JACKSON'S THEOREM FOR
COMPACT CONNECTED LIE GROUPS

Donald I. Cartwright and Krzysztof Kucharski

This is an announcement of results which will appear in detail in the J. Approx. Theory.

Let \( E \) be a Banach space of periodic functions on \( \mathbb{R} \), let \( f \in E \) and let \( n \geq 1 \) be an integer. A basic problem in approximation theory is to estimate the quantity

\[
\mathcal{E}_n(f) = \inf \{ \| f - t \|_E \},
\]

the infimum being taken over all trigonometric polynomials \( t \) of degree at most \( n \). Jackson's Theorem is the fundamental "direct theorem" here; it asserts that if the \( r \)-th derivative \( f^{(r)} \) exists in \( E \) (in the appropriate sense) and if \( E \) is suitable, then \( \mathcal{E}_n(f) \leq C_r n^{-r} \omega_1(n^{-1}, f^{(r)}) = o(n^{-r}) \) (see [6]). More precise versions of Jackson's Theorem provide estimates \( \mathcal{E}_n(f) \leq C_r \omega_r(n^{-1}, f) \) for any \( f \in E \), where \( \omega_r(t, f) \) is the \( r \)-th modulus of continuity of \( f \).

Jackson's Theorem extends in a straightforward way to periodic functions of \( k \) variables (i.e., functions on the group \( \mathbb{T}^k \)), and it is natural to ask whether it also applies to functions on nonabelian groups. We can prove that Jackson's Theorem is true for any compact connected Lie group:

**THEOREM** Let \( G \neq \{1\} \) be any compact connected Lie group. Let \( E \) denote one of the spaces \( C(G) \) or \( L^p(G) \), \( 1 \leq p < \infty \), and let \( r \geq 1 \) be an integer. Then there is a constant \( C_r \) and for each integer \( n \geq 1 \) there is a central trigonometric polynomial \( K_n \) of degree \( \leq n \) such that

\[
\| f - K_n \ast f \|_E \leq C_r \omega_r(\frac{1}{n}, f)
\]

for each \( f \in E \).

Here a central trigonometric polynomial of degree \( \leq n \) is a linear combination of the characters \( \chi_\gamma \), where \( \gamma \in \hat{K} \cap I^* \) and \( ||\gamma|| \leq n \) (The dual object \( \hat{G} \) of \( G \) may be identified with a semilattice \( \hat{K} \cap I^* \) as in [1, p. 242], and \( ||.|| \) is a norm
obtained from an inner product on \( g \) which is invariant under the adjoint action of \( G \) on \( g \).) Let \( f \in E \), where \( E = C(G) \) or \( L^p(G) \), \( 1 \leq p < \infty \). The \( r \)-th modulus of continuity \( \omega_r(t, f) \) of \( f \) is defined as follows: For any integer \( r \geq 1 \) and for \( t > 0 \), let

\[
\omega_r(t, f) = \sup \{ \| \Delta_{\exp X} f \|_E : X \in g \text{ and } \| X \| \leq t \}.
\]

Here

\[
(\Delta_h^r f)(x) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(h^{-j} x)
\]

for \( x, h \in G \).

Johnen [5] proved this theorem in the special case \( r = 2 \), but our method is quite different from his. The kernels \( K_n \) are related to the \( \Phi_n \) of [3], but even more to those used in [6] and [7] in proving the \( T^k \) case.

As an application of our theorem, we use the sharp estimates for the Lebesgue constants recently obtained by Giulini and Travaglini [4] to give "best possible" criteria for the norm convergence of Fourier series of functions on \( G \). Let \( E = C(G) \) or \( L^1(G) \). For \( f \in E \) and \( n \geq 1 \), \( s_n f = \sum_{\gamma \in C_n} d_{\gamma} \chi_{\gamma} \ast f \) is called the \( n \)-th spherical [resp. polyhedral] partial sum of the Fourier series \( \sum_{\gamma \in \hat{K} \cap I^*} d_{\gamma} \chi_{\gamma} \ast f \) of \( f \) if \( C_n = \{ \gamma \in \hat{K} \cap I^* : \| \gamma + q \| \leq n \} \) [resp. \( C_n = \{ \gamma \in \hat{K} \cap I^* : \gamma \leq n \omega \} \), where \( \omega \in K \cap I^* \) is fixed]. Giulini and Travaglini [4] showed that the Lebesgue constants

\[
\sup \{ \| s_n f \|_E : \| f \|_E \leq 1 \} = \| \sum_{\gamma \in C_n} d_{\gamma} \chi_{\gamma} \|_1
\]

for spherical partial sums satisfy

\[
c_1 n^{(d-1)/2} \leq \| \sum_{\gamma \in C_n} d_{\gamma} \chi_{\gamma} \|_1 \leq c_2 n^{(d-1)/2}
\]

for \( d = \dim G \) and for suitable constants \( c_1, c_2 > 0 \), while for polyhedral sums similar inequalities hold, but with \( (d - 1)/2 \) replaced by \( |R_+| \). We can now state a refinement of the Proposition in [4].

**PROPOSITION** Let \( G \) be a semisimple compact connected Lie group and let \( E = C(G) \) or \( L^1(G) \).

1. If \( f \in E \) and \( \omega_r(t, f) = o(t^{(d-1)/2}) \) as \( t \to 0 \) for some integer \( r \geq (d-1)/2 \), then the spherical partial sums \( s_n f \) converge to \( f \) in \( E \).

2. There exists \( F \in E \) such that \( \omega_r(t, F) = O(t^{(d-1)/2}) \) as \( t \to 0 \) but for which \( s_n F \) does not converge to \( F \) in \( E \). In fact, if \( 0 \leq s < (d - 1)/2 \) is an integer, we may choose \( F \in E^{(s)} \) with \( \omega_{r-s}(t, F^{(s)}) = O(t^{(d-1)/2-s}) \) for all \( r \geq (d-1)/2 \).
The corresponding result holds for polyhedral partial sums with \((d - 1)/2\) replaced by \(|R_+|\) throughout.

REFERENCES


