OSCILLATORY INTEGRALS

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We are interested in the behaviour at infinity of the Fourier transform $\hat{\mu}$ of the surface measure μ living on a smooth compact hypersurface S in $\mathbb{R}^{n+1}$, and in the transform $(f\mu)^\wedge$ of the product of certain functions f on S and μ. The aim of this work is to see how the decay of $\hat{\mu}$ and $(f\mu)^\wedge$ reflect the geometry of S. To simplify the statements of results, we assume that S is analytic.

The earliest and most important result on the decay of $(f\mu)^\wedge$ at infinity comes from the principle of stationary phase: in almost every direction σ in S^n,

$$(f\mu)^\wedge(\rho\sigma) = O(\rho^{-n/2}) \qquad \text{as} \quad \rho \to +\infty.$$

More precisely, if σ is a generic direction, which means that the (finitely many) points $s_1, s_2, \ldots, s_K$ of S to which σ is normal are points of non-zero Gaussian curvature $\mathcal{K}$, then for smooth enough f (C^1 will do),

$$(1) \qquad (f\mu)^\wedge(\rho\sigma) = \Sigma_{k=1}^{K}\, c(k)\, f(s_k)\, |\mathcal{K}(s_k)|^{-1/2}\, e^{-i\rho\sigma . s_k}\, \rho^{-n/2} + o(\rho^{-n/2}) \qquad \text{as} \quad \rho \to +\infty.$$

The constants $c(k)$ depend on the dimension n of S, and on whether σ is an inward or outward normal at s_k, relative to the principal curvatures. If σ is a non-generic direction, so that there is a point

s_k with normal vector σ where $\mathcal{K}(s_k) = 0$, then $(f\mu)^\wedge(\rho\sigma)$ decays slower that $\rho^{-n/2}$ as $\rho \to +\infty$, at least if $f(s_k) \neq 0$, due to the presence of an asymptotic term of the form

$$(2) \qquad c(k)\ f(s_k)\ e^{-i\rho\sigma . s_k}\ \rho^{-\alpha}\ \log^\beta(\rho)\ ;$$

the indices α (a positive rational) and β (a non-negative integer) depend on the nature of S near s_k. Amongst the important papers on this problem, we mention the work of B. Malgrange [5] and A.N. Varchenko [7], where the existence of an asymptotic expansion is proved, and α and β computed for many examples.

It can be shown that, if f is smooth enough and if S is convex and has no points of Gaussian curvature 0, then there exists a constant C so that

$$(3) \qquad |(f\mu)^\wedge(\rho\sigma)| \leq C(1+\rho)^{-n/2} \qquad \forall\sigma \in S^n,\ \forall\rho \in \mathbb{R}^+$$

(see C.S. Herz [3] and W. Littman [4]); it would appear from (1) that, in the general case, if $f = |\mathcal{K}|^{1/2}$, then the inequality (3) should still hold, as then in each generic direction the asymptotic decay is uniformly controlled. However, this is false, for at least two reasons (see Cowling and G. Mauceri [2], for one of these).

We now summarise several problems about the decay of $(f\mu)^\wedge$ at infinity which we consider important:

(a) Describe the decay of $(f\mu)^\wedge(\rho\sigma)$ as $\rho \to +\infty$ in terms of the geometry of the points on S having normal vector σ, as in (2) above.

(b) Find uniform estimates (as σ varies) for the decay of $(f\mu)^\wedge(\rho\sigma)$ as $\rho \to +\infty$, as in (3) above.

(c) What is the relation, if any, between the slowest decay (as σ varies) in the asymptotic expansions for $(f\mu)^\wedge(\rho\sigma)$, and the decay of $(f\mu)^\wedge(\rho\sigma)$ in a uniform estimate? Are these the same?

(d) If we take f to be a power of the Gaussian curvature, $|\mathcal{K}|^\theta$ say, find estimates for $(f\mu)^\wedge$, and find the smallest value of θ for which $(f\mu)^\wedge(\rho\sigma)$ decay uniformly as $\rho^{-n/2}$ as $\rho \to +\infty$, as in (3).

There has been some progress on these questions recently: C.D. Sogge and E.M. Stein [6] showed that, if $\theta = 2n$, then there is a constant C so that

$$|(|\mathcal{K}|^\theta \mu)^\wedge(\rho\sigma)| \le C\rho^{-n/2} \qquad \forall\sigma \in S^n,\ \forall\rho \in \mathbf{R}^+,$$

while M. Cowling and G. Mauceri [2] obtained this inequality when $\theta = [n/2] + 2$ ([ ] denotes the integer part function), under the additional hypothesis of the convexity of S. We believe, however, that $\theta = 1$ suffices for convex S, and perhaps $\theta = 2$ suffices in general.

J. Bruna, A. Nagel, and S. Wainger [1] considered the surface measure μ on a convex hypersurface S, and established the existence of a constant C so that

$$|\hat{\mu}(\rho\sigma)| \le C[|\mathrm{Cap}^+(\rho^{-1},\sigma)| + |\mathrm{Cap}^-(\rho^{-1},\sigma)|] \qquad \forall\sigma \in S^n,\ \forall\rho \in \mathbf{R}^+,$$

where $|T|$ indicates the surface area of a subset T of S, and $\mathrm{Cap}^+(\lambda,\sigma)$ (respectively $\mathrm{Cap}^-(\lambda,\sigma)$) denote the caps of S at the points s^+ (respectively s^-) of S where σ is an outward (respectively inward) normal, of height λ:-

$$\mathrm{Cap}^+(\lambda,\sigma) = \{s \in S: \sigma.s^+ - \sigma.s \in [0,\lambda]\}$$

$$\mathrm{Cap}^-(\lambda,\sigma) = \{s \in S: \sigma.s - \sigma.s^- \in [0,\lambda]\},$$

where of course s^+ (respectively s^-) are the points on S where $\sigma.s$ is maximised (respectively minimised).

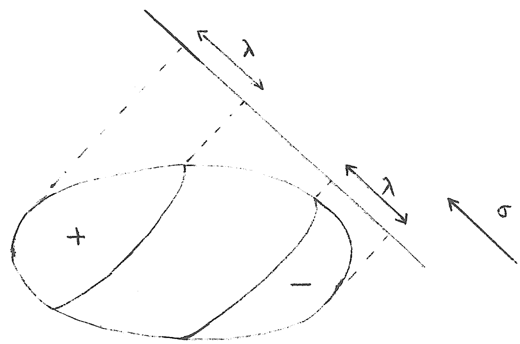


Diagram 1: $Cap^+(\lambda,\sigma)$ and $Cap^-(\lambda,\sigma)$.

Since the leading terms of the asymptotic expansion for $\hat{\mu}(\rho\sigma)$ are

$$e^{-i\rho\sigma.s^+}\,|Cap^+(\rho^{-1},\sigma)| \quad\text{and}\quad e^{-i\rho\sigma.s^-}\,|Cap^-(\rho^{-1},\sigma)|$$

(see Varchenko [7]), it follows that the slowest rate of decay in an asymptotic expansion will control $|\hat{\mu}(\rho\sigma)|$, uniformly in σ.

The contribution which we offer here may or may not be significant in the long run. We believe that a modification of the arguments of Bruna, Nagel and Wainger should show that

$$(4) \qquad |(\mathcal{K}\mu)^\wedge(\rho\sigma)| \le C[|Cap_1^+(\rho^{-1},\sigma)| + |Cap_1^-(\rho^{-1},\sigma)|],$$

where $|Cap_1^+(\lambda,\sigma)|$ denotes the area of $Cap^+(\lambda,\sigma)$ relative to the new surface measure $\mathcal{K}\mu$, and $|Cap_1^-(\lambda,\sigma)|$ is defined analogously. If (4) does hold, then, for some possibly different C,

$$|(\mathcal{K}\mu)^\wedge(\rho\sigma)| \le C\,\rho^{-n/2} \qquad \forall\sigma \in S^n,\ \forall\rho \in \mathbb{R}^+$$

as a consequence of the following theorem.

THEOREM. *Let* S *be a compact convex analytic hypersurface in* $\mathbf{R}^{n+1}$. *Then there is a constant* C *so that*

$$|\mathrm{Cap}_1^+(\lambda,\sigma)| \le C\,\lambda^{n/2} \qquad \forall\sigma \in S^n,\ \forall\lambda \in \mathbf{R}^+.$$

We shall sketch the main features of the proof. Let $g\colon S \to S^n$ denote the Gauss map : for s in S with outward unit normal σ, $g(s) = \sigma$. Gauss showed that

$$|\mathrm{Cap}_1^+(\lambda,\sigma)| = |g(\mathrm{Cap}^+(\lambda,\sigma))|.$$

We choose coordinates in $\mathbf{R}^{n+1}$ so that $x_1,\ldots,x_n$ are coordinates for the tangent plane $T_{s^+}(S)$ to S at s^+, so that s^+ lies at the origin, and so that $\sigma = (0,\ldots,0,-1)$. Then the part of the surface near s^+ is the graph of a convex analytic function $f = f_{s^+}$, defined on $T_{s^+}(S)$, of radius of convergence at least one, say, and satisfying $f(0) = 0$ and $\nabla f(0) = 0$, i.e.

$$\{\underline{x} \in \mathbf{R}^{n+1} : x_{n+1} = f(x_1,\ldots,x_n),\ \Sigma_{j=1}^n |x_j|^2 < 1\} \subset S.$$

Now $$|g(\mathrm{Cap}^+(\lambda,\sigma))| \le |\{\tau \in S^n : \tau_{n+1} \le M(\lambda)\}|,$$

where $$M(\lambda) = \max\{g_{n+1}(s) : s \in \mathrm{Cap}^+(\lambda,\sigma)\}.$$

For λ small enough, any element $\bar{s}$ of $\mathrm{Cap}^+(\lambda,\sigma)$ which satisfies

$$g_{n+1}(\bar{s}) = M(\lambda)$$

can be written in the form $(\bar{\underline{x}}, f(\bar{\underline{x}}))$, for some $\bar{\underline{x}}$ in $T_{s^+}(S)$ of length at most $1/2$, and then

$$M(\lambda) = g_{n+1}(\bar{\underline{x}}) = -(1+|\nabla f(\bar{\underline{x}})|^2)^{-1/2}.$$

Choose $\theta(\lambda)$ in $(0,\pi/2)$ so that $\cos(\theta(\lambda)) = (1+|\nabla f(\bar{\underline{x}})|^2)^{-1/2}$, then

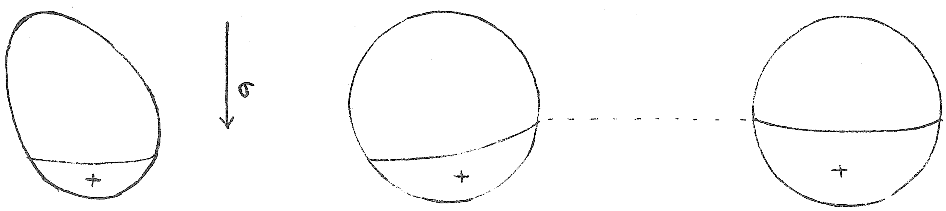
$$|\{\tau \in S^n : \tau_{n+1} \le M(\lambda)\}| \le C'_n\, \theta(\lambda)^n$$

$$= C'_n\, [\arccos((1+|\nabla f(\bar{\underline{x}})|^2)^{-1/2})]^n$$

$$\le C_n\, |\nabla f(\bar{\underline{x}})|^n.$$

In summary, then,

$$|Cap_1^+(\lambda,\sigma)| \le C_n \sup\{|\nabla f(\underline{x})|^n : f(\underline{x}) \le \lambda\}.$$



$Cap^+(\lambda,\sigma)$ $\qquad$ $g(Cap^+(\lambda,\sigma))$ $\qquad$ $\{\tau\in S^n : \tau_{n+1} \le M(\lambda)\}$

Diagram 2: The geometry of Theorem 1

Thus it suffices to prove that, for some constant C,

(5) $$|\nabla f_s(\underline{x})| \le C\, f_s(\underline{x})^{1/2},$$

for all f_s in the family of functions $\mathcal{F}$ which arise by considering the part of the surface near s as the graph of a function f_s defined on $T_s(S)$, as s varies in S, and for all $\underline{x}$ in $T_s(S)$ with

$\|\underline{x}\| < 1$. For each unit vector $\underline{e}$ in $T_s(S)$, we may write

$$f_s(t\underline{e}) = \Sigma_{m=0}^{\infty} a_m(\underline{e})t^m \qquad \forall t \in (-1,1),$$

where, for some (S-dependent) constants $K > 0$, $\delta > 0$, and P in $\mathbb{N}$,

(i) $\quad a_0(\underline{e}) = 0 \qquad \forall \underline{e} \in T_s(S),$

(ii) $\quad a_1(\underline{e}) = 0 \qquad \forall \underline{e} \in T_s(S),$

(iii) $\quad |a_m(\underline{e})| \le K \qquad \forall \underline{e} \in T_s(S),$

and (iv) $\quad \max\{|a_m(\underline{e})| : 2 \le m \le P\} \ge \delta \qquad \forall \underline{e} \in T_s(S).$

[This last condition is established by compactness: if it were false, then for all P in $\mathbb{N}$ there would be elements of $T(S)$, $\underline{e}_P$ say, so that

$$|a_m(\underline{e}_P)| = 0 \qquad m = 0,1,\ldots,P$$

and then there would be some $\underline{e}$ in $T(S)$ with

$$a_m(\underline{e}) = 0 \qquad \forall m \in \mathbb{N}.]$$

We may make a further simplification: by restricting attention to two-dimensional subspaces of $T_s(S)$ containing x and $\nabla f_s(\underline{x})$, it suffices to consider a compact family $\bar{\mathcal{F}}$ of convex analytic functions of two real variables, centred at the origin, with radius of convergence at least one 1 satisfying the two dimensional analogues of the above conditions (i) - (iv); again, we must prove that

$$|\nabla f(\underline{x})|^2 \le Cf(\underline{x}) \qquad \forall f \in \bar{\mathcal{F}}$$

for all $\underline{x}$ in the unit ball in $\mathbb{R}^2$ centred at the origin.
The first step in proving this is choosing a suitable coordinate system. For each 2-plane π in $T_s(S)$, we choose unit vectors $\underline{e}_1$ and $\underline{e}_2$ so that $\underline{e}_2$ is orthogonal to $\underline{e}_1$, and $\underline{e}_1$ is the "direction of slowest

growth" of f_s. This direction is obtained as follows: we write, for any unit vector $\underline{e}$ in π,

$$f_s(t\underline{e}) = \Sigma_{m=0}^{\infty} a_m(\underline{e})t^m ,$$

then either there are positive integers p and q, with $p > q \geq 2$, and a direction $\underline{e}_1$ so that

$$\min\{m : a_m(\underline{e}) \neq 0\} = \begin{cases} p & \text{if } \underline{e} = \pm\underline{e}_1 \\ q & \text{otherwise} \end{cases},$$

or
$$\min\{m : a_m(\underline{e}) \neq 0\} = p$$

for all $\underline{e}$ in π. In the former case, $\underline{e}_1$ is the "direction of shortest growth"; in the latter we choose $\underline{e}_1$ so that

$$0 < a_p(\underline{e}_1) \leq a_p(\underline{e}) \qquad \forall \underline{e} \in \pi,$$

$\underline{e}$ and $\underline{e}_1$ being unit vectors.

Now for f in $\bar{\mathcal{F}}$, we may write

$$f(x\underline{e}_1 + y\underline{e}_2) = \Sigma_{m,n=0}^{\infty} a_{mn}x^m y^n,$$

where $a_{mn} = 0$ if $m/p + n/q < 1$, by convexity, and

$$|a_{mn}| \leq K$$

and
$$\max\{|a_{m0}| : 2 \leq m \leq P\} \geq \delta$$

and
$$\max\{|a_{0n}| : 2 \leq n \leq P\} \geq \delta .$$

Let $p = \min\{m : |a_{m0}| \geq \delta\}$ and $q = \min\{n : |a_{0n}| \geq \delta\}$.

We now apply induction on $p + q$. One possibility is that

$$a_{m0} = 0 \quad \text{if} \quad m < p$$

$$a_{0n} = 0 \quad \text{if} \quad n < q$$

and (by convexity)
$$a_{mn} = 0 \quad \text{if} \quad m/p + n/q < 1.$$

The difficulty in proving (5) lies when $\|\underline{x}\|$ is small; here we estimate

$$|\nabla f(x\underline{e}_1 + y\underline{e}_2)|^2 = |\Sigma_{m,n=0}^{\infty} a_{mn} m x^{m-1} y^n|^2 + |\Sigma_{m,n=0}^{\infty} a_{mn} n x^m y^{n-1}|^2$$

$$\leq C[x^p + y^q]^{2(1-1/q)},$$

[by majorising the geometric mean by the arithmetic mean, for small m and n]. To estimate f, we let

$$f^{\#}(x\underline{e}_1 + y\underline{e}_2) = \Sigma_{m/p+n/q=1} a_{mn} x^m y^n;$$

we show that $f^{\#}(x\underline{e}_1 + y\underline{e}_2) \geq \delta_1(x^p + y^q)$, by observing that the quotient function must be bounded away from 0 by convexity and homogeneity arguments; and then it follows that

$$f(x\underline{e}_1 + y\underline{e}_2) \geq \delta_2(x^p + y^q)$$

for small x and y.

The other possibility is that some a_{mn} are non-zero for some m and n with $m/p + n/q < 1$. Now a dilation argument, like that employed by Cowling and Mauceri [2], (but in 2-variables), enables us to reduce to a case with smaller $p+q$. ☐

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