On the Frobenius Reciprocity Theorem for Square Integrable Representations of Nonunimodular Groups

*RAY A. KUNZE**

In the unimodular case, the Frobenius reciprocity theorem for irreducible square integrable representations asserts that certain intertwining spaces are canonically isomorphic; the essential analytic point is that square integrability implies the continuity of functions in particular subspaces of L^2 spaces on which the group acts and leads to a characterization of these subspaces in terms of reproducing kernels. In the nonunimodular case this is no longer true. There is a canonical isomorphism between proper subspaces of the intertwining spaces, one of which is uniformly dense in the full intertwining space.

The results in this paper were motivated by observations made in connection with the problem of constructing explicit unitary half-space models of ladder representations for the Lorentz groups $SO(1,n+1)$. That is joint work with J.E. Gilbert and K. Davis; It will appear elsewhere.

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1. PRELIMINARIES. Let P be a locally compact group with a fixed right Haar measure dx. Let Δ be the modular function on P, defined so that $\Delta(x)dx$ is a left Haar measure. Suppose M is a compact subgroup of P and that σ is an irreducible representation of M with representation space $\mathcal{H}_\sigma$. Here and throughout representation will mean continuous unitary representation. Let ρ be the representation of P induced by σ. By definition, $\rho(y)$ $(y \in G)$ is right translation by y on the space $\mathcal{H}_\rho = L^2(P,\sigma)$ of square integrable maps $f:P \to \mathcal{H}_\sigma$ such that

(1.1) $$f(mx) = \sigma(m)f(x)$$

for all (m,x) in $M \times P$.

Now suppose π is an irreducible square integrable representation of P with representation space $\mathcal{H}_\pi$. When the results of [1] are formulated in the present context, this means that π is unitarily equivalent to a subrepresentation of the right regular representation of P. Equivalently, π has a nonzero square integrable matrix entry

$$E(\phi,\gamma):x \to (\pi(x)\phi|\gamma),\ x \in P.$$

Moreover, there exists a unique self-adjoint positive operator D in $\mathcal{H}_\pi$, called the <u>formal degree of</u> π, with the following properties.

(1.2) For $\phi \neq 0$, $E(\phi,\gamma)$ is square integrable if and only if $\gamma \in \operatorname{dom} D^{-\frac{1}{2}}$.

(1.3) For ϕ,ψ in $\mathcal{H}_\pi$ and γ,δ in $\operatorname{dom} D^{-\frac{1}{2}}$

$$\int(\pi(x)\phi|\gamma)(\pi(x)\psi|\delta)dx = (\phi|\psi)(D^{-\frac{1}{2}}\delta|D^{-\frac{1}{2}}\gamma) .$$

$$\pi(a)D\,\pi(a^{-1}) = \Delta(a^{-1})D, \quad a \in P . \tag{1.4}$$

If P is unimodular, D is a scalar operator which may be identified with the usual formal degree. But otherwise D is an unbounded operator with a dense $\pi(P)$ invariant domain.

Let $\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ denote the space of continuous linear maps of $\mathcal{H}_\pi$ to $\mathcal{H}_\rho$ that intertwine π and ρ. Similarly, let $\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$ denote the space of continuous linear maps of $\mathcal{H}_\pi$ to $\mathcal{H}_\sigma$ that intertwine the restriction of π to M and σ. In the unimodular case, these two intertwining spaces are canonically, even isometrically, isomorphic, [2]. Here we investigate the extent to which this remains true in the nonunimodular case.

2. THE SPACE $^\circ\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$. The distinctive feature of the unimodular case is that for any $U \in \mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$, $U(\mathcal{H}_\pi)$ is a Hilbert space, relative to the L^2 norm, of continuous maps of P to $\mathcal{H}_\sigma$ in which point evaluations are continuous [2, Theorem 1]. Because π is irreducible, Schur's lemma implies that U^*U is a scalar multiple of the identity; this in turn implies that U is a scalar multiple of an isometry. Thus, it is always true that $U(\mathcal{H}_\pi)$ is a closed subspace of $L^2(P,\sigma)$. But in the nonunimodular case, it is not always true that $U(\mathcal{H}_\pi)$ may be identified with a space of continuous functions in which point evaluations are continuous. Let $^\circ\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ denote the subspace of $\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ consisting of the maps U for which $U(\mathcal{H}_\pi)$ is a space of continuous functions in which point evaluations are continuous.

3. THEOREM. *Suppose* $U \in {}^{\circ}\mathrm{Hom}_P(\mathcal{H}_\pi, \mathcal{H}_\rho)$. *Then the equation*

$$A\phi = (U\phi)(1), \quad \phi \in \mathcal{H}_\pi \tag{3.1}$$

defines an operator A *in* $\mathrm{Hom}_M(\mathcal{H}_\pi, \mathcal{H}_\sigma)$ *such that for arbitrary in* P *and* ϕ *in* $\mathcal{H}_\pi$

$$(U\phi)(x) = A\pi(x)\phi \; . \tag{3.2}$$

Moreover, the adjoint A^* *of* A *has the property that*

$$A^*(\mathcal{H}_\sigma) \subset \mathrm{dom}\, D^{-\frac{1}{2}}. \tag{3.3}$$

Proof. It is clear that (3.1) defines a continuous linear map A of $\mathcal{H}_\pi$ to $\mathcal{H}_\sigma$. Let $m \in M$ and $\phi \in \mathcal{H}_\pi$. Then

$$A\pi(m)\phi = (U\pi(m)\phi)(1) = (\rho(m)U\phi)(1) = (U\phi)(m) = \sigma(m)A\phi \; .$$

Thus, $A \in \mathrm{Hom}_M(\mathcal{H}_\pi, \mathcal{H}_\sigma)$; moreover

$$(U\phi)(x) = (\rho(x)U\phi)(1) = (U\pi(x)\phi)(1) = A\pi(x)\phi$$

for all ϕ in $\mathcal{H}_\pi$ and x in P . To prove (3) let $\alpha \in \mathcal{H}_\sigma$ and $\phi \in \mathcal{H}_\pi$. Then, since $U\phi \in L^2(P,\sigma)$, (2) implies

$$\int |(A\pi(x)\phi|\alpha)|^2 \, dx < \infty \; .$$

Now $(A\pi(x)\phi|\alpha) = (\pi(x)\phi|A^*\alpha)$, and by (1.2), $E(\phi,A^*\alpha)$ is square integrable for all ϕ iff $A^*\alpha \in \operatorname{dom} D^{-\frac{1}{2}}$. Thus, A^* maps $\mathcal{H}_\sigma$ into $\operatorname{dom} D^{-\frac{1}{2}}$.

4. THE SPACE ${}^\circ\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$. Let ${}^\circ\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$ denote the subspace of $\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$ consisting of the operators A such that $A^*(\mathcal{H}_\sigma) \subset \operatorname{dom} D^{-\frac{1}{2}}$. If $A \in {}^\circ\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$ then $D^{-\frac{1}{2}}A^*$ is a continuous linear map of $\mathcal{H}_\sigma$ to $\mathcal{H}_\pi$. It follows that the equation

$$(4.1) \qquad (A|B) = \operatorname{tr}(D^{-\frac{1}{2}}A^*)^*(D^{-\frac{1}{2}}B^*)$$

defines an inner product on ${}^\circ\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$. If U and V are operators in $\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\sigma)$, then from Schur's lemma one sees that there is a unique scalar $(U|V)$ such that

$$(4.2) \qquad V^*U = (U|V)I_{\mathcal{H}_\pi} .$$

It follows that the pairing

$$U,V \to (U|V)$$

is an inner product on $\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ such that $(U|U)$ is the square of the operator norm of U . Since $\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ is complete relative to the operator norm, it results that $\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ is a Hilbert space when equipped with the inner product defined by (2) .

5. THEOREM. *For each* A *in* ${}^\circ\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$, (3.2) *defines an operator* $U = U_A$ *in* ${}^\circ\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ *such that the map*

$$A \to U_A$$

is an isometry of $^\circ\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$ *onto* $^\circ\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$.

Proof. Suppose $A \in {}^\circ\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$. Let $\phi \in \mathcal{H}_\pi$ and define f on P by

$$f(x) = A\pi(x)\phi \ .$$

Then f is continuous and

$$f(mx) = A\pi(m)\pi(x)\phi = \sigma(m)A\pi(x)\phi = \sigma(m)f(x)$$

for all (m,x) in $M \times P$. If $x \in \mathcal{H}_\sigma$ then

$$(f(x)|\alpha) = (\pi(x)\phi|A^*\alpha) \ .$$

Since $A^*\alpha \in \mathrm{dom}\, D^{-\frac{1}{2}}$, (1.2) implies

$$\int|(f(x)|\alpha)|^2\, dx < \infty \ .$$

Because $\mathcal{H}_\sigma$ is finite dimensional, it follows that $f \in L^2(P,\sigma)$. Thus, (3.2) defines a linear map $U = U_A$ of $\mathcal{H}_\pi$ into $L^2(P,\sigma)$.

Now suppose A and B lie in $^\circ\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$, let $\varepsilon_1,\ldots,\varepsilon_n$ be an orthonormal base for $\mathcal{H}_\sigma$, and let ϕ and ψ be vectors in $\mathcal{H}_\pi$. Then

$$\begin{aligned}(U_A\phi|U_B\psi) &= \int (A\pi(x)\phi|B\pi(x)\psi)dx \\ &= \sum_i \int(\pi(x)\phi|A^*\varepsilon_i)\overline{(\pi(x)\psi|B^*\varepsilon_i)}dx \\ &= \sum_i (\phi|\psi)(D^{-\frac{1}{2}}B^*\varepsilon_i|D^{-\frac{1}{2}}A^*\varepsilon_i)\end{aligned}$$

by (1.3) . Therefore

$$(\mathrm{U}_A\phi|\mathrm{U}_B\psi) = (\phi|\psi)(A|B) \; . \tag{5.1}$$

It follows, in particular, that

$$\|\mathrm{U}_A\phi\|_2^2 = \|A\|^2\|\phi\|^2 \tag{5.2}$$

so that U_A is an isometry multiplied by $\|A\|$. In addition

$$(\mathrm{U}_A\pi(y)\phi)(x) = A\pi(xy)\phi = \rho(y)(\mathrm{U}_A\phi)(x)$$

for all x,y in P and ϕ in $\mathcal{H}_\pi$. If $\|A\|_\infty$ denotes the operator norm of A , then

$$\|\mathrm{U}_A\phi(x)\| = \|A\pi(x)\phi\| \le \|A\|_\infty\|\phi\| \; .$$

Hence, (5.2) implies that point evaluations are continuous. Thus $\mathrm{U}_A \in {}^\circ\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$. The map

$$A \to \mathrm{U}_A \; , \quad A \in {}^\circ\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$$

is evidently linear. It is surjective by Theorem 3. By (1)

$$(\mathrm{U}_B^*\mathrm{U}_A\phi|\psi) = (A|B)(\phi|\psi)$$

for all A,B in ${}^\circ\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$ and ϕ,ψ in $\mathcal{H}_\pi$. It now follows from (4.2) that $A \to \mathrm{U}_A$ is an isometry.

6. EXAMPLE. Consider the special case in which M is the identity subgroup and σ the identity representation of M on $\mathcal{H}_\sigma = \mathbb{C}$. Then $\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$ is just the dual of $\mathcal{H}_\pi$. The elements of ${}^\circ\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$ are the linear functionals determined by the vectors γ in dom $D^{-\frac{1}{2}}$. In this

case, $\mathcal{H}_\rho = L^2(P)$ and ρ is the right regular representation. Specifically, if $\gamma \in \operatorname{dom} D^{-\frac{1}{2}}$ and γ^* is the corresponding linear functional, then U_{γ^*} in ${}^\circ\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ is defined by

$$(U_{\gamma^*}\phi)(x) = (\pi(x)\phi|\gamma), \; x \in P.$$

For this case, the results proved in (9) and (10) were obtained by other methods in [4].

7. EXAMPLE. In general there is a distinction between ${}^\circ\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ and $\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$. To see this, consider the affine group

$$P = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} : \; a,b \in \mathbb{R} \text{ and } a > 0 \right\}$$

of the line. Then the measure

(7.1) $$d(a,b) = \frac{dadb}{a^2}$$

is right invariant, and the modular function is given by

(7.2) $$\Delta(a,b) = a \quad .$$

As is well known, the formula

(7.3) $$(\pi(a,b)\phi)(t) = e^{ia^{-1}bt} \; \phi(a^{-1}t)$$

defines an irreducible square integrable unitary representation π of P on

$$\mathcal{H}_\pi = L^2((0,\infty), \frac{dt}{t}) \quad .$$

In fact, let ϕ and ψ be any elements of $\mathcal{H}_\pi$ and set

$$h(a,b) = (\pi(a,b)\phi|\psi) \quad .$$

Then since

$$H(a,b) = \int_0^\infty e^{ibt}\ \phi(t)\bar{\psi}(at)\ \frac{dt}{t}\ .$$

it follows by Plancherel's theorem that

$$\frac{1}{2\pi}\int_{\infty}^{\infty} |h(a,b)|^2 dt = \int_0^\infty |\phi(t)|^2 |\psi(at)|^2 t^{-2}\ dt$$

and hence from (1) that

$$\frac{1}{2\pi}\int |h(a,b)|^2 d(a,b) = \int_0^\infty |\phi(t)|^2 t^{-2}\ [\int_0^\infty |\psi(at)|^2\ a^{-1}\ \frac{da}{a}\ ]\ dt$$

$$= \int_0^\infty |\phi(t)|^2\ \frac{dt}{t}\ \int_0^\infty (|\psi(a)|^2 a^{-1})\ \frac{da}{a}\quad .$$

From this it is easy to see that the formal degree D is multiplication by the independent variable; thus γ in $\mathcal{H}_\pi$ is in dom $D^{-\frac{1}{2}}$ iff

$$\int_0^\infty t^{-1}\ |\gamma(t)|^2\ \frac{dt}{t} < \infty$$

and $(D^{-\frac{1}{2}}\gamma)(t) = t^{-\frac{1}{2}}\gamma(t)$ for all such γ. This is a situation in which (6) applies.

We shall construct a Cauchy sequence in $^\circ\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ that converges to an operator outside $^\circ\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$. For this purpose, let γ_n be the characteristic function of the interval [1,n] and set

$$U_n = U_{\gamma_n^*}\ ,\ n = 1,2,\ldots\ .$$

Then it follows from (5) or by direct computation that for the norms in (4.1) and (4.2)

$$\|U_n - U_m\|^2 = \|\gamma_n^* - \gamma_n^*\|^2$$

$$= (D^{-\frac{1}{2}}(\gamma_n - \gamma_m) | D^{-\frac{1}{2}}(\gamma_n - \gamma_m))$$

$$= \int_m^n \frac{dt}{t^2} \qquad (n > m).$$

Thus, $\|U_n - U_m\|^2 \to 0$ as $m,n \to \infty$. Let $U = \lim_n U_n$. Then U is not of the form U_{γ^*} with $\gamma \in \operatorname{dom} D^{-\frac{1}{2}}$. For if it were, then by (5)

$$\lim_{n\to\infty} \int_0^\infty |\gamma(t) - \gamma_n(t)|^2 \frac{dt}{t^2} = 0.$$

But this implies γ is equal a.e. to the characteristic function of the interval $[1,\infty)$ and hence that γ is not in $\mathcal{H}_\pi$.

The obvious question that one might ask at this point is settled by the following result.

8. THEOREM. *The subspace* ${}^\circ\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ *is dense in the full intertwining space* $\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$.

Proof. This is proved by an approximate identity argument. For this let N be an arbitrary compact neighborhood of the identity in P and 1_N its characteristic function. Set

$$e_N(y) = \int_M c\, 1_N(m^{-1} y\, m)dm,\ y \in P$$

where dm denotes normalized Haar measure on M and the constant c is chosen so that

$$\int e_N(y^{-1})dy = 1.$$

Then $e_N(mym^{-1}) = e_N(y)$ for all (m,y) in $M \times P$, and for each neighborhood N_1 of the identity there is a compact neighborhood N_2 of the identity such that supp $e_{N_2} \subset N_1$.

Now suppose $f \in L^2(P,\sigma)$. Then the convolution

$$(e_N * f)(x) = \int e_N(xy^{-1})f(y)dy = \int e_N(y^{-1})f(yx)dy$$

of e_N and f is a well defined continuous function from P to $\mathcal{H}_\sigma$. For $m \in M$

$$\begin{aligned}(e_N * f)(mx) &= \int e_N(y^{-1})\ f(y\ mx)dy \\ &= \int e_N(my^{-1}m^{-1})\ f(myx)dy \\ &= \ \sigma(m)(e_N * f)(x)\end{aligned}$$

since $\Delta(m^{-1}) = 1$ and $f(myx) = \sigma(m)\ f(yx)$. We also have

$$\begin{aligned}\|e_N * f\|_2 &\leq \int e_N(y-1)\left(\int \|f(yx)\|^2dx\right)^{1/2} dy \\ &= \|f\|_2 \int e_N(y)dy.\end{aligned}$$

Thus, $e_N * f \in L^2(P,\sigma)$. Moreover, standard arguments now apply and show that $e_N * f \to f$ in $L^2(P,\sigma)$ as $N \to 1$.

Next suppose that U is a non-zero operator in $\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ that is orthogonal to ${}^\circ\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$. Choose $\phi_0 \neq 0$ in $\mathcal{H}_\pi$. Then $U\phi_0 \neq 0$ and there exists a compact neighborhood N_0 of 1 in P such that $e_{N_0} * U\ {}_0 \neq 0$. For compact neighborhoods of 1 with $N \subset N_0$ define $U_N\colon \mathcal{H}_\pi \to L^2(P,\sigma)$ by

$$U_N\phi = e_N * U\phi.$$

Then since

$$\|U_N\phi\|_2 \leq \|e_N\|_1\ \|U\phi\|_2 \leq \|e_N\|_1\ \|U\|\|\phi\|$$

it follows that U_N is a non-zero continuous linear map of $\mathcal{H}_\pi$ into $\mathcal{H}_\rho = L^2(P,\sigma)$. In addition

$$U_N\pi(x)\phi = e_N * \rho(x)U\phi = \rho(x)\ U_N\phi$$

for all x in P and ϕ in $\mathcal{H}_\pi$. Thus $U_N \in \mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$. Now define $A_N\colon \mathcal{H}_\pi \to \mathcal{H}_\sigma$ by

$$A_N\phi = (U_N\phi)(1) = \int e_N(y^{-1})(U\phi)(y)dy.$$

Then one has

$$\begin{aligned}\|A_N\phi\| &\leq \int e_N(y^{-1})\|(U\phi)(y)\|dy \\ &\leq \|e_N\|_2\ \|U\phi\|_2 \\ &\leq \|e_N\|_2\ \|U\|\|\phi\|\end{aligned}$$

where $e_N(y) = e_N(y^{-1})$. In addition

$$A_N\ \pi(x)\phi = (U_N\pi(x)\phi)(1) = (U_N\phi)(x)$$

so that $A_N\pi(m) = \sigma(m)A_N$ for every m in M. Finally, since

$$(\pi(x)\phi|A_N^*\alpha) = (U_N\phi(x)|\alpha)$$

is square integrable for all α in $\mathcal{H}_\sigma$, it follows that $A_N \in {}^\circ\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$. Therefore, $U_N \in {}^\circ\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$. By assumption $(U|U_N) = 0$. Hence, by (4.2), $U_N^*\ U = 0$. This implies

$$(U\phi|U_N\phi) = 0$$

for every ϕ . But this is impossible because $U_N\phi \to U\phi$ as $N \to 1$. Therefore, $°\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$ is dense in $\mathrm{Hom}_P(\mathcal{H}_\pi,\mathcal{H}_\rho)$.

9. THE SPHERICAL FUNCTION Φ_A. Let A be a non-zero operator in $°\mathrm{Hom}_M(\mathcal{H}_\pi,\mathcal{H}_\sigma)$. Then by Theorem 5, $U_A(\mathcal{H}_\pi)$ is a Hilbert space, relative to the L^2-norm, of continuous vector valued functions for which the point evaluations

$$E_x\colon f \to f(x),\ x \in P$$

are continuous. Because $A \neq 0$ and σ is irreducible, A is surjective. Hence E_x is surjective for every x. Thus $U_A(\mathcal{H}_\pi)$ is completely determined by the corresponding operator valued reproducing kernel

$$Q_A(x,y) = E_xE_y^* \tag{9.1}$$

in the simple fashion described in [3]. But in the present context, Q_A may be described more explicitly in terms of A and the representation π. For this purpose, we define the operator valued spherical function $\Phi_A\colon P \to \mathrm{End}\,(\mathcal{H}_\sigma)$ by

$$\Phi_A(x) = \frac{1}{\|A\|^2}\ A\,\pi(x)\,A^*,\ x \in P \tag{9.2}$$

Then Φ_A is continuous and σ-spherical in the sense that

$$\Phi_A(m_1xm_2) = \sigma(m_1)\Phi_A(x)\sigma(m_2) \tag{9.3}$$

for all m_1, m_2 in M and x in P. Because π is unitary

$$\Phi_A(x^{-1})^* = \Phi_A(x),\ x \in P \tag{9.4}$$

and Φ_A is positive definite in the sense that

$$(9.5) \qquad \Sigma_{i,j}(\Phi_A(x_i\, x_j^{-1})\alpha_j \mid \alpha_i) \geq 0$$

for all finite sequences $x_1, x_2, \ldots$ in P and $\alpha_1, \alpha_2, \ldots$ in $\mathcal{H}_\sigma$.

10. THEOREM. *Let* A *be a non-zero operator in* ${}^\circ\mathrm{Hom}_M(\mathcal{H}_\pi, \mathcal{H}_\sigma)$. *Then the reproducing kernel for* $U_A(\mathcal{H}_\pi)$ *is given by*

$$(10.1) \qquad Q_A(x,y) = \Phi_A(xy^{-1})$$

The spherical function Φ_A *is square integrable on* P

$$(10.2) \qquad \Phi_A(x) = \int \Phi_A(xy^{-1})\, \Phi_A(y)dy$$

for all x *in* P, *and the map*

$$(10.3) \qquad f \to \Phi_A * f,\ f \in L^2(P,\sigma)$$

is the orthogonal projection of $L^2(P,\sigma)$ *onto* $U_A(\mathcal{H}_\pi)$.

Proof. By (9.1), $Q_A(x,y) = E_x E_y^*$. Now for ϕ in $\mathcal{H}_\pi$ and α in $\mathcal{H}_\sigma$, we have

$$(U_A\phi \mid E_y^*\alpha) = (A\pi(y)\phi \mid \alpha) = (\phi \mid \pi(y^{-1})A^*\alpha).$$

Thus, by (5.1)

$$(U_A\phi \mid E_y^*\alpha) = \frac{1}{\|A\|^2}\,(U_A\phi \mid U_A\pi(y^{-1})A^*\alpha).$$

This implies

$$E_y^* = \frac{1}{\|A\|^2} U_A \pi(y^{-1}) A^*, \quad y \in P \tag{10.4}$$

and (1) is an immediate corollary. Since

$$(\Phi_A(x)\alpha|\beta) = \|A\|^{-2}(\pi(x)A^*\alpha|A^*\beta),$$

the matrix entries of Φ_A are square integrable on P; hence, Φ_A is square integrable in the sense that

$$\int \operatorname{tr}(\Phi_A(x)^* \, \Phi_A(x))dx < \infty.$$

To prove $\Phi_A * \Phi_A = \Phi_A$, i.e., that (2) is valid note, that

$$(\Phi_A(x)\alpha|\beta) = \|A\|^{-2}(\pi(*x)A^*\alpha|\beta)$$

$$= \|A\|^{-4}(U_A\pi(x)A^*\alpha|U_A A^*\beta)$$

$$= \|A\|^{-4} \int (A\pi(yx)A^*\alpha|A\pi(y)A^*\beta)dy$$

$$= \int (\Phi_A(yx)\alpha|\Phi_A(y)\beta)dy \; .$$

Thus, by (9.4)

$$(\Phi_A(x)\alpha|\beta) = \int (\Phi_A(y^{-1})\Phi_A(yx)\alpha|\beta)dy$$

$$= \int (\Phi_A(xy^{-1})\Phi_A(y)\alpha|\beta)dy$$

for all α,β in $\mathcal{H}_\sigma$.

To prove (10.3), let $f \in L^2(P,\sigma)$. Then for $\alpha \in \mathcal{H}_\sigma$, (10.4) and (9.4) imply

$$(f|E_x^*) = \int (f(y)|\Phi_A(yx^{-1})\alpha)dy$$

$$= \int (\Phi_A(xy^{-1})f(y)|\alpha)dy \ .$$

If f is orthogonal to $U_A(\mathcal{H}_\pi)$, it follows that

$$\int \Phi_A(xy^{-1})f(y)dy = 0$$

for all x . On the other hand, if $f \in U_A(\mathcal{H}_\pi)$, then for arbitrary x in P and x in $\mathcal{H}_\sigma$

$$(f(x)|\alpha) = (E_x f|\alpha) = (f|E_x^* \alpha) \ .$$

Therefore, $f = \Phi_A * f$ when $f \in U_A(\mathcal{H}_\pi)$. Hence $f \to \Phi_A * f$ is the orthogonal projection of $L^2(P,\sigma)$ onto $U_A(\mathcal{H}_\pi)$.

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Department of Mathematics
University of Georgia
Athens, GA 30602 USA