EVOLUTION EQUATIONS OF PARABOLIC TYPE

A. Yagi

1. INTRODUCTION

The aim of this talk is to report some recent results concerning linear and nonlinear evolution equations of parabolic type in Banach spaces.

Let

(L) $du/dt + A(t)u = f(t) \quad , \; 0 < t \leqq T$

(Q) $du/dt + A(t,u)u = f(t,u) \quad , \; 0 < t \leqq T$

be linear and quasilinear evolution equations of parabolic type in a Banach space X respectively. By "parabolic type" we mean that $A(t)$ and $A(t,u)$ are all the infinitesimal generators of analytic linear semigroups on X (we do not necessarily assume that the domains of the operators $A(t)$ and $A(t,u)$ are dense subspaces of X, so the semigroups generated by them may not be of class C_0). The domains $\mathcal{D}(A(t))$ and $\mathcal{D}(A(t,u))$ of $A(t)$ and $A(t,u)$ are allowed to vary with t or u, but it is assumed that there exists a number $0 < h \leqq 1$ such that the domains $\mathcal{D}(A(t)^h)$ and $\mathcal{D}(A(t,u)^h)$ of the fractional powers $A(t)^h$ and $A(t,u)^h$ respectively are independent of t and u.

In Section 2 we shall study the linear equation (L) by constructing a fundamental solution (evolution operator) for (L).

In Section 3 we shall consider the initial value problem

$$(QI) \quad \begin{cases} du/dt + A(t,u)u = f(t,u) \ , \ 0 < t \leq T \\ u(0) = u_0 \end{cases}$$

of (Q) on the basis of the linear result for (L). Some remarks on application to the partial differential equation will be made in Section 4.

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2. LINEAR EQUATION (L)

Let $A(t)$ be defined on an interval $[0,T]$. We shall assume that:

(Li) $-A(t)$, $0 \leq t \leq T$, are the infinitesimal generators of analytic semi-groups on X.

(Lii) There exists a number $0 < h \leq 1$ such that the domains $D(A(t)^h)$ of the fractional powers are independent of t. Moreover, for $0 \leq r < h$ such that $1 = mh + r$ where $m \geq 1$ is an integer, the domains $D(A(t)^r)$ are also independent of t.

(Liii) $A(\cdot)^h$ and $A(\cdot)^r$ are Hölder continuous functions:

$$(1) \quad \|A(t)^h A(s)^{-h} - I\|_{L(X)} \leq C_h |t - s|^k \quad , \ 0 \leq s,t \leq T$$

$$(2) \quad \|A(t)^r A(s)^{-r} - I\|_{L(X)} \leq C_r |t - s|^k \quad , \ 0 \leq s,t \leq T$$

with some constant $k > 1 - h$.

Then we can prove:

THEOREM 1. *Under* (Li), (Lii) *and* (Liii), *a family of bounded operators* $U(t,s)$, $0 \leq s \leq t \leq T$, *on* X *can be constructed. The operators* $U(t,s)$ *(called evolution operators for* $A(t)$*) have the following properties:*

(a) $U(t,s)U(s,r) = U(t,r)$, $U(t,t) = I$ *for* $0 \leq r \leq s \leq t \leq T$ *;*

(b) $U(t,s)$ *is strongly continuous in* (t,s) *for* $0 \leqq s < t \leqq T$ *;*

(c) if $f \in C([0,T];X)$ *, then any strict solution* u *of* (L) *such that* $u \in C([0,T];Y) \cap C^1((0,T];X)$ *and* $A(\cdot)u(\cdot) \in C((0,T];X)$ *must be of the form*

$$(3) \qquad u(t) = U(t,0)u(0) + \int_0^t U(t,\tau)f(\tau)d\tau \ , \ 0 < t \leqq T$$

(here Y *denotes the space* X *equipped with the norm* $\|\cdot\|_Y = \|A(0)^{-1}\cdot\|_X$ *, which is weaker than the norm of* X*);*

(d) conversely, if $f \in C^\sigma([0,T];X)$ *,* $\sigma > 0$ *, then for any initial value* $u(0) \in X$ *, the function defined by (3) gives a strict solution of* (L) *on* $[0,T]$.

This result had been already obtained by Kato [4] in the case when $r = 0$ (or more precisely when $h = 1/m$ with some positive integer m). Therefore our result is a generalization of this for an arbitrary number h by adding the same kind of assumptions for $r = 1 - mh$ also. As may be the case in applications, these additional assumptions are fulfilled of course if the domains $D(A(t)^r)$ are independent of t for all $0 \leqq r \leqq h$ and if (2) holds for all $0 \leqq r \leqq h$.

Kato needed the assumption $h = 1/m$ in proving that his approximating operators $[(A(t)^h)_n]^m$ of $A(t)$, where $(A(t)^h)_n$ ($n = 1, 2, 3, \cdots$) are the Yosida approximation of $A(t)^h$, also satisfy the condition (Li) with certain uniformity in n. We use a different approximation of $A(t)$ – the Yosida approximation $A_n(t)$ of $A(t)$ itself, so it is immediate to verify (Li). But, on the contrary, establishing (1) and

(2) for the operators $A_n(t)$ becomes a very difficult problem; indeed some new techniques are required. Complete proofs will be published in [14].

3. QUASILINEAR EQUATION (Q)

Let $R > 0$, and let W be a normed subspace of X. The operators $A(t,u)$ are defined for all $0 \leqq t \leqq T$ and all $u \in W$ such that $\|u\|_W <$ R. $f(t,u)$ are also defined for all such (t,u). The initial value u_0 is from W and $\|u_0\|_W < R$. Assumptions we shall make are the following :

(Qi) $-A(t,u)$, $0 \leqq t \leqq T$ and $\|u\|_W < R$, are the infinitesimal generators of analytic semigroups on X.

(Qii) There exists a number $0 < h \leqq 1$ such that, for all $0 \leqq r \leqq h$, the domains $\mathcal{D}(A(t,u)^r)$ of the fractional powers $A(t,u)^r$ are independent of (t,u).

(Qiii) For all $0 \leqq r \leqq h$

$$\|A(t,u)^r A(s,v)^{-r} - I\|_{\mathcal{L}(X)} \leqq C_r\{|t - s|^k + \|u - v\|_W\}$$

hold with some constant $1 - h < k \leqq 1$.

(Qiv) For some $0 < \sigma \leqq 1$

$$\|f(t,u) - f(s,v)\|_X \leqq C\{|t - s|^\sigma + \|u - v\|_W\}.$$

(Qv) There exists $0 < \alpha < h$ such that $\mathcal{D}(A_0^\alpha) \subset W$ with norm continuity, where A_0 denotes the operator $A(0,u_0)$.

(Qvi) There exists $\alpha < \beta \leqq 1$ such that $u_0 \in \mathcal{D}(A_0^\beta)$.

(Qvii) Among α, β and h, the relation $1 - h < \beta - \alpha$ holds.

Then we can obtain:

THEOREM 2. *Under* (Qi) $\sim$ (Qvii), *there exists a unique local solution* $u \in C^{\gamma}([0,T_0];W) \cap C^1((0,T_0];X)$, $1 - h < \gamma < \beta - \alpha$, *for* (QI).

In order that this time local solution u can be extended to a time global one we need to make the following further assumptions:

(Qviii) $\{-A(t,u)\}_{0 \leqq t \leqq T}$, $\|u\|_W < R$, are stable families on X with a uniform stable constant $\{M,-\delta\}$, $\delta > 0$.

(Qix) For some $\rho > 0$

$$\|f(t,u)\|_X \leqq C\|u\|_X^{1+\rho}.$$

THEOREM 3. *Under* (Qi) $\sim$ (Qix), *if* $\|A_0^{\beta}u_0\|_X$ *is sufficiently small, then the local solution* u *on* $[0,T_0]$ *is extendable to a solution on* $[0,T]$.

The work in this section was done in collaboration with Kiyoko Furuya (Ochanomizu University). She [1,2] had already discussed local existence of solutions of (QI) in the case when $h = 1/m$ (m is an integer) on the basis of Kato's result [4].

These two theorems are proved by using Banach's fixed point theorem for contraction mappings. We first seek an appropriate function space $E \subset \{v \in C^{\gamma}([0,T];D(A_0^{\alpha}));\ v(0) = u_0\}$ and define a mapping Ψ such that, for $v \in E$, Ψv is a unique solution u of the linear problem

$$(LI)_v \quad \begin{cases} du/dt + A(t,v(t))u = f(t,v(t)), & 0 < t \leqq T \\ u(0) = u_0. \end{cases}$$

We next prove that Ψ is a contraction mapping from E into itself by replacing T in the proof of Theorem 2 by a sufficiently small number T_0 and by taking $A_0^{\beta}u_0$ sufficiently small in the proof of Theorem 3. For a detailed proof, see [3]

4. APPLICATION

The problem of determing the domain of definition of the fractional powers of elliptic differential operators has been studied by Seeley [7,8] and [11]. From them we can know for which number h the assumptions (Lii) and (Qii) are actually fulfilled. To the contrary, the assumptions (Liii) and (Qiii) can be verified for the present only for the case when X is an L^2-space, because regularity of functions of the form $A(\cdot)^h$ is known only when $A(t)$ ($0 \leqq t \leqq T$) are linear operators in a Hilbert space, see McIntosh [5,6] and [10,12,13].

Accordingly, we are able to handle, for example, the following problems in $L^2(\Omega)$.

$$\begin{cases} \partial u/\partial t - \sum_{i,j} \partial/\partial x_i \{a_{ij}(t,x)\partial u/\partial x_j\} - \sum_i a_i(t,x)\partial u/\partial x_i - a(t,x)u = \\ \qquad\qquad f(t,x) \quad \text{in} \quad (0,T] \times \Omega, \\ \sum_i b_i(t,x)\partial u/\partial x_i = 0 \quad \text{on} \quad (0,T] \times \partial\Omega , \\ u(0,x) = u_0(x) \quad \text{in} \quad \Omega , \end{cases}$$

where $\Omega \subset \mathbb{R}^n$.

$$\begin{cases} \partial u/\partial t - \sum_{i,j} \partial/\partial x_i \{a_{ij}(t,x,u)\partial u/\partial x_j\} = f(t,x,u,\nabla u) & \text{in} \quad (0,T] \times \Omega , \\ \sum_{i,j} a_{ij}(t,x,u) n_i(x) \partial u/\partial x_j = 0 & \text{on} \quad (0,T] \times \partial\Omega , \\ u(0,x) = u_0(x) & \text{in} \quad \Omega , \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $n = 1$ or 2.

REFERENCES

[1] K.Furuya, *Analyticity of solutions of quasilinear evolution equations*, Osaka J. Math. 18 (1981), 669-698.

[2] K.Furuya, *Analyticity of solutions of quasilinear evolution equations II*, Osaka J. Math. 20 (1983), 217-236.

[3] K.Furuya and A.Yagi, *Quasilinear evolution equations of parabolic type*, in preparation.

[4] T.Kato, *Abstract evolution equations of parabolic type in Banach and Hilbert spaces*, Nagoya Math. J. 5 (1961), 93-125.

[5] A.McIntosh, *Square roots of operators and applications to hyperbolic p.d.e.'s*, Proc. Centre Math. Anal., Australian National University 5 (1984), 124-136.

[6] A.McIntosh, *Square roots of elliptic operators*, J. Func. Anal. 61 (1985), 307-327.

[7] R.Seeley, *Norms and domains of the complex powers* A_B^z , Amer. J. Math. 93 (1971), 299-309.

[8] R.Seeley, *Interpolation in* L^p *with boundary conditions*, Studia Math. 44 (1972), 47-60.

[9] P.E.Sobolevskii, *Parabolic equations in Banach space with an un-*

bounded variable operator, a fractional power of which has a constant domain of definition, Soviet Math. Dokl. 2 (1961), 545-548.

[10] A.Yagi, *Differentiability of families of the fractional powers of self-adjoint operators associated with sesquilinear forms,* Osaka J. Math. 20 (1983), 265-284.

[11] A.Yagi, *Coincidence entre des espaces d'interpolation et des domaines de puissances fractionnaires d'opérateurs,* C. R. Acad. Sc. Paris, 299, I, 6 (1984), 173-176.

[12] A.Yagi, *Applications of the purely imaginary powers of operators in Hilbert spaces,* J. Func. Anal., to appear.

[13] A.Yagi, *Regularity of functions on an interval with values in the space of fractional powers of operators,* Lecture Notes in Math. Ser., Springer-Verlag, to appear.

[14] A.Yagi, *Generalization of Sobolevskii and Kato's theorem on parabolic evolution equations in Banach spaces,* in preparation.

[15] H.Tanabe, *Equations of evolution,* Pitman, London, 1979.

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan