ADDITIVE SET FUNCTIONS OF BOUNDED Φ–VARIATION

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The notion of a (point) function of finite p–variation was introduced by N. Wiener in [5]. It was extended by L.C. Young who considered, in [6], functions of finite Φ–variation, where Φ is an increasing function on $[0,\infty)$. Young also gave sufficient conditions for the existence of the Stieltjes integral

$$\int_a^b f\,\mathrm{d}g \tag{0.1}$$

in terms of the Φ–variation of the function f and Ψ–variation of the function g in the interval $[a,b]$. Such integrals were from this point of view subsequently studied by several authors including Young himself. Using a very interesting idea, A. Beurling improved Young's condition in [1]. However, this idea does not seem easy to generalize so as to cover additive set functions in abstract spaces; it uses the fact that (0.1) is essentially integral with respect an additive set functions defined on sub–intervals of $[a,b]$. In this note, methods remotely akin to that of Beurling are presented for introducing and studying integrals with respect to set functions on semialgebras in abstract spaces. The interest in such enterprise stems from various, seemingly unrelated, sources: stochastic fields (processes with multidimensional time–parameter), spectral theory, Feynman integral and, possibly, others.

1. Let $\mathcal{Q}$ be a semialgebra of sets in a space Ω. That is, $\mathcal{Q}$ is a semiring (cf. [2],4.6) such that $\Omega \in \mathcal{Q}$. By a partition will be understood a finite family of pair–wise disjoint sets from $\mathcal{Q}$ whose union is equal to Ω. The set of all such partitions is denoted by Π. If the partition $\mathcal{P}'$ is a refinement of the partition $\mathcal{P}$, we write $\mathcal{P} \prec \mathcal{P}'$.

We shall abuse the notation by writing $f(X) = \{f(\omega)\colon \omega \in X\}$ for any function f on Ω and a set $X \subset \Omega$. Furthermore, the same symbol will be used to denote a subset of Ω and its characteristic function.

Let $\mathcal{P} \in \Pi$. By a $\mathcal{P}$–simple function is meant a function f on Ω which is constant on every set belonging to $\mathcal{P}$, so that, in the introduced conventions,

$$f = \sum_{X \in \mathcal{P}} f(X)X. \tag{1.1}$$

If $\Xi \subset \Pi$, then by $\operatorname{sim}(\Xi)$ is denoted the family of all functions f for which there exists a partition $\mathcal{P} \in \Xi$ such that f is $\mathcal{P}$–simple. If the set of partitions Ξ is directed (by the relation of refinement) then $\operatorname{sim}(\Xi)$ is an algebra of functions on Ω. We write $\operatorname{sim}(\mathcal{Q}) = \operatorname{sim}(\Pi)$.

Let E be a Banach space and let $\mu\colon \mathcal{Q} \to E$ be an additive set function.

For a function $f \in \operatorname{sim}(\mathcal{Q})$, given by (1.1), let

$$\mu(f) = \int_\Omega f \,\mathrm{d}\mu = \sum_{X \in \mathcal{P}} f(X)\mu(X).$$

The additivity of μ implies that the element $\mu(f)$ of the space E, called of course the integral of f with respect to μ, is uniquely determined by the function f. The set function $f\mu\colon \mathcal{Q} \to E$ defined by $(f\mu)(X) = \mu(Xf)$, $X \in \mathcal{Q}$, is called the indefinite integral of f with respect to μ. Clearly, if the function f is given by (1.1), then

$$(f\mu)(X) = \sum_{Y \in \mathcal{P}} f(Y)\mu(X \cap Y)$$

for every $X \in \mathcal{Q}$. Also, $\mu(f) = (f\mu)(\Omega)$.

By a Young–Orlicz gauge we shall mean a continuous, strictly increasing and convex function, Φ, on $[0,\infty)$ such that

(i) $s^{-1}\Phi(s) \to 0$ as $s \to 0+$, and $s^{-1}\Phi(s) \to \infty$ as $s \to \infty$; and

(ii) there exists a number $k > 0$ such that $\Phi(2s) \le k\,\Phi(s)$ for every $s \in [0,\infty)$.

The requirement (ii) represents what is called the (Δ_2) condition for both the small and the large s.

Let $\Phi(s) = s$ for every $s \in [0,\infty)$, or else let Φ be a Young–Orlicz gauge. For a set $X \in \mathcal{Q}$ and a partition $\mathcal{P} \in \Pi$, let

$$v_{\Phi}(\mu,\mathcal{P};X) = \sum_{Y \in \mathcal{P}} \Phi(|\mu(X \cap Y)|) .$$

Let, further, $\Xi \subset \Pi$ and

$$v_{\Phi}(\mu,\Xi;X) = \sup \{v_{\Phi}(\mu,\mathcal{P};X) \colon \mathcal{P} \in \Xi\}$$

for every $X \in \mathcal{Q}$.

The set function $v_{\Phi}(\mu,\Xi)$, that is, $X \mapsto v_{\Phi}(\mu,\Xi;X)$, $X \in \mathcal{Q}$, is called the Φ–variation of μ with respect to the family of partitions Ξ .

If $v_{\Phi}(\mu,\Xi;\Omega) < \infty$, then the set function μ is said to have finite Φ–variation with respect to the set of partitions Ξ . The family of all additive set functions $\xi\colon \mathcal{Q} \to E$ which have finite Φ–variation with respect to Ξ will be denoted by $\mathrm{BV}^{\Phi}(\Xi,E)$.

In the case when $\Phi(s) = cs^p$, for some real constants $c > 0$ and $p \geq 1$ and every $s \in [0,\infty)$, we shall write simply $v_p(\mu,\mathcal{P};X)$, $v_p(\mu,\Xi;X)$ and $\mathrm{BV}^p(\Xi,E)$ instead of $v_{\Phi}(\mu,\mathcal{P};X)$, $v_{\Phi}(\mu,\Xi;X)$ and $\mathrm{BV}^{\Phi}(\Xi,E)$, respectively. Similar conventions will also be used, without explicit mention, in other symbols denoting some objects depending on Φ introduced later on.

Finally, by $\mathrm{BV}^{\infty}(\Xi,E)$ is denoted the vector space of all additive set functions $\mu\colon \mathcal{Q} \to E$ for which there exists a constant k (depending on μ) such that $|\mu(X)| \leq k$ for every X belonging to a partition $\mathcal{P}$ from Ξ .

Let $V_1(\Xi,\xi) = v_1(\xi,\Xi;\Omega)$ for every $\xi \in \mathrm{BV}^1(\Xi,E)$. Then the functional $\xi \mapsto V_1(\Xi,\xi)$, $\xi \in \mathrm{BV}^1(\Xi,E)$, is a norm under which the space $\mathrm{BV}^1(\Xi,E)$ is complete. Similarly, let

$$V_{\infty}(\Xi,\xi) = \sup\{|\xi(X)| \colon X \in \mathcal{P} \in \Xi\}$$

for every $\xi \in \mathrm{BV}^{\infty}(\Xi,E)$. Then the functional $\xi \mapsto V_{\infty}(\Xi,\xi)$, $\xi \in \mathrm{BV}^{\infty}(\Xi,E)$, is a norm making the space $\mathrm{BV}^{\infty}(\Xi,E)$ complete.

If Φ is an arbitrary Young–Orlicz gauge, then a norm can still be naturally introduced in the space $\mathrm{BV}^{\Phi}(\Xi,E)$. In fact, it is usually done in at least two ways. Thus let

$$V_{\Phi}(\Xi,\xi) = \inf\{k : k > 0, v_{\Phi}(k^{-1}\xi ,\Xi;\Omega) \leq 1\}$$

for every $\xi \in \mathrm{BV}^{\Phi}(\Xi,E)$. Secondly, given a set function $\xi \in \mathrm{BV}^{\Phi}(\Xi,E)$ and a partition $\mathcal{P}$, let

$$V^{\circ}_{\Phi}(\mathcal{P},\xi) = \sup \sum_{X \in \mathcal{P}} \beta(X)\,|\mu(X)|\ ,$$

where the supremum is taken over all functions $\beta\colon \mathcal{P} \to [0,\infty)$ such that

$$\sum_{X \in P} \Psi(\beta(X)) \leq 1\ ,$$

Ψ being the gauge complementary to Φ (cf.[4],0), and then

$$V^{\circ}_{\Phi}(\Xi,\xi) = \sup\{ V^{\circ}_{\Phi}(\mathcal{P},\xi) \colon \mathcal{P} \in \Xi\}\ .$$

According to the following proposition (cf.[4],3.31), the introduced norms are equivalent.

PROPOSITION 1.1. *The functionals* V_{Φ} *and* V°_{Φ} *are norms on the space* $\mathrm{BV}^{\Phi}(\Xi,E)$ *such that*

$$V_{\Phi}(\Xi,\xi) \leq V^{\circ}_{\Phi}(\Xi,\xi) \leq 2\,V_{\Phi}(\Xi,\xi)$$

for every $\xi \in \mathrm{BV}^{\Phi}(\Xi,E)$.

The space $\mathrm{BV}^{\Phi}(\Xi,E)$ *is complete in each of these norms.*

2. Let $\Xi \subset \Pi$ be a directed set of partitions and

$$\mathcal{Q}_{\Xi} = \{\emptyset\} \cup \bigcup_{\mathcal{P} \in \Xi} \mathcal{P} .$$

Let $\mu \in \mathrm{BV}^{\Phi}(\Xi,E)$. Then $f\mu \in \mathrm{BV}^{\Phi}(\Xi,E)$ for every function $f \in \mathrm{sim}\,(\Xi)$. The closure of the vector space $\{f\mu : f \in \mathrm{sim}\,(\Xi)\}$ in $\mathrm{BV}^{\Phi}(\Xi,E)$ is denoted by $\mathrm{BV}^{\Phi}(\Xi,\mu)$. Then $\mathrm{BV}^{\Phi}(\Xi,\mu)$ is a Banach space, being a closed subspace of $\mathrm{BV}^{\Phi}(\Xi,E)$.

In the case of a (positive) measure, λ , the space $\mathrm{BV}^{1}(\Pi,\lambda)$ consists of all measures absolutely continuous with respect to λ . Furthermore, the elements of $\mathrm{BV}^{1}(\Pi,\lambda)$ are canonically associated with certain functions (more accurately, equivalence classes of functions) on Ω , namely the λ–integrable ones. So, the space $\mathrm{BV}^{1}(\Pi,\lambda)$ is identified with $L^{1}(\lambda)$. In this section, those set functions μ , belonging to $\mathrm{BV}^{\Phi}(\Xi,E)$, are isolated for which an analoguous identification of $\mathrm{BV}^{\Phi}(\Xi,\mu)$ with a space of (equivalence classes of) functions on Ω is possible.

PROPOSITION 2.1. *Let L and α be constants such that $0 < L < \infty$, $0 < \alpha \le \infty$ and*

$$\Phi(s)\Phi(t) \le L\Phi(st)$$

for every $s \in [0,\infty)$ and $t \in [0,\alpha)$.

Let $\mu \in \mathrm{BV}^{\Phi}(\Xi,E)$ be a set function such that $|\mu(X)| < \alpha$ for every $X \in \mathcal{Q}_{\Xi}$. Let λ be a measure in the space Ω such that $\lambda(X) \le v_{\Phi}(\mu,\Xi;X)$ for every $X \in \mathcal{Q}_{\Xi}$. Then

$$\int_{\Omega} \Phi(|f|)\mathrm{d}\lambda \le L v_{\Phi}(f\mu,\Xi;\Omega) \tag{2.1}$$

for every function $f \in \mathrm{sim}(\Xi)$.

The set function μ will be called (Φ,Ξ)–closable if the following statement holds:

If $f_j \in \mathrm{sim}(\Xi)$, $j = 1,2,\ldots,$ are functions such that

(2.2)
$$\sum_{j=1}^{\infty} V_\Phi(\Xi, f_j\mu) < \infty$$

and

(2.3)
$$\sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every $\omega \in \Omega$ for which

(2.4)
$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty ,$$

then

$$\lim_{n\to\infty} V_\Phi(\Xi, \sum_{j=1}^{n} f_j\mu) = 0 .$$

Assume that the set function μ is (Φ,Ξ)–closable. By $\mathcal{L}(\mu,\Phi,\Xi)$ is denoted the family of all functions f on Ω for which there exist functions $f_j \in \mathrm{sim}(\Xi)$, $j = 1,2,\ldots,$ satisfying condition (2.2), such that

(2.5)
$$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

for every $\omega \in \Omega$ for which the inequality (2.4) holds. By (2.2), the sequence $\{f_j\mu\}_{j=1}^{\infty}$ is then absolutely summable in the space $\mathrm{BV}^\Phi(\Xi,E)$ and so, we define

$$f\mu = \sum_{j=1}^{\infty} f_j\mu .$$

This definition is unambiguous because μ is assumed to be (Φ,Ξ)–closable. It is then immediate that

$$(2.6) \qquad V_{\Phi}(\Xi,f\mu) = \inf \sum_{j=1}^{\infty} V_{\Phi}(\Xi,f_j\mu) ,$$

where the infimum is taken over all choices of the functions $f_j \in \mathrm{sim}(\Xi)$, $j = 1,2,\ldots$, satisfying condition (2.2), such that (2.5) holds for every $\omega \in \Omega$ for which (2.4) does.

Functions belonging to $\mathcal{L}(\mu,\Phi,\Xi)$ will be called (μ,Φ,Ξ)–integrable. For each such function f we define

$$\mu(f) = \int_{\Omega} f \, \mathrm{d}\mu = \int_{\Omega} f(\omega)\mu(\mathrm{d}\omega) = (f\mu)(\Omega) .$$

Using the definition of $\mathcal{L}(\mu,\Phi,\Xi)$ and formula (2.6), the proof of the following proposition – the Beppo Levi theorem – is straightforward. (Cf. [3], Proposition 1.5.)

PROPOSITION 2.2. *If* $f_j \in \mathcal{L}(\mu,\Phi,\Xi)$, $j = 1,2,\ldots$, *are functions satisfying condition* (2.2) *and* f *is a function on* Ω *such that the equality* (2.5) *holds for every* $\omega \in \Omega$ *for which the inequality* (2.4) *does, then* $f \in \mathcal{L}(\mu,\Phi,\Xi)$ *and*

$$\lim_{n\to\infty} V_{\Phi}(\Xi, f\mu - \sum_{j=1}^{n} f_j\mu) = 0 .$$

This proposition implies, among other things, that $\mathrm{BV}^{\Phi}(\Xi,\mu) = \{f\mu\colon f \in \mathcal{L}(\mu,\Phi,\Xi)\}$. Hence, if we identify any functions f and g from $\mathcal{L}(\mu,\Phi,\Xi)$ such that $V_{\Phi}(\Xi,(f-g)\mu) = 0$, then the resulting space, denoted by $L(\mu,\Phi,\Xi)$, equipped with the norm $f \mapsto V_{\Phi}(\Xi,f\mu)$ is linearly isometric with $\mathrm{BV}^{\Phi}(\Xi,\mu)$.

Now, assuming that λ is a finite measure in the space Ω and f a λ–integrable function, let

$$M_{\lambda}(f,X) = \frac{1}{\lambda(X)} \int_{X} f \, \mathrm{d}\lambda$$

if X is a λ–measurable set such that $\lambda(X) > 0$, and $M_{\lambda}(f,X) = 0$ if $\lambda(X) = 0$. Furthermore, let

$$M_\lambda(f,\mathcal{P}) = \sum_{X\in\mathcal{P}} M_\lambda(f,X)X$$

for any finite partition $\mathcal{P}$ of Ω into λ–measurable sets. So, if X is a measurable set, then $M_\lambda(f,X)$ is a well–defined number and, if $\mathcal{P}$ is a partition of Ω into λ–measurable sets, then $M_\lambda(f,\mathcal{P})$ is a $\mathcal{P}$–simple function.

If f is a $\mathcal{P}$–simple function and $\mathcal{P} \prec \mathcal{P}'$, then $M_\lambda(f,\mathcal{P}') = f$.

PROPOSITION 2.3. *Under the assumptions of Proposition* 2.1, *if*

$$V_\Phi(\Xi,M_\lambda(f,\mathcal{P})\mu) \le KV_\Phi(\Xi,f\mu)\ ,$$

for some number $K > 0$, *every function* $f \in \mathrm{sim}(\Xi)$ *and every partition* $\mathcal{P} \in \Xi$, *then the set function* μ *is* (Φ,Ξ)*–closable and, for every function* $f \in \mathcal{L}(\mu,\Phi,\Xi)$, *the equality*

$$f\mu = \lim_{\mathcal{P}\in\Xi} M_\lambda(f,\mathcal{P})\mu$$

holds in the sense of the norm–convergence in the space $\mathrm{BV}^\Phi(\Xi,E)$.

For the description of an interesting class of examples, let $p \ge 1$ and $\Phi(s) = s^p$, for $s \in [0,\infty)$. Let $\mathcal{R}$ be the algebra of sets generated by $\mathcal{Q}$, that is, $\mathcal{R}$ is the family of sets whose characteristic functions are $\mathcal{Q}$–simple. A set function $\mu: \mathcal{Q} \to E$ will be termed p–scattered if the set function λ defined by

$$\lambda(X) = |\mu(X)|^p$$

for every $X \in \mathcal{R}$, is σ–additive and, hence, generates a measure in Ω . In that case,

$$V_p(\Pi,f\mu) = \left[\int_\Omega |f|^p \mathrm{d}\lambda\right]^{1/p} \tag{2.7}$$

for every $f \in \mathrm{sim}(\mathcal{Q})$. It then follows that μ is (p,Π)–closable, $\mathcal{L}(\mu,p,\Pi)$ consists precisely of all functions f on Ω over that $f|f|^{p-1}$ is λ–integrable and (2.7) holds for every such function f.

3. There are set functions of interest in analysis, belonging to $\mathrm{BV}^{\Phi}(\Xi,E)$, which are not (Φ,Ξ)–closable. In this section, an integration scheme, modelled on Proposition 2.3, applicable to some such set functions, will be described.

Let $\mu\colon \mathcal{Q} \to E$ be an additive set function. Let $\Xi \subset \Pi$ be a directed set of partitions containing the coarsest partition, $\{\Omega\}$. Let λ be a finite measure in the space Ω such that every set belonging to the semialgebra $\mathcal{Q}$ is λ–measurable.

Let $\mathcal{K} = \mathcal{K}(\mu,\Xi,\lambda)$ be the family of all functions $f \in \mathcal{L}^1(\lambda)$ such that

$$\delta(f) = \sup\{V_\infty(\Xi, M_\lambda(f,\mathcal{P})\mu) \colon \mathcal{P} \in \Xi\} < \infty . \tag{3.1}$$

Let $\mathcal{J} = \mathcal{J}(\mu,\Xi,\lambda)$ be the family of all functions $f \in \mathcal{K}$ such that the net $\{M_\lambda(f,\mathcal{P})\mu\}_{\mathcal{P} \in \Xi}$ is convergent in the space $\mathrm{BV}^\infty(\Xi,E)$. For every $f \in \mathcal{J}$, the limit of the net $\{M_\lambda(f,\mathcal{P})\mu\}_{\mathcal{P} \in \Xi}$ is denoted by $f\mu$, that is,

$$f\mu = \lim_{\mathcal{P} \in \Xi} M_\lambda(f,\mathcal{P})\mu . \tag{3.2}$$

This notation is legitimate because the following proposition implies that the element $f\mu$ of the space $\mathrm{BV}^\infty(\Xi,E)$ is uniquely determined by the function f.

PROPOSITION 3.1. *The family of functions, $\mathcal{K}$, is a vector space and the functional, δ, defined by (3.1) for every $f \in \mathcal{K}$ is a seminorm which makes $\mathcal{K}$ complete. The family $\mathcal{J}$ is a closed subspace of $\mathcal{K}$ and* $\mathrm{sim}(\Xi)$ *is a dense subspace of $\mathcal{J}$. The correspondence $f \mapsto f\mu$ is a linear map from $\mathcal{J}$ into* $\mathrm{BV}^\infty(\Xi,E)$ *such that $V_\infty(\Xi,f\mu) \le \delta(f)$ for every $f \in \mathcal{J}$.*

Let B be the Banach space of all bounded nets of real or complex numbers, indexed by the elements of Ξ, with the norm defined by

$$\|\beta\| = \sup\{|\beta_{\mathcal{P}}| \colon \mathcal{P} \in \Xi\} ,$$

for every $\beta = \{\beta_{\mathcal{P}}\}_{\mathcal{P} \in \Xi}$ in B. Let C be the subspace of B consisting of those elements which are convergent. Let LIM be a continuous linear functional on B of the norm equal to 1 such that

$$\mathrm{LIM}\beta = \lim_{\mathcal{P}\in\Xi} \beta_{\mathcal{P}}$$

for every $\beta = \{\beta_{\mathcal{P}}\}_{\mathcal{P}\in\Xi}$ belonging to C.

Now, for every $f \in \mathcal{K}$, we define the set function $f\mu_{\mathrm{LIM}}: \mathcal{Q}_{\Xi} \to E''$ by letting

$$\langle (f\mu_{\mathrm{LIM}})(X), x' \rangle = \mathrm{LIM}\{\langle (M_{\lambda}(f,\mathcal{P})\mu)(X), x' \rangle\}_{\mathcal{P} \in \Xi}$$

for every $X \in \mathcal{Q}_{\Xi}$ and $x' \in E'$. The element $(f\mu_{\mathrm{LIM}})(\Omega)$ of the space E'' may be denoted by

$$\mu_{\mathrm{LIM}}(f) = \int_{\Omega} f \, \mathrm{d}_{\mathrm{LIM}}\mu = \int_{\Omega} f(\omega)\mu(\mathrm{d}_{\mathrm{LIM}}\omega) \,. \tag{3.3}$$

Then, clearly, the map $f \mapsto \mu_{\mathrm{LIM}}(f)$ of $\mathcal{K}$ into E'' is linear and $|\mu_{\mathrm{LIM}}(f)| \le \delta(f)$ for every $f \in \mathcal{K}$. Furthermore, it is clear that, if $f \in \mathcal{J}$, then $(f\mu_{\mathrm{LIM}})(X) = (f\mu)(X)$, for every $X \in \mathcal{Q}_{\Xi}$, where $f\mu$ is defined by (3.2). Hence, for the functions in $\mathcal{J}$, it is not necessary to indicate the functional LIM in the notation (3.3).

Assume now that Ξ is the set of all terms of a sequence, $\{\mathcal{P}_n\}_{n=0}^{\infty}$, of partitions such that $\mathcal{P}_0 = \{\Omega\}$ and $\mathcal{P}_n \prec \mathcal{P}_{n+1}$, for $n = 0,1,2,\dots$. For a function $f \in \mathcal{L}^1(\lambda)$, let $f_0 = M_{\lambda}(f,\mathcal{P}_0)$ and

$$f_n = M_{\lambda}(f - M_{\lambda}(f,\mathcal{P}_{n-1}),\mathcal{P}_n)$$

for $n = 1,2,\dots$. Then

$$\sum_{j=0}^{n} f_j = M_{\lambda}(f,\mathcal{P}_n)$$

for every $n = 0,1,2,\dots$. Hence, if the partial sums of the sequence $\{f_n\mu\}_{n=0}^{\infty}$ are bounded in the space $\mathrm{BV}^{\infty}(\Xi,E)$, then $f \in \mathcal{K}(\mu,\Xi,\lambda)$. If the sequence $\{f_n\mu\}_{n=0}^{\infty}$ is (simply) summable in $\mathrm{BV}^{\infty}(\Xi,E)$, then $f \in \mathcal{J}(\mu,\Xi,\lambda)$.

To express a sufficient condition for the function f to belong to $\mathcal{J}$, let φ and ψ be monotonic functions on $[0,\infty)$ such that $\varphi(0) = \psi(0) = 0$, and

$$|\mu(X)| \leq \varphi(\lambda(X)) \quad \text{and} \quad \sup\{|f(\omega_1) - f(\omega_2)| : \omega_1 \in X, \omega_2 \in X\} \leq \psi(\lambda(X))$$

for every $X \in \mathcal{Q}_{\Xi}$. If φ and ψ are the least such functions, it is apt to call them the moduli of continuity of μ and f , respectively, with respect to λ and $\mathcal{Q}_{\Xi}$.

PROPOSITION 3.2. *If*

$$\sum_{n=0}^{\infty} \sum_{Z \in \mathcal{P}_n} \psi(\lambda(Z)) \sum_{Y \in \mathcal{P}_{n+1}} \varphi(\lambda(Z \cap Y)) < \infty$$

then $f \in \mathcal{J}(\mu,\Xi,\lambda)$. *In fact, the sequence* $\{f_n\mu\}_{n=0}^{\infty}$ *is absolutely summable in the space* $\mathrm{BV}^{\infty}(\Xi,E)$.

The condition of this proposition is satisfied, for example, if there exists an integer $k > 0$ such that, for every $n = 0,1,2,\dots,$ the partition $\mathcal{P}_{n+1}$ is obtained by dividing every set in $\mathcal{P}_n$ into k disjoint parts of equal measure λ , and $\varphi(s) = s^{1/p}$, $\psi(s) = s^{1/q}$, for every $s \geq 0$, where $p > 1$, $q > 1$ and $p^{-1} + q^{-1} > 1$.

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