# BANACH SPACES WITH MANY PROJECTIONS

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A fundamental question in operator theory is: how rich is the collection of operators on a given Banach space? For classical spaces, especially Hilbert space, a well developed theory of operators exists. However a Banach space may have far fewer operators than one might expect. Shelah [S] has constructed a nonseparable Banach space $X$, for which the space of operators with separable range has codimension one in $B(X)$. An interesting open problem is whether there is a Banach space $X$ for which the space of compact operators has finite codimension in $B(X)$.

For many classical Banach spaces, the projections generate $B(X)$. For example, every operator on a Hilbert space is a linear combination of at most ten self-adjoint projections [M]. This is false for $\ell_\infty$, since every projection thereon, unless of finite rank, has nonseparable range [L2]. It is still unknown whether every Banach space admits a nontrivial projection. (By nontrivial we mean that both the range and the null space are infinite dimensional.) In this report we will be interested in a particular type of projection.

Let $X$ be a Banach space and let $M$ be a closed subspace of $X$. By a linear extension operator (LEO) we mean a linear mapping $T : M^* \to X^*$ such that, for each $f$ in $M^*$, $Tf$ is a norm-preserving extension of $f$. A routine exercise shows that there exists a LEO from $M^*$ to $X^*$ if, and only if, $M^o$ is the kernel of a contractive projection on $X^*$. Most subspaces of most Banach spaces admit no LEO. To obtain reasonable results we must restrict our attention to a smaller class of Banach spaces.

An Asplund space is a Banach space for which every separable subspace has a separable dual. This is equivalent to every subspace having the same density character as its dual [Ph]. We use this property to show that every non-separable Asplund space has many subspaces (in fact, an uncountable increasing family of them) which admit LEOs. Hence the dual of an Asplund space (that is, a dual space with the Radon-Nikodym Property [Ph]) admits many projections. Moreover these projections can be chosen to be well behaved in a certain sense - in particular they all commute.

In the second section we consider the problem of renorming Asplund spaces and their duals. The existence of many projections is a useful tool for this, and other problems. It is well known that any space with an equivalent Frechet smooth norm is automatically an Asplund space. The converse remains open. Examples, [E] and [Tl], show that it may not be possible to renorm an Asplund space so that the dual norm is reasonably convex. We conjecture that the dual of every Asplund space can be renormed so as to be locally uniformly convex, although the known proofs all require some additional smoothness hypothesis. Such renormings need not of course be dual renormings.

In the final section, we examine some particular Asplund spaces, illustrating the extent and limitations of previous results.

By the density character , dens $X$, of a Banach space $X$, we mean the least cardinality of any dense subset of $X$. Thus $X$ is separable if and only if dens $X = \omega$. We will identify cardinal numbers with their initial ordinals. The reader is assumed to be familiar with transfinite induction arguments, and the basic smoothness and convexity properties of Banach spaces [D].

## 1. Asplund spaces have lots of LEOs.

Our first few results are valid for arbitrary Banach spaces. We begin with a rather technical result.

**Lemma 1.** *Let $X$ be a Banach space, $M$ a finite dimensional subspace, $k$ a positive integer, $\varepsilon$ a positive real number and $G$ a finite subset of $X^*$. Then there is a finite dimensional subspace $Z$ containing $M$ such that for every subspace $E$ satisfying* $\dim E/M \leq k$ *we can find an operator $T : E \to Z$ such that $T$ fixes $M$, $\|T\| < 1 + \varepsilon$ and $|f(x) - f(Tx)| < \varepsilon\|x\|$ for all $x \in E$, $f \in G$.*

**Proof.** The special case when $G$ is empty is just [L1, Lemma 1]. A slight modification of that proof yields the conclusion for any finite $G$. Alternatively, the result may be viewed as a specialization of [AL, Lemma 2].

The reasons for introducing $G$ into Lemma 1 will become apparent in the next few proofs. In particular it allows us to conclude that $T_\alpha(M_\alpha^*) \subseteq T_\beta(M_\beta^*)$ for $\alpha < \beta$ in Theorem 4, and thus $P_\beta P_\alpha = P_\alpha$ for $\alpha < \beta$ in theorem 5. Tacon [Tc] obtained similar results under the additional assumption that $X$ was smooth. The extra complication in Lemma 1 allows us to avoid any such smoothness assumption on $X$, at least in this section.

**Proposition 2.** *Let $X$ be any Banach space, $N$ a separable subspace of $X$, and $F$ a separable subspace of $X^*$. Then $X$ has a separable subspace $M$ containing $N$, which admits a LEO $T : M^* \to X^*$ satisfying $T(M^*) \supseteq F$.*

**Proof.** Let $(f_i)$ be a sequence dense in $F$, and $(m_i)$ a sequence dense in $N$. Beginning with $M_0 = \{0\}$, we define an increasing sequence of subspaces $M_n$ of $X$ as follows: $M_n$ is the subspace $Z$ given by Lemma 1 when $M = M_{n-1} + < m_n >$, $k = n$, $\varepsilon = \frac{1}{n}$ and $G = \{f_1, f_2, \cdots, f_n\}$. Clearly $M = \overline{\bigcup M_n}$ is separable and contains $N$. Let $D_n$ be the collection of subspaces $E$ of $X$ which contain $M_n$ and satisfy $\dim E/M_n \leq n$. Then for each $E \in D_n$, there is an operator $T_E : E \to M_n$ satisfying $|f_i(x) - f_i(T_E x)| \leq \|x\|/n$ for $x \in E$, $1 \leq i \leq n$ and $\|T_E\| \leq 1 + \frac{1}{n}$. Clearly $\bigcup_{n=1}^\infty D_n$ is, under inclusion, a directed set, so let $U$ be an ultrafilter thereover. Note that, for each $x \in X$, $T_E x$ is defined for all sufficiently large $E$, and may be considered as an element of $M^{**}$. Thus we may define $T : M^* \to X^*$ by $(Tf)(x) = (w^* - \lim_U T_E x)(f)$. (Alternatively, we could have defined $T$ by the well known *Lindenstrauss compactness argument.*) A routine argument shows that $T$ is a LEO and that $T(f_i|M) = f_i$ for every $i$. Hence $T(M^*) \supseteq F$.

Taking $N = \{0\}$ in Proposition 2, and assuming that $X$ is an Asplund space, verifies a claim made by W.B.Johnson [DU,p38].

**Lemma 3.** *Let $N$ be a subspace of $X$, $F$ a subspace of $X^*$, with* dens$F \leq$ dens$N$. *Then there is a subspace $M$ of $X$ containing $N$, and a LEO $T : M^* \to X^*$ such that* dens $M =$ dens $N$ *and* $T(M^*) \supseteq F$.

**Proof.** We establish this by a transfinite induction on $\mu =$ dens $N$. The base case, $\mu = \omega$, is given by Proposition 2. So assume $\mu > \omega$, and let $\{x_\alpha : \alpha < \mu\}$ and $\{f_\alpha : \alpha < \mu\}$ be dense in $N$ and $F$ respectively. The inductive hypothesis gives us, for each $\alpha < \mu$, a subspace $M_\alpha$ containing $\{x_\alpha\} \cup \bigcup_{\beta<\alpha} M_\beta$, and a LEO $T_\alpha : M_\alpha^* \to X^*$ such that dens $M_\alpha \leq \alpha$ and $T_\alpha(M_\alpha^*) \supseteq \{f_\beta : \beta < \alpha\}$. Set $M = \overline{\bigcup_{\alpha<\mu} M_\alpha}$, and let $R_\alpha : M^* \to M_\alpha^*$ be the restriction operators. Since the unit ball of $B(M^*, X^*)$ is compact in the weak* operator topology, we may take $T$ to be any limit point of the net $(T_\alpha R_\alpha)$.

**Theorem 4.** *Let $X$ be an Asplund space, with* dens $X = \mu$. *Then there exist subspaces $M_\alpha$ of $X$ and LEOs $T_\alpha : M_\alpha^* \to X^*$ $(\omega \leq \alpha < \mu)$ such that*

*(i)* $M_\alpha \subset M_\beta$ *if* $\alpha < \beta$,

*(ii)* dens $M_\alpha \leq \alpha$, *for all* $\alpha$,

*(iii)* $T_\alpha(M_\alpha^*) \subset T_\beta(M_\beta^*)$ *if* $\alpha < \beta$, *and*

*(iv)* $M_\alpha = \overline{\bigcup_{\beta<\alpha} M_\beta}$ *whenever $\alpha$ is a limit ordinal.*

**Proof.** Predictably we construct $T_\alpha$ and $M_\alpha$ inductively. Proposition 2 gives us suitable $M_\omega$ and $T_\omega$. Now suppose that $M_\beta$ and $T_\beta$ are given, for all $\beta < \alpha$. If $\alpha$ is a successor ordinal, we may set $N = M_{\alpha-1}$ and $F = T_{\alpha-1}(M_{\alpha-1}^*)$. Since $X$ is an Asplund space, we have dens $F =$ dens $M_{\alpha-1}^* =$ dens $M_{\alpha-1} =$ dens $N$. Thus Lemma 3 yields a suitable $M_\alpha$ and $T_\alpha$. If $\alpha$ is a limit ordinal, we define $M_\alpha$ by (iv) and obtain $T_\alpha$ as in the previous proof.

**Theorem 5.** *Let $Y$ be any dual space with the Radon-Nikodym property, and let* $\mu =$dens $Y$. *Then there exist norm-one projections $P_\alpha$ on $Y$ $(\omega \leq \alpha < \mu)$ such that*

*(i)* $P_\alpha P_\beta = P_\beta P_\alpha$, *for all $\alpha$ and $\beta$*

*(ii)* $P_\alpha(Y) \subset P_\beta(Y)$ *if* $\alpha < \beta$

*(iii)* dens $P_\alpha(Y) \leq \alpha$, *for all* $\alpha$.

**Proof.** We have $Y = X^*$ for some (not necessarily unique) Asplund space $X$. Let $T_\alpha$ and $M_\alpha$ be as given by Theorem 4, and let $R_\alpha : X^* \to M_\alpha^*$ be the restriction maps. We set $P_\alpha = T_\alpha R_\alpha$ and leave the reader to check the details.

## 2. Attempts at Renorming.

Long sequences of projections, such as those given by Theorem 5, have become an established tool for studying weakly compactly generated spaces [AL]. Since each *factor* $(P_{\alpha+1} - P_\alpha)(Y)$ has strictly smaller density character than the original space, the possibility presents itself of proving results by transfinite induction.

A routine diagram chasing argument shows that, in the notation of Theorems 4 and 5, $(P_{\alpha+1} - P_\alpha)(Y) \cong (M_{\alpha+1}/M_\alpha)^*$ for every $\alpha$. Thus if $Y$ is a dual space with the Radon-Nikodym property, so also is each $(P_{\alpha+1} - P_\alpha)(Y)$. Thus one might expect transfinite induction arguments to flow fairly smoothly.

However, there is still one problem. To prove the results we are interested in, we also need to know that, for each $f \in Y^*$, the map $\alpha \mapsto P_\alpha f$ is norm continuous. This is equivalent to the requirement that $P_\alpha(Y)$ is the norm closure of $\bigcup_{\beta<\alpha} P_\beta(Y)$, whenever $\alpha$ is a limit ordinal. This is known to be the case for WCG spaces. Without this requirement, the decomposition may not be genuine, in the sense that the closed linear span of $\bigcup_\alpha (P_{\alpha+1} - P_\alpha)(Y)$ might not be all of $Y$. We know of no dual space with the RNP for which this fails, but it seems only possible to prove it under additional hypotheses.

Put simply, the idea is this: Given a limit ordinal $\alpha$, and a functional $f \in P_\alpha(Y)$ which attains its norm at some $x \in M_\alpha$, we may approximate $x$ arbitrarily closely by some $y \in M_\beta$, where $\beta < \alpha$. Let $g \in P_\beta(Y)$ be a functional which supports $y$. Given suitable continuity of the support mapping, i.e. reasonable smoothness properties for $X$, one then finds that $g$ is close to $f$. And so $P_\alpha(Y) = \overline{\bigcup_{\beta<\alpha} P_\beta(Y)}$ as required.

Well, the idea might be simple, but the details are the important things. Several authors have investigated smoothness properties which would yield the required continuity of $\alpha \mapsto P_\alpha f$. The most careful analysis so far has been made by Fabian, [F1] and [F2], who showed the following.

**Theorem 6.** *Let $X$ be an Asplund space which admits a non-trivial Gateaux smooth bump function. Then the LEOs in Theorem 4 may be chosen so that $T_\alpha(X^*) = \overline{\bigcup_{\beta<\alpha} T_\beta(X^*)}$ whenever $\alpha$ is a limit ordinal.*

The renorming result, implicit in Fabian's work, is then immediate.

**Theorem 7.** *Let $Y$ be a dual space with RNP, some predual of which admits a Gateaux smooth bump function. Then $Y$ has an equivalent locally uniformly convex norm.*

**Proof.** Let $Y = X^*$, with $M_\alpha, T_\alpha, P_\alpha$ as usual. We note that $P_\alpha(Y) \cong M_\alpha^*$, and that $M_\alpha^*$ is a dual space with RNP, whose predual $M_\alpha$, being a subspace of $X$, has a Gateaux smooth bump function. Moreover dens $P_\alpha Y <$ dens $Y$, so arguing inductively, we may suppose that each $P_\alpha Y$ has an equivalent locally uniformly convex norm. The other hypotheses of [Z] are easily seen to be satisfied, and the conclusion follows.

## 3. Examples of Asplund Spaces.

Asplund spaces may be divided into four mutually exclusive categories.

(i) Banach spaces with separable duals,

(ii) nonseparable reflexive spaces,

(iii) spaces $\mathcal{C}(K)$, where $K$ is compact, scattered, but not metrizable,

(iv) exotic examples.

It is well known [D] that all examples in (i) and (ii) may be renormed so that their duals are locally uniformly convex. It is surprising that spaces in category (iii) have been ignored for so long. However, this situation has recently been rectified by Deville [Dv] and Talagrand [Tl]. Between them, they have proved the following.

**Theorem 8.** *Let $K$ be a compact scattered space and $\alpha$ any ordinal. Then we have the following.*

*(i) $\mathcal{C}[0,\alpha]$ admits an equivalent Frechet smooth norm.*

*(ii) $\mathcal{C}[0,\alpha]$ admits an equivalent norm, under which its dual is strictly convex, if and only if $\alpha$ is countable.*

*(iii) If the $\alpha^{th}$ derived set of $K$ is empty, and $\alpha$ is countable (and also in some other cases), then $\mathcal{C}(K)$ may be renormed so that its dual is locally uniformly convex.*

These results do not completely settle the situation for all spaces in category (iii). For example, it is not known whether $\mathcal{C}([0,\aleph_1]\times[0,\aleph_1])$ admits a Frechet smooth norm. We note that the dual of any space in category (iii) is $l_1(K)$, which is well known to have an equivalent locally uniformly convex norm.

We come to category (iv). We have only been able to think of three sporadic Banach spaces which are also Asplund - the Johnson-Lindenstrauss space [JL] and the long James space ([E] or [B]), and its dual. (Note that the predual of the James tree space, and certain James-Lindenstrauss spaces, fall into category (i).) We consider these three spaces in turn.

The Johnson-Lindenstrauss space - call it $X$ - contains an uncomplemented subspace $M \cong c_0$ such that $H = X/M$ is a nonseparable Hilbert space. It follows from the lifting property of $\ell_1$ that $X^* \cong \ell_1 \oplus H$. A long sequence of projections, as in Theorem 5, is then easy to construct explicitly. However these projections cannot be chosen to be weak$^*$ continuous. Indeed a peculiarity of the dual is that $\ell_1$ is weak$^*$ dense in $X^*$. This shows that the operator in our basic Proposition 2 cannot be chosen weak$^*$ continuous. If it could, then by Sobczyk's theorem, $M$

would be complemented in $X$. Nonetheless $X$ can be renormed so that $X^*$ is locally uniformly convex [JL].

The long James space, $J(\eta)$, was first studied by Edgar [E]. A more detailed exposition of Edgar's work was later given by Bourgin [B]. Let $\eta$ be an uncountable ordinal. (If $\eta$ is countable, then $J(\eta)$ falls into the uninteresting category (i).) Given a function $f : [0,\eta] \to \mathbb{R}$ we define $\|f\|$ by $\|f\|^2 = \sup \sum_{i=1}^{n} |f(\alpha_i) - f(\alpha_{i-1})|^2$, where the sup is taken over all finite subsets of $[0,\eta]$ satisfying $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. Then $J(\eta)$ is defined as the set of all continuous functions $f : [0,\eta] \to \mathbb{R}$ for which $f(0) = 0$ and $\|f\|$ is finite. Letting $g_\alpha$ be the characteristic function of $(\alpha, \eta]$, it can be verified that $\{g_\alpha : \alpha < \eta\}$ is a transfinite basis for $J(\eta)$. If $e_\alpha \in J(\eta)^*$ are the evaluation functionals, then $\{e_\alpha : \alpha < \eta\}$ is a basis for $J(\eta)^*$. Moreover, $\overline{\text{lin sp}}\ \{e_\alpha : \alpha < \eta,\ \alpha \textit{ is a successor ordinal}\}$ is a subspace of $J(\eta)^*$ whose dual is naturally isomorphic to $J(\eta)$. Thus $\{e_\alpha : \alpha < \eta,\ \alpha \textit{ is a successor ordinal}\}$ is a basis for a predual, $J(\eta)_*$. One can verify that every separable subspace of $J(\eta)$ or $J(\eta)_*$ has separable dual, so both of these spaces are Asplund spaces.

A little manipulation with ordinal numbers shows that $J(\eta)^{**} \cong J(\eta+1) \cong J(\eta)$, and so all the even duals and preduals of $J(\eta)$ are isomorphic. Likewise the odd duals and preduals of $J(\eta)$ are all isomorphic to one another. Thus there are only two spaces of real interest here, $J(\eta)$ and $J(\eta)_*$. They are both Asplund spaces with RNP, but are highly nonreflexive.

Define projections $P_\alpha$ on $J(\eta)$ by

$$(P_\alpha f)(\beta) = \begin{cases} f(\beta), & \text{for} \beta \le \alpha; \\ f(\alpha), & \text{for } \beta > \alpha. \end{cases}$$

It is routine to verify that the $P_\alpha$ satisfy all the conclusions of Theorem 6. Of course the $P_\alpha$ are just the basis projections corresponding to $g_\alpha$. Note that as $\beta \uparrow \alpha$, $g_\beta \xrightarrow{w^*} g_\alpha$. Thus $P_\alpha$ is weak* continuous precisely when $\alpha$ is not a limit ordinal.

It follows from [Tr] that $J(\eta)_*$ and $J(\eta)$ have equivalent locally uniformly convex norms. The standard argument for making this a dual renorming for $J(\eta)$ cannot be applied, since not every $P_\alpha$ is weak* continuous. In fact Edgar [E] showed that $J(\eta)$ does not admit any such dual renorming. Adapting the argument of Talagrand [Tl, Theorem 3] gives us a stronger result. The argument is so delightfully simple that it bears repeating.

**Proposition 9.** *Let $X$ be any Banach space, and $\eta$ an uncountable ordinal. Suppose $X^*$ contains a subset weak* homeomorphic to $[0,\eta]$. Then $X^*$ is not strictly convex.*

**Proof.** Let $\{g_\alpha : \alpha < \eta\}$ be the given subset of $X^*$. We show that, under any equivalent dual norm $|||-|||$, $X^*$ contains an isometric copy of the positive cone of $\ell_1$.

Since $|||g_\alpha|||$ is a lower semicontinuous function of $\alpha$, it must be constant on some closed uncountable set $S \subset [0,\eta]$. Scaling, we may suppose that $|||g_\alpha||| = 1$ for every $\alpha \in S$. Let us denote by $\alpha^n$ the $n^{th}$ successor in $S$ of any such $\alpha$.

If $X^*$ does not contain the positive cone of $\ell_1$, then for any $\alpha \in S$, there exist $\varepsilon, \lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbf{Q}^+$ such that $|||\sum_{i=1}^n \lambda_i g_{\alpha^i}||| \leq \sum_{i=1}^n \lambda_i - \varepsilon$. But then we can fix $\varepsilon, \lambda_1, \cdots, \lambda_n$ so that $T = \{\alpha \in S : |||\sum \lambda_i g_{\alpha^i}||| \leq \sum \lambda_i - \varepsilon\}$ is uncountable. Select a sequence $(\alpha_1, \alpha_2, \cdots)$ in $T$ for which $\alpha_{m+1} > \alpha_m^n$ for all $m$. Clearly $\alpha = \sup \alpha_m \in S$ and $|||g_\alpha||| = 1$. Now $g_\alpha = w^* - \lim_{m \to \infty} g_{\alpha_m^i}$ for each $i$, and so

$$\sum \lambda_i = |||\sum \lambda_i g_\alpha||| \leq \liminf_{m \to \infty} |||\sum \lambda_i g_{\alpha_m^i}||| \leq \sum \lambda_i - \varepsilon$$

which is impossible.

The hypotheses of Proposition 9 are satisfied, of course, by $C[0,\eta]$, $J(\eta)$ and $J(\eta)_*$. The question of whether or not $J(\eta)_*$ or $J(\eta)$ admit equivalent Frechet smooth norms remains totally open.

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