INVARIANT DIFFERENTIAL OPERATORS ON SOME LIE GROUPS

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1. INTRODUCTION

A differential operator P on a Lie group G is said to be left (or right) invariant by G if it commutes with the action of G by left (or right) translations. We shall only consider linear differential operators. The algebra of left invariant linear differential operators on G is identified with the complexified universal enveloping algebra $U(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G. Bi-invariant operators (i.e. operators which are both left and right invariant) then correspond to the elements of the centre $Z(\mathcal{G})$ of $U(\mathcal{G})$.

We study the problem of the existence of fundamental solutions for left invariant differential operators on Lie groups. We recall that if P is a differential operator on G, a fundamental solution for P is a distribution $E \in \mathcal{D}'(G)$ on G satisfying the equation $PE = \delta_G$, where δ_G is the Dirac distribution at the origin e_G of G.

Left invariant operators on a Lie group in general do not possess a global fundamental solution, but under additional conditions either on the operator or on the group, one can prove the existence of such solutions. For a review of the results we refer the reader for example to [1]. The present paper is a continuation of [1] and [2] and reports on some recent developments.

We first consider the case of a semidirect product $H \rtimes K$, where H and K are two Lie groups, K is compact connected and acts on H. In the next section we define the partial Fourier coefficients of a differential operator on $G = H \rtimes K$ and describe the action of K on the elements of the universal enveloping algebra $U(\mathfrak{h} \oplus \mathfrak{k})$ of $H \rtimes K$. In the following section we use this partial Fourier transform to study the existence of fundamental solutions for left G-invariant, right K-invariant differential operators and to prove a necessary condition for the existence of a solution. In particular, we apply these results to the operator $D = \Delta_H - \mu\omega_k$ on the n-dimensional Euclidean motion group $M(n) = \mathbb{R}^n \rtimes SO(n)$, where Δ_H denotes the Laplacian on $\mathbb{R}^n$ and ω_k is the Casimir operator on $SO(n)$. In this case we give explicitly a fundamental solution for D on $M(n)$.

In the last section we consider the Casimir operator on the group $SO(n,1)$. The Cartan motion group associated to $SO(n,1)$ is the n-dimensional Euclidean motion group $M(n)$. Using the method of [2] and the results of the previous section we are able to construct a fundamental solution for the Casimir operator on $SO(n,1)$.

2. FOURIER TRANSFORM ON A SEMIDIRECT PRODUCT

Let H and K be two Lie groups, K compact connected. Assume that K acts on H by

$$\sigma : H \times K \to H$$
$$(h,k) \mapsto \sigma_k(h)$$

and that this action is such that the map

$$K \to \mathrm{Aut}(H)$$
$$k \mapsto \sigma_k$$

is a group homomorphism.

We consider the semidirect product $G = H \rtimes K$ associated with this action. The multiplication law is given by

$$(h,k)(h',k') = (h\sigma_k(h'),kk')$$

for all $h,h' \in H$ and $k,k' \in K$.

EXAMPLE: CARTAN MOTION GROUPS

Let G be a noncompact semisimple Lie group, connected, with finite centre and let K be a maximal compact subgroup of G.

Let $\mathcal{G}$ and $\mathcal{k}$ be the Lie algebras of G and K respectively, Ad the adjoint representation of K on G and V a vector complement of $\mathcal{k}$ in $\mathcal{G}$ such that $Ad(K)V \subset V$.

The semidirect product $V \rtimes K$ relative to this action is called the Cartan motion group associated to the pair (G,K).

The multiplication law in $V \rtimes K$ is given by

$$(v,k)(v',k') = (v+k.v',kk')$$

where $v,v' \in V$, $k,k' \in K$, $k.v' = Ad(k)(v')$ and $\exp(k.v') = k(\exp v')k^{-1}$.

In particular if $G = PSL(2,\mathbb{R})$ and $K = SO(2)$, then $V = \mathbb{R}^2$ and the corresponding Cartan motion group is the Euclidean motion group $M(2)$.

Since the group K is compact, we have a partial Fourier transform on G. We use this Fourier transform to translate the problem on G to an equivalent problem on the group H.

Let us first define the partial Fourier coefficients of a differential operator.

Let $\hat{K}$ denote the dual of K.

Let P be a left invariant differential operator on the semidirect product G. The partial Fourier coefficients P_Λ, $\Lambda \in \hat{K}$, of P are defined by

$$(P_\Lambda f)(h) = P(f \otimes \Lambda)(h, e_K)$$

for every $f \in \mathcal{D}(H)$ and $\Lambda \in \hat{K}$, where e_K is the unit element of K.

They are left invariant differential operators on H with coefficients in $\mathrm{End}(H_\Lambda)$ (H_Λ is the representation space of Λ).

In order to describe them more precisely, let us express them in terms of vector fields.

Let $\mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of H and K and let $(X_1,\ldots,X_n)$ and $(T_1,\ldots,T_p)$ be bases of $\mathfrak{h}$ and $\mathfrak{k}$ respectively. Then $(X_1,\ldots,X_n,T_1,\ldots,T_p)$ is a basis of the Lie algebra $\mathfrak{h} \oplus \mathfrak{k}$ of $H \rtimes K$.

Let $X_{H_1},\ldots,X_{H_n}, T_{K_1},\ldots,T_{K_p}, X_{G_1},\ldots,X_{G_n}, T_{G_1},\ldots,T_{G_p}$ denote the corresponding left invariant vector fields on H,K and G respectively. For $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$ and $\beta = (\beta_1,\ldots,\beta_p) \in \mathbb{N}^p$, we set

$$X_H^\alpha = X_{H_1}^{\alpha_1}\cdots X_{H_n}^{\alpha_n}, \quad X_G^\alpha = X_{G_1}^{\alpha_1}\cdots X_{G_n}^{\alpha_n},$$

and

$$T_K^\beta = T_{K_1}^{\beta_1}\cdots T_{K_p}^{\beta_p}, \quad T_G^\beta = T_{G_1}^{\beta_1}\cdots T_{G_p}^{\beta_p}.$$

For each $k \in K$, let s_k be the derivative of the map $\sigma_k : H \to H$.

We have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\exp} & H \\ s_k \big\downarrow & & \big\downarrow \sigma_k \\ \mathfrak{h} & \xrightarrow{\exp} & H \end{array}$$

i.e. for every $k \in K$ and $X \in \mathfrak{h}$,

$$\exp(s_k(X)) = \sigma_k(\exp X).$$

LEMMA 1 *(i) For all* $X \in \mathfrak{h}$, $T \in \mathfrak{k}$, $f \in \mathcal{D}(H)$ *and* $g \in C^\infty(K)$ *we have*

$$X_G(f \otimes g)(h,k) = [s_k(X)_H f](h)g(k)$$

and

$$T_G(f \otimes g)(h,k) = f(h)(T_K g)(k)$$

(ii) For all $X \in \mathfrak{h}$, $f \in \mathcal{D}(H)$, $k \in K$ *and* $\alpha \in \mathbb{N}$, *we have*

$$[s_k(X)_H]^\alpha f = X_H^\alpha(f \circ \sigma_k) \circ \sigma_{k^{-1}}$$

PROOF (i) For $X \in \mathfrak{h}$ we have

$$X_G(f \otimes g)(h,k) = \frac{d}{dt}\Big|_{t=0} (f \otimes g)((h,k)(\exp tX, e_K))$$

$$= \frac{d}{dt}\Big|_{t=0} (f \otimes g)(h\sigma_k(\exp(tX)),k)$$

$$= \frac{d}{dt}\Big|_{t=0} (f \otimes g)(h \exp[ts_k(X)],k)$$

$$= \frac{d}{dt}\Big|_{t=0} \{f(h \exp[t\ s_k(X)])g(k)\}$$

$$= \left\{\frac{d}{dt}\Big|_{t=0} f(h \exp[t\ s_k(X)])\right\}g(k)$$

$$= [s_k(X)_H f](h)g(k) ,$$

and, for $T \in \mathfrak{h}$

$$T_G(f \otimes g)(h,k) = \frac{d}{dt}\Big|_{t=0} (f \otimes g)((h,k)(e_H, \exp(tT)))$$

$$= \frac{d}{dt}\Big|_{t=0} (f \otimes g)(h, k\exp(tT))$$

$$= \frac{d}{dt}\Big|_{t=0} \{f(h)g(k\exp(tT))\}$$

$$= f(h)\frac{d}{dt}\Big|_{t=0} g(k\exp(tT))$$

$$= f(h)(T_K g)(k)$$

(ii) For $X \in \mathfrak{h}$, we have

$$[s_k(X)_H f](h) = \frac{d}{dt}\Big|_{t=0} f(h\exp[ts_k(X)])$$

But
$$\exp[ts_k(X)] = \sigma_k(\exp(tX))$$

and since the map $k \mapsto \sigma_k$ is a homomorphism,

$$h\sigma_k(\exp(tX)) = \sigma_{kk^{-1}}(h)\sigma_k(\exp tX) = \sigma_k(\sigma_{k^{-1}}(h)\exp(tX)).$$

So

$$[s_k(X)_H f](h) = \frac{d}{dt}\Big|_{t=0} f(h\exp[ts_k(X)])$$

$$= \frac{d}{dt}\Big|_{t=0} f(\sigma_k(\sigma_{k^{-1}}(h)\exp(tX)))$$

$$= \frac{d}{dt}\Big|_{t=0} (f \circ \sigma_k)(\sigma_{k^{-1}}(h)\exp(tX))$$

$$= [X_H(f \circ \sigma_k)](\sigma_{k^{-1}}(h)).$$

Thus we have

$$s_k(X)_H f = X_H(f \circ \sigma_k) \circ \sigma_{k^{-1}}$$

and, by induction, for $\alpha \in \mathbb{N}$

$$[s_k(X)_H]^\alpha f = X_H^\alpha(f \circ \sigma_k) \circ \sigma_{k^{-1}}.$$ ❑

Let $P \in U(\mathcal{G})$. We can now calculate explicitly the partial Fourier coefficients P_Λ of P.

According to the Poincaré-Birkhoff-Witt theorem, P can be written

$$P = \sum_{\alpha\in\mathbf{N}^n,\beta\in\mathbf{N}^p} a_{\alpha\beta}\, X_G^\alpha\, T_G^\beta$$

where the $a_{\alpha\beta}$'s are complex numbers.

For $f \in \mathcal{D}(H)$ and $\Lambda \in \hat{K}$, we have

$$P(f \otimes \Lambda)(h,k) = [\sum_{\alpha,\beta} a_{\alpha\beta}\, X_G^\alpha T_G^\beta (f \otimes \Lambda)](h,k)$$

and applying lemma 1

$$P(f \otimes \Lambda)(h,k) = [\sum_{\alpha,\beta} a_{\alpha\beta} X_G^\alpha (f \otimes \Lambda) T_\Lambda^\beta](h,k)$$

where $T_\Lambda = \frac{d}{dt}\Lambda(\exp(tT))\big|_{t=0}$ is an endomorphism of H_Λ.

So

$$P(f \otimes \Lambda)(h,k) = [\sum_{\alpha,\beta} a_{\alpha\beta}\, X_G^\alpha (f \otimes \Lambda)](h,k) T_\Lambda^\beta$$

and applying once again lemma 1 we get

$$P(f \otimes \Lambda)(h,k) = \sum_{\alpha,\beta} a_{\alpha\beta} (s_k(X)_H^\alpha f)(h)\Lambda(k) T_\Lambda^\beta .$$

Now

$$\begin{aligned}(P_\Lambda f)(h) &= P(f \otimes \Lambda)(h,e_K)\\ &= [\sum_{\alpha,\beta} a_{\alpha\beta} (s_k(X)_H^\alpha f)(h)\Lambda(k) T_\Lambda^\beta]\big|_{k=e_K}\\ &= \sum_{\alpha,\beta} a_{\alpha\beta} (X_H^\alpha f)(h) T_\Lambda^\beta ,\end{aligned}$$

hence

$$P_\Lambda = \sum_{\alpha,\beta} a_{\alpha\beta}\, X_H^\alpha\, T_\Lambda^\beta .$$

We define as well the partial Fourier coefficients of a distribution on G.

Let U be an open subset in H and E be a distribution on $U \rtimes K$. The partial Fourier coefficients $\hat{E}(h,\Lambda)$ of E are defined as follows:

$$\langle \hat{E}(h,\Lambda), f(h) \rangle = \langle E(h,k), f(h)\Lambda(k) \rangle$$

for all $\Lambda \in \hat{K}$ and $f \in \mathcal{D}(U)$.

They are distributions on U with values in $\mathrm{End}(H_\Lambda)$. Let $\mathcal{D}'(U,\mathrm{End}(H_\Lambda))$ denote the space of distributions on U with values in $\mathrm{End}(H_\Lambda)$.

3. A NECESSARY CONDITION

Applying this partial Fourier transform to the differential equation $PE = \delta_G$, we get the following theorem:

THEOREM 2 *Let* P *be a left* G-*invariant, right* K-*invariant linear differential operator on* G, *and let* U *be an open subset in* H. *Then* P *has a fundamental solution on* $U \rtimes K$ *if and only if, for each* $\Lambda \in \hat{K}$, *there exists a distribution* $E_\Lambda \in \mathcal{D}'(U,\mathrm{End}(H_\Lambda))$ *on* U *such that*

(1) $$^t[(^tP)_\Lambda]E_\Lambda = \delta_H \mathrm{Id}_{H_\Lambda}$$

and

(2) *for every compact subset* $C \subset U$, *there exists a constant* $A > 0$ *and positive integers* a *and* b *such that*

$$\|\langle E_\Lambda, f \rangle\|_{HS} \le AN(\Lambda)^a \|f\|_b$$

for all $\Lambda \in \hat{K}$ *and* $f \in \mathcal{D}(U)$ *with* supp $f \subset C$.

Here δ_H is the Dirac distribution on H at e_H, N is a positive function on $\hat{K}$, the semi-norms $\|.\|_b$, $b \in \mathcal{N}$, define the topology of $\mathcal{D}(H)$ and, if u is an endomorphism of a vector space, $\|u\|_{HS}$ denotes the Hilbert-Schmidt norm of u, i.e. $\|u\|_{HS} = \sqrt{tr(uu^*)}$, where u^* is the adjoint of u and $tr(uu^*)$ is the trace of uu^*.

This result has been proved in detail in [1], in the case of Cartan motion groups. The same proof based, on one hand, on the calculation of the Fourier transform of the distribution PE, for some $E \in \mathcal{D}'(G)$, and, on the other hand, on a characterization of distributions on G, works for the semidirect product $G = H \rtimes K$.

We can then use this theorem to give a necessary condition for the existence of a fundamental solution for left G-invariant, right K-invariant differential operators on the semidirect product $G = H \rtimes K$.

A canonical coordinate neighbourhood U of e_H in H is an open neighbourhood of e_H such that there exists an open neighbourhood W of 0 in $\mathfrak{h}$ such that the exponential map is an analytic diffeomorphism of W onto U. Such neighbourhoods exist (cf. [4], proposition 1.6, p.104).

In that case, let $(X_1,\dots,X_n)$ be a basis of $\mathfrak{h}$ and $(x_1,\dots,x_n)$ the corresponding system of coordinates.

Then, for any left invariant linear differential operator P on G, the operators ${}^t[({}^tP)_\Lambda]$, $\Lambda \in \hat{K}$, can be written on U

$${}^t[({}^tP)_\Lambda] = Q_\Lambda\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)$$

where Q_Λ is a polynomial of n variables, with coefficients in $\mathrm{End}(H_\Lambda)$.

If M is a matrix in $\mathrm{End}(H_\Lambda)$, ${}^{co}M$ denotes the comatrix of M, that is the transpose of the matrix of the cofactors of M. We have

$$ {}^{co}M.M = M.{}^{co}M = \det M.\mathrm{Id}_{H_\Lambda} $$

Let $M(\xi)$ be a matrix in $\mathrm{End}(H_\Lambda)$ such that its coefficients with respect to an orthonormal basis of H_Λ are polynomials $m_{ij}(\xi)$, then $\tilde{M}(\xi)$ denotes the matrix whose coefficients with respect to this basis are $\tilde{m}_{ij}(\xi)$ where

$$ \tilde{m}_{ij}(\xi) = \left(\sum_{\alpha \in \mathbb{N}} |m_{ij}^{(\alpha)}(\xi)|^2 \right)^{1/2} $$

(this definition does not depend on the choice of the orthonormal basis in H_Λ).

THEOREM 3 *Let* H *and* K *be two Lie groups,* K *compact connected,* $G = H \rtimes K$ *and* U *be a canonical coordinate neighbourhood of* e_H *in* H. *Let* P *be a right* K*-invariant left* G*-invariant linear differential operator on* G. *The following condition:*

$$ \exists A > 0, \quad \exists a \in \mathbb{N}, \quad \forall \Lambda \in \hat{K} $$

$$ \det Q_\Lambda(\xi) \not\equiv 0 \quad \text{and} \quad \left\| \frac{({}^{co}Q_\Lambda(0))^{\sim}}{(\det Q_\Lambda(0))^{\sim}} \right\|_{HS} \leq A N(\Lambda)^a $$

is a necessary condition for the existence of a fundamental solution for P *on* $U \rtimes K$.

PROOF Let P be as in the theorem, and assume that P has a fundamental solution on $U \rtimes K$. Then applying theorem 2, for each $\Lambda \in \hat{K}$, there exists a distribution $E_\Lambda \in \mathcal{D}'(U,\mathrm{End}(H_\Lambda))$ on U satisfying (1) and (2).

Since U is a canonical coordinate neighbourhood of e_H in H, there exists an open neighbourhood of 0 in $\mathfrak{h}$ such that the exponential map is an analytic diffeomorphism of W onto U. Use of this diffeomorphism and the same arguments as in [1] give the necessity of the condition.

This condition generalizes the necessary and sufficient condition of Cerezo and Rouvière [3] for solvability of left invariant differential operators on a direct product $\mathbb{R}^n \times K$.

EXAMPLE Let $H = \mathbb{R}^n$ and $K = SO(n)$. It is interesting to apply the previous theorems to the differential operator on $M(n) = H \rtimes K$ given by $D = \Delta_H - \mu\omega_K$, where μ is a positive constant, Δ_H denotes the Laplace operator in the H-variables and ω_K denotes the Casimir on K. This operator is K-bi-invariant on $H \rtimes K$.

Letting Λ be a highest weight for $SO(n)$, the partial Fourier coefficient D_Λ of D is the operator on H given by

$$D_\Lambda = (\Delta_H + \mu\beta(\Lambda))\mathrm{Id}_{H_\Lambda}$$

where $\beta(\Lambda) = (\Lambda+\delta,\Lambda+\delta)-(\delta,\delta)$ and 2δ is the sum of positive roots (cf. [6], p.188). One sees easily that ${}^t[({}^tD)_\Lambda] = D_\Lambda$.

We shall prove

PROPOSITION 4 *A K-bi-invariant fundamental solution for D is given by*

$$F^\mu(v,k) = \sum_{\Lambda\in\hat{K}} d(\Lambda)(\sqrt{\mu\beta(\Lambda)})^{n-2}F(\sqrt{\mu\beta(\Lambda)}\,r)\chi_\Lambda(k) \tag{3}$$

where r *denotes* $|v|$. *The function* F *is given by*

$$F(t) = \frac{(-1)^{\frac{n}{2}-1}}{2^{\frac{n}{2}+1}\pi^{\frac{n}{2}-1}}\, t^{-\frac{(n-2)}{2}}\, N_{\frac{n-2}{2}}(t) ,$$

where N_n *is the Neumann function, and* χ_Λ *is the character of* Λ, *that is* $\chi_\Lambda(k) = \mathrm{tr}[\Lambda(k)]$ *and* $d(\Lambda) = \dim H_\Lambda$.

PROOF The operator $\Delta_H + \mu\beta(\Lambda)$, considered as a differential operator on $\mathbf{R}^n$ has a classical rotation-invariant fundamental solution E^μ_Λ given, for example by Trèves [5], p.259. Setting $r = |v|$, one has

$$E^\mu_\Lambda(v) = \frac{(-\sqrt{\mu\beta(\Lambda)})^{\frac{n-2}{2}}}{2^{\frac{n}{2}+1}\pi^{\frac{n}{2}-1}}\, r^{-\frac{n-2}{2}}\, N_{\frac{n-2}{2}}(\sqrt{\mu\beta(\Lambda)}\, r)$$

$$= (\sqrt{\mu\beta(\Lambda)})^{n-2}\, F(\sqrt{\mu\beta(\Lambda)}\, r) .$$

(Notice that $\sup|F(t)| < \infty$, cf [7], p.375).

Thus, the operator D_Λ has as fundamental solution $\underline{E}^\mu_\Lambda(v) = E^\mu_\Lambda(v)\,\mathrm{Id}_{H_\Lambda}$. In order to show that (3) defines a K-bi-invariant distribution (on $H \rtimes K$), it is necessary to show that condition (2) of theorem 2 holds. In order to do this, let $f \in \mathcal{D}(C)$ where $C \subset \mathbf{R}^n$ is a symmetric compact set and let $\Lambda \in SO(n)^\wedge$. Then

$$\|\langle \underline{E}^\mu_\Lambda, f\rangle\|_{HS} = (\sqrt{\mu\beta(\Lambda)})^{n-2} \left| \int_C F(\sqrt{\mu\beta(\Lambda)}\, r) f(v)\, dv \right| \|\mathrm{Id}_{H_\Lambda}\|_{HS}$$

$$\leq d(\Lambda)^{1/2} (\sqrt{\mu\beta(\Lambda)})^{n-2} \mathrm{vol} C \|f\|_0 \sup|F(t)| .$$

By the Weyl dimension formula $d(\Lambda)^{1/2}$ grows polynomially in Λ, so we have finished. ❑

4. TRANSFERRING FUNDAMENTAL SOLUTIONS

In a previous paper [2] it was shown how to "transfer" fundamental solutions of a family of differential operators on a Cartan motion group to obtain a fundamental solution on a semisimple group G. We shall consider here the case where $G = SO(n,1)$ for which the associated Cartan motion group is $\mathbb{R}^n \rtimes SO(n)$, the semidirect product of the preceding example.

The Casimir operator for $SO(n,1)$ is given by [6], p.169, by $D = \Delta_H - \omega_K$ where Δ_H is the Laplacian on the "$\mathbb{R}^n$" part of $SO(n,1)$ and ω_K is the Casimir operator on the $SO(n)$ part. We are going to use proposition 6 of [2] to find a K-bi-invariant fundamental solution for D on G.

The differential operators $\phi_\lambda^{-1}(D)$ on $\mathbb{R}^n \rtimes SO(n) = M(n)$ are $D_\lambda = \lambda^2\Delta_V - \omega_K$, almost exactly those considered in the preceding example. As seen in the proposition 4, this operator has a K-bi-invariant fundamental solution $\tilde{F}_\lambda(v,k)$ given by

$$\tilde{F}_\lambda(v,k) = \lambda^{-2}F^{\lambda^{-2}}(v,k)$$

$$= \lambda^{-2} \sum_{\Lambda\in\hat{K}} d(\Lambda)\,[\sqrt{\beta(\Lambda)}\,]^{n-2} F(\sqrt{\beta(\Lambda)}\tfrac{r}{\lambda})\chi_\Lambda(k) \quad .$$

One calculates an expression for the distribution $\tilde{E}_\lambda$ by using

$$\langle\tilde{E}_\lambda, f\rangle = \langle\tilde{F}_\lambda, f \circ \pi_\lambda\rangle$$

$$= \lambda^{-n} \sum_{\Lambda\in\hat{K}} d(\Lambda)\,(\sqrt{\beta(\Lambda)}\,)^{n-2} \int_K\int_V F(\sqrt{\beta(\Lambda)}\tfrac{r}{\lambda})\chi_\Lambda(k) f(\exp\tfrac{v}{\lambda}.k)\,dvdk.$$

Putting $\frac{v}{\lambda} = u$ in the v-integration, one obtains

$$<\tilde{E}_\lambda, f> = \sum_{\Lambda \in \hat{K}} d(\Lambda)(\sqrt{\beta(\Lambda)})^{n-2} \int_K \int_V F(\sqrt{\beta(\Lambda)}|u|)\chi_\Lambda(k) f(\exp u.k)\,du\,dk.$$

Since this expression is independent of λ, we see that $\tilde{E}_\lambda$ does indeed converge to a distribution E, given by the right hand side of the above expression. It is clear that E is K-bi-invariant.

We have shown, by an application of proposition 6 of [2]:

THEOREM 5 *The distribution* E *defined by*

$$<E, f> = \sum_{\Lambda \in SO(n)^\wedge} d(\Lambda)(\sqrt{\beta(\Lambda)}) \int_K \int_V F(\sqrt{\beta(\Lambda)}|u|) f(\exp u.k)\,du\,\chi_\Lambda(k)\,dk$$

defines a K-*bi-invariant fundamental solution of the Casimir operator on* SO(n,1).

These methods also work for the other rank 1 semisimple groups.

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