AN INTRODUCTION TO THE ABEL TRANSFORM

R.J.Beerends

This paper is intended as an introduction to the Abel transform for the non-specialist. Nevertheless it contains some of the essential ideas which enables us to present some recent results on this transform in the last section.

The Abel transform plays an important role in the theory of the spherical Fourier transform on symmetric spaces of the noncompact type. Its role is analogous to the role of the Radon transform in the theory of the ordinary Fourier transform on $\mathbb{R}^n$. Therefore we first present the example of the ordinary Fourier transform in section 1. Then we turn to the example of $SL(2,\mathbb{R})$ (sections 2 and 3). Here we give an explicit expression for the Abel transform and review some of the results and applications. This will serve as motivation and as prototype for further research. In the last section we present some recent results.

1. Fourier and Radon transform on $\mathbb{R}^n$

In order to explain the role of the so-called Abel transform in harmonic analysis on semisimple Lie groups, we first take a look at the ordinary Fourier transform on $\mathbb{R}^n$. Let $\mathcal{D}(\mathbb{R}^n)$ denote the space of all C^∞-functions on $\mathbb{R}^n$ with compact support. For $f \in \mathcal{D}(\mathbb{R}^n)$ we consider its Fourier transform

$$(1.1) \qquad (\mathcal{F}f)(\lambda) = f^*(\lambda) = \int_{\mathbb{R}^n} f(x)\, e^{-i<x,\lambda>}\, dx \quad , \quad \lambda \in \mathbb{R}^n \ ,$$

where $<\cdot,\cdot>$ denotes the standard inner product on $\mathbb{R}^n$. Put $\lambda=\rho\omega$, $\rho \in \mathbb{R}^+$, $\omega \in S^{n-1}$ (the unit sphere in $\mathbb{R}^n$), then

$$f^*(\rho\omega) = \int_{-\infty}^{\infty} \int_{<y,\omega>=r} f(y)\, e^{-i\rho<y,\omega>}\, dy\, dr \ ,$$

where dy is the Lebesgue measure on the hyperplane $\{y \in \mathbb{R}^n | <y,\omega> = r\}$. Consequently

$$(1.2) \qquad f^*(\rho\omega) = \int_{-\infty}^{\infty} f^\wedge(\omega,r)\, e^{-i\rho r}\, dr \ ,$$

with

$$(1.3) \qquad f^\wedge(\omega,r) = \int_{<y,\omega>=r} f(y)\, dy \quad , \quad (\omega,r) \in S^{n-1}\times \mathbb{R} \ ,$$

The function $f^\wedge$ is called the Radon transform of f and (1.2) shows that the Fourier transform on $\mathbb{R}^n$ equals the one-dimensional Fourier transform of the Radon transform. Thus the Radon transform and Fourier transform are closely connected.

One shows that

$$\left(\frac{\partial f}{\partial x_i}\right)^\wedge(\omega,r) = \omega_i \frac{\partial f^\wedge}{\partial r}(\omega,r) \quad , \quad (\omega,r) \in S^{n-1}\times \mathbb{R} \ ,$$

and hence

$$(1.4) \qquad (Lf)^\wedge(\omega,r) = \frac{\partial^2}{\partial r^2} f^\wedge(\omega,r) \ , \quad \text{where} \quad L = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \ ,$$

i.e. L is the Laplacian on $\mathbb{R}^n$. The so-called "transmutation property" (1.4) is very useful in harmonic analysis: it can reduce problems concerning L to the simpler operator $\dfrac{\partial^2}{\partial r^2}$.

We now consider the ordinary Fourier transform (1.1) from a group-theoretical point of view. This will enable us to give a natural definition of the spherical Fourier transform on noncompact semisimple Lie groups. So, let $G = M(n)$ denote the group of all isometries of $\mathbb{R}^n$, i.e. $M(n)$ is the semidirect product $O(n)\cdot\mathbb{R}^n$ where $\mathbb{R}^n$ acts as translation group and $K = O(n)$ is the orthogonal group. We can view $X = \mathbb{R}^n$ as the quotient $G/K = M(n)/O(n)$ ($O(n)$ fixes the origin $o \in \mathbb{R}^n$, so we map a coset gK, $g \in G$ to $g\cdot o$). The Laplacian L is invariant under the action of $M(n)$: $L(f \circ M) = Lf \circ M$, $f \in \mathcal{D}(\mathbb{R}^n)$, $M \in M(n)$. The kernel $e^{-i<x,\lambda>}$ on $\mathbb{R}^n\times \mathbb{R}^n$ in (1.1) is an eigenfunction of L.

The radial functions form an important class of functions on $\mathbb{R}^n$. These functions only depend on the distance $r=|x|$ from the origin and are precisely the $O(n)$-invariant functions on $\mathbb{R}^n$. One then studies the eigenfunctions of the radial part $\Delta(L)$ of the Laplacian L, i.e. eigenfunctions of the operator

$$(1.5) \qquad \Delta(L) = \frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr}$$

Instead of the kernel $e^{-i<\lambda,x>}$ (eigenfunctions of L) one uses Bessel functions (eigenfunctions of $\Delta(L)$) as kernel in order to define the "spherical" Fourier transform of a radial function. The resulting transform is called the Hankel transform. Of course $\Delta(L)$, Bessel functions and Hankel transform still make sense if we replace n-1 in (1.5) by an arbitrary real parameter μ. The "group-cases" $G = M(n)$, which correspond to $\mu=n-1$, $n \in \mathbb{N}$ can now be considered as special cases of a more general theory. We will return to this point of view later on.

We shall refer to the ordinary Fourier transform on $\mathbb{R}^n$ as "the Euclidean case". Using $SL(2,\mathbb{R})$ as a typical example of a "noncompact semisimple Lie group", we will show how one can generalize the theory of the spherical Fourier transform to these groups.

2. The spherical Fourier transform

Let G be a noncompact semisimple Lie group with finite center. A typical example is $G = SL(2,\mathbb{R})$, the group of all 2×2 real matrices with determinant 1:

$$G = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{R} \ , \ ad-bc=1\}.$$

As analogue of the subgroup $O(n)$ in the Euclidean case, one takes a maximal compact subgroup K in G and studies the homogeneous space $X = G/K$. For $SL(2,\mathbb{R})$ we have $K = SO(2)$, so

$$K = \{\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \mid \phi \in [0,2\pi)\}.$$

Let $SL(2,\mathbb{R})$ act on the upper half-plane $X = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ by fractional linear transformations:

$$g\cdot z = \frac{az+b}{cz+d} \quad \text{if} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

(note that $\operatorname{Im}(g\cdot z) = y|cz+d|^{-2}$ so $g\cdot z \in X$). Then K is precisely the stabilizer of $i \in X$: $K = \{g \in G \mid g\cdot i=i\}$. We can identify G/K and X using the map $gK \longrightarrow g\cdot i$; so the point i will be the origin in the space X.

In the Euclidean case one takes as metric $ds^2= dx_1^2+\ldots+dx_n^2$, i.e. the length of a curve $\gamma(t)=(\gamma_1(t),\ldots,\gamma_n(t))$, $a \le t \le b$ equals $\int_a^b (\sum_{i=1}^n (\gamma_i'(t))^2)^{\frac12}\, dt$. This metric is invariant under the action of $M(n)$. On the upper half-plane X we take as metric $ds^2= y^{-2}(dx^2+dy^2)$ $(z=x+iy)$, i.e. the length of a curve $z(t)=x(t)+iy(t)$, $a \le t \le b$ is defined as

$\int_a^b y^{-1}(t)(x'(t)^2 + y'(t)^2)^{1/2}\, dt$. As in the Euclidean case, this metric is invariant under the action of $SL(2,\mathbb{R})$: if $g\cdot(x+iy) = u(x,y)+iv(x,y)$ then $(du^2+dv^2)/v^2 = (dx^2+dy^2)/y^2$. With this metric X is often called the Poincaré or Lobatchevsky upper half-plane and it is a model for hyperbolic geometry. The geodesics (the curves minimizing the Poincaré arc length) are straight lines or circles orthogonal to the x-axis. For a general Lie group as above one takes a G-invariant metric on the homogeneous space $X = G/K$. With this metric one associates a G-invariant differential operator L_X. The central role of the Laplacian L on $\mathbb{R}^n$ in the Euclidean case is taken over by this operator L_X, which is now called the Laplace-Beltrami operator. For the case $G = SL(2,\mathbb{R})$ one has

$$L_X = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) .$$

Instead of radial functions on $\mathbb{R}^n$ we study left K-invariant functions on $X = G/K$, so functions on X such that $f(kx)=f(x)$ for all $k \in K$, $x \in X$. Of course one can also consider these functions as K-biinvariant functions on G. We restrict ourselves to the case of K-biinvariant functions on G, which is often called the spherical case. In contrast to the ordinary Fourier transform on $\mathbb{R}^n$, the spherical case is easier than the general case. For $G = SL(2,\mathbb{R})$ we thus consider left $SO(2)$-invariant functions on the upper half-plane X. As in the Euclidean case we introduce coordinates (t,ϕ) on X which are suitable for the analysis of left K-invariant functions, i.e. which are the analogue of polar coordinates. It can be shown that any point $z \in X$ can be written as

$$z = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot i \quad , \tag{2.1}$$

where $t \geq 0$ and $0 \leq \phi < 2\pi$. Put

$$A = \{a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R}\} \quad , \quad \mathfrak{a} = \{\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mid t \in \mathbb{R}\} \ ,$$

then $A = \exp \mathfrak{a}$, where $\exp\colon \mathfrak{a} \longrightarrow A$ is the usual matrix exponentiation. From (2.1) it follows that a K-biinvariant function on G is completely determined by its restriction to A. In the coordinates (t,ϕ) the Laplace-Beltrami operator can be written as

$$L_X = \frac{\partial^2}{\partial t^2} + \frac{\operatorname{ch} t}{\operatorname{sh} t}\frac{\partial}{\partial t} + \frac{1}{\operatorname{sh}^2 t}\frac{\partial^2}{\partial \phi^2} .$$

So for the spherical Fourier transform we have to consider eigenfunctions of the radial part $\Delta(L_X)$ of the operator L_X where

$$\Delta(L_X) = \frac{\partial^2}{\partial t^2} + \frac{\operatorname{ch} t}{\operatorname{sh} t}\frac{\partial}{\partial t} .$$

The differential equation $\Delta(L_X)\psi = \lambda\psi$ is well-known and its solutions are given by Legendre functions. In general one studies the eigenfunctions ϕ_λ (corresponding to an eigenvalue depending on λ) of the "radial part" $\Delta(L_X)$ of the Laplace-Beltrami operator L_X. One then defines the spherical Fourier transform as integration over G using ϕ_λ as kernel.

Let $\mathcal{D}(G//K)$ denote the space of all C^∞-functions on G with compact support and K-biinvariant. If $f \in \mathcal{D}(G//K)$ then $\tilde{f}$ will denote its spherical Fourier transform. As in the case $G = SL(2,\mathbb{R})$ one can show that a K-biinvariant function on G is completely determined by its restriction to $A = \exp \mathfrak{a}$, where $\mathfrak{a}$ is a finite dimensional Euclidean space. The dimension of $\mathfrak{a}$ is called the rank of $X = G/K$. If $h \in \mathcal{D}(A)$ then $(h \circ \exp)^*$ will denote the Euclidean Fourier transform of $h \circ \exp$. Then one can show that the following analogue of (1.2) holds:

$$\tilde{f} = (F_f \circ \exp)^* , \tag{2.2}$$

where F_f is the so-called Abel transform of f. The function F_f is an element of $\mathcal{D}(A)$. So in the context of the spherical Fourier transform on noncompact semisimple Lie groups, the Abel transform is the analogue of the Radon transform (1.3). One also has the analogue of (1.4), i.e. the "transmutation property"

$$F_{\Delta(L_X)f} = (L_A - c)\, F_f , \tag{2.3}$$

where L_A is the ordinary Laplacian on A, c is a positive constant and f is restricted to A. The basic identities (2.2) and (2.3) show that the Abel transform is an important object in the theory of the spherical Fourier transform. Since the Euclidean Fourier transform is well-known one can try e.g. to obtain the inversion of the spherical Fourier transform $f \longrightarrow \tilde{f}$ by use of (2.2) and an explicit inversion of the transform $f \longrightarrow F_f$. In fact it was Godement [6] who first used this method for the case $SL(2,\mathbb{R})$.

3. The Abel transform for SL(2,ℝ)

Let N be the subgroup of $G = SL(2,\mathbb{R})$ defined by

$$N = \{n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}\}.$$

If $f \in \mathcal{D}(G//K)$ then the Abel transform F_f of f is the function on A defined by

$$(3.1) \qquad F_f(a_t) = \pi^{-1} e^t \int_{\mathbb{R}} f(a_t n_x)\, dx \quad , \quad a_t \in A .$$

In (3.1) we actually integrate over the subgroup N. Since f is right K-invariant we can also view f as a function on the upper half-plane X and then we integrate over the orbit $a_t N \cdot i$ (recall that i is the origin in X). Since $a_t \cdot z = e^{2t} z$ for $z \in X$ it follows that

$$a_t N \cdot i = \{e^{2t} i + x \mid x \in \mathbb{R}\} ,$$

and so the orbits $a_t N \cdot i$ are just the straight lines orthogonal to the y-axis. The transform (3.1) is the analogue for $SL(2,\mathbb{R})$ of the Radon transform (1.3). A hyperplane in $\mathbb{R}^n$ is orthogonal to a family of parallel lines, i.e. orthogonal to a family of parallel geodesics. In (3.1) integration over hyperplanes is replaced by integration over the orbits $a_t N \cdot i$, which are orthogonal to the family of parallel geodesics consisting of the straight lines orthogonal to the x-axis. Let us now calculate the integral (3.1) explicitly. First we remark that one can show that $F_f(a_t) = F_f(a_{-t})$ (one uses the K-biinvariance of f). So we can restrict ourselves to the case $t \geq 0$. From (2.1) it follows that every $g \in G$ can be written as $g = k' a_s k$ with $k, k' \in K$, $s \geq 0$. Here s is uniquely determined by g. In particular we can write for fixed $t \geq 0$

$$a_t n_x = k' a_s k \quad , \quad s \geq 0 .$$

If we multiply $a_t n_x$ by its transpose (denoted by tr) then it follows that

$${}^{tr}n_x \cdot a_t^2 \cdot n_x = \begin{pmatrix} e^{2t} & e^{2t}x \\ e^{2t}x & e^{2t}x^2 + e^{-2t} \end{pmatrix} = k^{-1} \cdot a_s^2 \cdot k = k^{-1} \cdot \begin{pmatrix} e^{2s} & 0 \\ 0 & e^{-2s} \end{pmatrix} \cdot k ,$$

where s depends on x (t is fixed). Taking traces we obtain

$$x^2 = 2(\text{ch } 2s - \text{ch } 2t)e^{-2t} ,$$

so in particular $\operatorname{ch} 2s \geq \operatorname{ch} 2t$, and thus $s \geq t$. Also note that $a_t n_x$ and $a_t n_{-x}$ determine the same value of s and by the K-biinvariance of f it then follows that $f(a_t n_x) = f(a_s) = f(a_t n_{-x})$. Hence we can replace the integral over $\mathbb{R}$ in (3.1) by twice the integral over $[0,\infty)$. Since

$$dx = 2^{\frac{1}{2}} \cdot e^{-t} (\operatorname{ch} 2s - \operatorname{ch} 2t)^{-\frac{1}{2}} \operatorname{sh} 2s \, ds$$

and $f(a_t n_x) = f(a_s)$ we obtain

$$F_f\left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right) = 2\pi^{-1} 2^{\frac{1}{2}} \cdot \int_{s \geq t} f\left(\begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}\right) \cdot (\operatorname{ch} 2s - \operatorname{ch} 2t)^{-\frac{1}{2}} \operatorname{sh} 2s \, ds.$$

By abuse of notation we put $h(2t) = h\left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right)$. Then it follows that

$$(3.2) \qquad F_f(t) = \pi^{-1} 2^{\frac{1}{2}} \cdot \int_{s=t}^{\infty} f(s) \ (\operatorname{ch} s - \operatorname{ch} t)^{-\frac{1}{2}} \ d(\operatorname{ch} s) \quad , \ t \geq 0.$$

As was noted by Godement [6] this integral equation is a version of Abel's classical ($\pm$ 1830) integral equation which is defined for $f \in C^{\infty}(\mathbb{R})$ with compact support in $[y,\infty)$ $(y \in \mathbb{R})$ by

$$(3.3) \qquad g(y) = \pi^{-\frac{1}{2}} \int_y^{\infty} f(x) \ (x-y)^{-\frac{1}{2}} \ dx \ .$$

In his papers [1,2] Abel introduced this integral equation, which is generally considered as the first integral equation in history, in relation to the following problem. A particle starts from rest at a point on a smooth curve, which lies in a vertical plane, and slides down the curve (without friction) to its lowest point. The time of descent depends on the shape of the curve. The problem is to determine the curve for which the time of descent is a given function $f(h)$ of the height h of the starting point. The solution of the resulting integral equation was obtained by Abel using two different methods in [1] and [2]; many classical textbooks on integral equations start with the solution of Abel's integral equation. The solution of (3.3) is given by

$$f(x) = -\pi^{-\frac{1}{2}} \int_x^{\infty} g'(y) \ (y-x)^{-\frac{1}{2}} \ dy \qquad (g' = \frac{dg}{dy}).$$

For the case $G = SL(2,\mathbb{R})$ one then obtains the inversion formula

$$f(t) = -2^{\frac{1}{2}} \int_t^\infty F_f'(s)\ (\mathrm{ch}\ s - \mathrm{ch}\ t)^{-\frac{1}{2}}\ ds\ , \quad t \geq 0\ .$$

In particular

$$f(0) = -\int_0^\infty F_f'(s)\ (\mathrm{sh}\ \tfrac{1}{2}s)^{-1}\ ds\ .$$

and since F_f' is an odd function on $\mathbb{R}$ (recall that F_f is even) we have

$$f(0) = -\int_{\mathbb{R}} F_f'(s)\ (\mathrm{sh}\ \tfrac{1}{2}s)^{-1}\ ds\ .$$

An exercise in contour integration gives

$$\int_{\mathbb{R}} \sin \lambda s\ (\mathrm{sh}\ \tfrac{1}{2}s)^{-1}\ ds = 2\pi \tanh \pi\lambda\ ,$$

which can also be interpreted as $\mathcal{F}((\mathrm{sh}\ \frac{1}{2}s)^{-1})(\lambda) = -2\pi i \tanh \pi\lambda$. So

$$\begin{aligned} f(0) &= 2\pi i \int_{\mathbb{R}} F_f'(s)\ \mathcal{F}^{-1}(\tanh \pi\lambda)(s)\ ds = i \int_{\mathbb{R}} \int_{\mathbb{R}} F_f'(s) \tanh \pi\lambda\ e^{i\lambda s}\ d\lambda ds \\ &= -i \int_{\mathbb{R}} \tanh \pi\lambda\ \left(\int_{\mathbb{R}} F_f'(s) e^{-i\lambda s}\ ds\right) d\lambda = \int_{\mathbb{R}} (F_f)^*(\lambda)\ \lambda \tanh \pi\lambda\ d\lambda\ . \end{aligned}$$

For the last equality we used integration by parts and definition (1.1) for n=1; for the second equality we used the inversion formula for the Euclidean Fourier transform (1.1). Hence by (2.2)

$$f(0) = \int_{\mathbb{R}} (F_f)^*(\lambda)\ \lambda \tanh \pi\lambda\ d\lambda = \int_{\mathbb{R}} \tilde{f}(\lambda)\ \lambda \tanh \pi\lambda\ d\lambda\ . \tag{3.4}$$

Here we have obtained the Plancherel measure $\lambda \tanh \pi\lambda$ (i.e. the measure which gives the inversion of the spherical Fourier transform) for the case $G = SL(2,\mathbb{R})$.

With (3.2) in mind we now generalize the Abel transform as follows. For an even function in $\mathcal{D}(\mathbb{R})$ we define its Abel transform by

$$F_f^\alpha(t) = \frac{2^{3\alpha+\frac{1}{2}}\ \Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{s=t}^\infty f(s)\ (\mathrm{ch}\ s - \mathrm{ch}\ t)^{\alpha-\frac{1}{2}}\ d(\mathrm{ch}\ s)\ , \quad t \geq 0, \tag{3.5}$$

where Γ is the usual gamma function. This integral transform is a version of a Weyl fractional integral transform. If $\alpha=0$ then we regain (3.2) which is the Abel transform for $G = SL(2,\mathbb{R})$. For $G = SL(2,\mathbb{C})$ we obtain (3.5) with $\alpha=\frac{1}{2}$ (the proof is analogous to the one given for $G = SL(2,\mathbb{R})$). It is possible to obtain (3.5) as the Abel transform of a certain noncompact semisimple Lie group G of rank one for $\alpha = 0,\frac{1}{2},1,1\frac{1}{2},\dots$.

Now define the operator W_μ ($\mathrm{Re}\ \mu > 0$), which is a version of Weyl's fractional integral transform, by

$$(W_\mu f)(t) = \frac{1}{\Gamma(\mu)} \int_t^\infty (\mathrm{ch}\ s - \mathrm{ch}\ t)^{\mu-1} f(s)\, d(\mathrm{ch}\ s) ,$$

where $f \in \mathcal{D}(\mathbb{R})$ and even. Note that

$$F_f^\alpha = c_1(\alpha)\, W_{\alpha+\frac{1}{2}} f , \tag{3.6}$$

where $c_1(\alpha)$ is a constant depending on α. This is the motivation to use (3.6) as definition for the transform $f \longrightarrow F_f^\alpha$ for arbitrary $\alpha \in \mathbb{C}$, $\mathrm{Re}\ \alpha > -\frac{1}{2}$. Now W_μ has an analytic continuation to all complex μ: if $k = 0,1,2,\dots$ and $\mathrm{Re}\ \mu > -k$ then

$$(W_\mu f)(t) = \frac{(-1)^k}{\Gamma(\mu+k)} \int_t^\infty (\mathrm{ch}\ s - \mathrm{ch}\ t)^{\mu+k-1} \frac{d^k f(s)}{d(\mathrm{ch}\ s)^k}\, d(\mathrm{ch}\ s) .$$

Note that $(W_{-k} f)(t) = (-1)^k \dfrac{d^k f(t)}{d(\mathrm{ch}\ t)^k}$, $k = 1,2,\dots$. The operator W_μ is a bijection of $\{f \in \mathcal{D}(\mathbb{R}),\ \text{even}\}$ onto itself and the inverse is given by $W_{-\mu}$. Hence

$$f = c_2(\alpha)\, W_{-\alpha-\frac{1}{2}} F_f^\alpha .$$

These results can be found in [8,§3] or [9,§5.3]. In particular the Abel transform can be inverted by a differential operator if 2α is odd. Also note that

$$\frac{d}{d\mathrm{ch}\ t} \circ W_\mu = W_\mu \circ \frac{d}{d\mathrm{ch}\ t} = - W_{\mu-1} \qquad (\mathrm{Re}\ \mu > 1) ,$$

so if we put $D = \dfrac{d}{d\mathrm{ch}\ t}$ then we obtain from (3.6) that

$$(3.7) \qquad F^{\alpha}_{Df} = c_3(\alpha)\, F^{\alpha-1}_{f} \quad , \quad \mathrm{Re}\,\alpha > \tfrac{1}{2} \; .$$

This anticipates the analogous results in section 4 for a rank-two case.

Based on the explicit expression (3.6) for the Abel transform, Koornwinder [8] (also see [9,§7]) obtained the Plancherel and a Paley-Wiener theorem for the so-called Jacobi transform, which can be considered as a generalization of the spherical Fourier transform on rank-one symmetric spaces of the noncompact type.

It was Godement [6] who first used the method of an explicit inversion of the transform $f \longrightarrow F_f$ in order to obtain the Plancherel theorem for $G = SL(2,\mathbb{R})$.

Property (3.7) enabled Takahashi [13] to obtain an inversion of the Abel transform for the "generalized Lorentz group" $G = SO_o(1,m+1)$. Again this result was used to obtain the Plancherel measure ([13,Ch I,§4]).

As stated before, Koornwinder [8] then generalized not only to arbitrary rank-one but also to arbitrary complex parameters.

The rank-one group-case can also be found in Lohoué-Rychener [10], where the inversion is used to solve the heat equation on symmetric spaces of the noncompact type and of rank one.

Unaware of the results in [8,10], Rouvière [12] also determined an explicit inversion of the Abel transform for the rank-one symmetric spaces of the noncompact type. He used $SU(2,1)$-reduction. The results were used to obtain the Plancherel measure, the spherical functions and Paley-Wiener theorems.

The case where G is complex (and arbitrary rank) is also well-known. In [5] Gangolli determined the explicit inversion of the Abel transform for complex G. He used this result to obtain the analogue of the Gauss kernel $(4\pi t)^{-\frac{1}{2}} e^{-\frac{1}{4}x^2 t^{-1}}$ on the real line, i.e. he obtained the fundamental solution of the heat equation on G/K. The complex case can also be found in [12]. A completely different method for $G = SL(n,\mathbb{C})$ occurs in Aomoto [3] and Hba [7].

Aomoto [3] also treated the case $G = SL(n,\mathbb{R})$ and gave an inversion formula for $n = 3$. However this inversion is far from being explicit.

Combining the results of Koornwinder on the Weyl fractional integral transforms with the explicit formula for the spherical functions, Meaney [11] obtained an explicit inversion of the Abel transform for $G = SU(p,q)$.

4. Further results on the Abel transform

In this section we assume that the reader is familiar with root systems associated with symmetric spaces of the noncompact type. In section 3 the associated root system was of type A_1. Recently we showed that an analogue of (3.7) holds for the rank-two case corresponding to the root system of type A_2 and in particular we obtained an inversion by a differential operator if $\alpha = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \ldots$. The differential operator D involved was found by Vretare [14] in the context of orthogonal polynomials in several variables and was called "lowering" operator since the parameter is shifted from α to $\alpha-1$. If we put $m = 2\alpha+1$ then m corresponds to the root multiplicity for the group cases. In order to prove (3.7) in the rank-two case A_2 for values of the parameter which do not correspond to a group-case, we need an explicit formula for the Abel transform as in (3.5). Aomoto [3] obtained an integral representation for the cases $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$ and we extended his results to the case $SU^*(2n)$ [4, Ch. III]. For $n = 3$ this integral representation can be used to prove the analogue of (3.7) for the rank-two case A_2. Here we will not give this integral representation (see [4, Ch. III] for details). We write F_f^m to emphasize the dependance on m ($m \in \mathbb{C}$, $\mathrm{Re}\ m > 0$). Let $\mathfrak{a}$ denote the hyperplane in $\mathbb{R}^3$ orthogonal to the vector $e_1+e_2+e_3$ (e_1, e_2, e_3 standard basis). The root system of type A_2 can be identified with the set $\Sigma = \{\pm(e_1-e_2), \pm(e_1-e_3), \pm(e_2-e_3)\}$ in $\mathfrak{a}$. For Σ we take as basis $\Delta = \{e_1-e_2, e_2-e_3\}$. Let Σ^+ be the set of positive roots with respect to Δ. The Weyl group W of Σ is isomorphic to the symmetric group S_3. Write e^λ for the function on $\mathfrak{a}$ which sends $\mu \in \mathbb{R}^3$ to $e^{\langle\lambda,\mu\rangle}$, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product. Also write ∂_λ for the derivative in the direction of λ. Put

$$\delta = \prod_{\alpha \in \Sigma^+} (e^{\alpha} - e^{-\alpha}) \quad .$$

Let $D(m)$ be the differential operator on $\mathfrak{a}^+ = \{(x_1,x_2,x_3) \in \mathfrak{a} \mid x_1>x_2>x_3\}$ defined by

$$(4.1) \qquad D(m) = \delta^{-1} \Big(\prod_{\alpha \in \Sigma^+} \partial_\alpha + \tfrac{1}{2}(m-2) \sum_{\alpha \in \Sigma^+} \epsilon(\alpha)\, \partial_\alpha \circ \coth\alpha \circ \partial_\alpha \Big),$$

where $\epsilon(\alpha) = \prod_{\beta>0,\, \beta\neq\alpha} \langle\alpha,\beta\rangle$. Then one has the following analogue of (3.7) (see [4, Ch. IV] for details).

Theorem

Let $D(m)$ *be given by* (4.1) *and* f *a* W*-invariant element in* $\mathcal{D}(\mathfrak{a})$. *Then*

$$F^m_{D(m)f} = const.\ F^{m-2}_f \quad on \quad \mathfrak{a}^+,\ \mathrm{Re}\ m > 2\ ,$$

and

$$F^2_{D(2)f} = const.\ f \quad on \quad \mathfrak{a}^+\ .$$

In particular the Abel transform can be inverted by the differential operator $D(m,m) = D(m) \circ D(m-2) \circ \ldots \circ D(2)$ *if* m *is even, i.e.*

$$F^m_{D(m,m)f} = const.\ f \quad on \quad \mathfrak{a}^+,\quad m\ even\ .$$

The values $m = 2,4$ *or* 6 *correspond to the group-cases* $SL(3,\mathbb{C})$, $SU^*(6)$ *and* $E_{6(-26)}$ *respectively, and for these cases one has*

$$F^m_{D(m,m)f} = D(m,m)F^m_f = const.\ f\ .$$

Remarks

1. We have also been able to prove inversion of the Abel transform by a differential operator for $SU^*(8)$, which has A_3 as associated root system ([4, Appendix 1]).
2. Inversion of the Abel transform for $m=4$ and root system A_2 has recently also been obtained by Hba, using a different method which was introduced in [7] for $G = SL(n,\mathbb{C})$. One can show that his sixth order differential operator which inverts the Abel transform is indeed equal to $D(4,4)$.
3. In a forthcoming paper we will show that the generalized Abel transform F^m_f (m arbitrary) also satisfies the transmutation property (2.3).

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Centre for Mathematical Analysis

The Australian National University

G.P.O. Box 4 Canberra

ACT 2601 Australia