mathbb{R}} f_\varepsilon(t)dt \in \{T\}''$, for every $\varepsilon > 0$. Choose a sequence $\varepsilon(n)\to 0$. Then the definition of Bochner integral together with (1) imply that $\Phi(f)$, being equal to the limit (in $L(X)$) of the sequence $\{\int_{\mathbb{R}} f_{\varepsilon(n)}(t)dt\}_{n=1}^\infty$, belongs to $\{T\}''$ as required.

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