On the representation theory of SU(2,1).

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Abstract. In their paper on the Szegö map, Knapp and Wallach [KW] remarked that in the case of those discrete series representations of SU(2,1) which occur as the subquotient of three principal series representations, their methods provided only two of these. That is, the Szegö map built from the highest weight vector misses some occurrences of discrete series representations as quotients. In this note we use the extension of Szegö maps due to Gilbert, Stanton, Kunze, and Tomas [GKST] to investigate these further cases.

INTRODUCTION

In all that follows we will assume that G is the noncompact semisimple Lie group SU$(2,1)$, consisting of 3×3 complex matrices g which satisfy

$$g\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}g^* = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}.$$

In G there is the maximal compact subgroup K consisting of $G\cap \mathrm{U}(3)$ and this can be identified with U(2), since

$$K = \left\{\begin{pmatrix}u&0\\0&\det(u^*)\end{pmatrix} : u\in \mathrm{U}(2)\right\}.$$

We fix in K the maximal torus T consisting of all its diagonal elements. This is also a Cartan subgroup for G. The Lie algebras of these groups are denoted by $\mathcal{G}$, $\mathcal{K}$, and $\mathcal{T}$. The complexifications are denoted by $\mathcal{G}_{\mathbf{C}}$,$\mathcal{K}_{\mathbf{C}}$, and $\mathcal{T}_{\mathbf{C}}$. The vector space $\mathcal{T}_{\mathbf{C}}$ is $\mathbf{C}^3_0 = \{\mathrm{x}\in\mathbf{C}^3 : x_1+x_2+x_3=0\}$. The group K has a central part K_z and a semisimple part K_s, where

$$K_z = \{t\in T : t_{11}=t_{22}\} \quad\text{and}\quad K_s=\{k\in K : k_{33}=1\}.$$

Notice that when we identify K with U(2), K_s corresponds to SU(2). An irreducible unitary representation of K is completely determined by its restrictions to these two subgroups. Hence, $\widehat{K}$ is parameterized by two integers m and n, such that $n\ge 0$ and $m-n$ is even, where m determines the character of the centre and n determines the corresponding representation of SU(2). More precisely, suppose $(\tau,\mathcal{H})$ is an irreducible of representation K. The centre K_z will act by

$$\tau\begin{pmatrix}e^{i\theta}&0&0\\0&e^{i\theta}&0\\0&0&e^{-2i\theta}\end{pmatrix}\phi = e^{im\theta}\phi,\quad \forall\phi\in\mathcal{H}. \tag{1}$$

The integer n corresponds to the action of $K_s \cap T$ on a highest weight vector, say ψ. That is, for $0 \leq \theta \leq 2\pi$,

$$\tau \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \psi = e^{in\theta}\psi. \tag{2}$$

The requirement that $m - n$ is even follows from having both (1) and (2) valid on $K_s \cap K_z$. When dealing with an irreducible unitary representation $(\pi, \mathcal{V})$ of a noncompact semisimple Lie group such as G it is often useful to concentrate attention on the subspace of K-finite vectors in $\mathcal{V}$, say $\mathcal{V}_K$. Recall that an element $\mathbf{v}$ of $\mathcal{V}$ is said to be K-finite if the span of $\pi(K)\mathbf{v}$ is finite dimensional in $\mathcal{V}$. This is an example of of a $(\mathcal{G}, K)$-module [**Vo**]. The set of K-types of this representation of G are the irreducible unitary representations of K which occur in $(\pi|_K, \mathcal{V}_K)$. Vogan has defined the notion of **minimal** K**-type** of a $(\mathcal{G}, K)$-module, see [**Vo**] Definition 5.4.18, and proved the following theorem.

THEOREM(VOGAN). *Suppose that $(\pi, \mathcal{V})$ is an irreducible unitary representation of G. If τ is a minimal K-type of $(\pi, \mathcal{V})$, then τ has multiplicity one in $\pi|_K$.*

As Vogan remarks, this suggests that minimal K-types play a role in the representation theory of noncompact semisimple Lie groups similar to that of dominant integral weights in the compact group case. This also suggests that in studying unitary representations of G, one direction to follow is to start with an irreducible unitary representation τ of K and find irreducible $(\mathcal{G}, K)$-modules which have τ as a minimal K-type. After that, one must still see if it is possible to equip such a module with a unitary structure.

INVARIANT DIFFERENTIAL OPERATORS AND $(\mathcal{G}, K)$-MODULES.

The $(\mathcal{G}, K)$-modules we consider are the spaces of K-finite elements of the kernels of Schmid operators, certain first order invariant differential operators acting on vector-valued functions on G with the property that the K-types which occur in their kernels can be controlled, [**Sc**] and [**HP**]. Before defining them, we need to fix some Lie algebra notation. Let $\mathcal{P}$ be the complement of $\mathcal{K}$ in $\mathcal{G}$, so that the Cartan decomposition of the complexification is $\mathcal{G}_{\mathbf{C}} = \mathcal{K}_{\mathbf{C}} + \mathcal{P}_{\mathbf{C}}$ and $\mathcal{P}_{\mathbf{C}}$ is invariant under the action of $\mathrm{Ad}(K)$. Let $\Phi_{\mathcal{K}}$ denote the roots for (K, T) and Φ the roots for (G, T). When we identify $\mathcal{T}_{\mathbf{C}}$ with $\mathbf{C}_0^3$, these are the functionals α_{ij} given by $\alpha_{ij}(\mathbf{x}) = x_i - x_j$ with $i \neq j$. Fix an ordering on $\Phi_{\mathcal{K}}$ so that the set of positive compact roots is $\{\alpha_{12}\}$. There are three compatible orderings of Φ :

$$\begin{aligned} \Phi^+(0) &= \{\alpha_{12}, \alpha_{31}, \alpha_{32}\}; \\ \Phi^+(1) &= \{\alpha_{12}, \alpha_{13}, \alpha_{32}\}; \\ \Phi^+(2) &= \{\alpha_{12}, \alpha_{13}, \alpha_{23}\}. \end{aligned}$$

For $\ell = 0, 1$ or 2, let $\Phi_{\mathcal{S}}^+(\ell) = \Phi^+(\ell) \backslash \Phi_{\mathcal{K}}^+$ be the set of positive noncompact roots. If $E \subset \Phi$ let $\langle E \rangle = \sum_{\alpha \in E} \alpha$, with the special functionals

$$\rho_{\mathcal{K}} = \frac{1}{2}\langle \Phi_{\mathcal{K}}^+ \rangle \quad \text{and} \quad \rho_{\mathcal{P}}(\ell) = \frac{1}{2}\langle \Phi_{\mathcal{P}}^+(\ell) \rangle.$$

Now fix a dominant integral weight λ for K with respect to Φ_K^+, with corresponding representation $(\tau_\lambda, \mathcal{V}_\lambda)$ and consider the decomposition of $(\tau_\lambda \otimes \mathrm{Ad}, \mathcal{V}_\lambda \otimes \mathcal{P}_{\mathbb{C}})$ into $K-$invariant subspaces. The only possible highest weights which can occur are of the form $\lambda + \alpha$ where α is a noncompact root and the multiplicity $m(\lambda, \alpha)$ of such a space is at most one. For $0 \leq \ell \leq 2$ let P_ℓ denote the projection

$$P_\ell : (\tau_\lambda \otimes \mathrm{Ad}, \mathcal{V}_\lambda \otimes \mathcal{P}_{\mathbb{C}}) \to \sum_{\alpha \in \Phi_{\mathcal{P}}^+(\ell)} m(\lambda, -\alpha)\mathcal{V}_{\lambda-\alpha}.$$

The space of smooth $\tau_\lambda-$covariant functions on G is denoted by $C^\infty(G, \tau_\lambda)$ and there is the canonical first order invariant differential operator

$$: C^\infty(G, \tau_\lambda) \to C^\infty(G, \tau_\lambda \otimes \mathrm{Ad}),$$

as described in [**KW**]. The **Schmid operator** is

$$\mathcal{D}_\ell = P_\ell \circ \quad .$$

As in [**Me**],part 2.5, we will say that such a pair (λ, ℓ) satisfy condition $(\sharp)$ provided:

$$(i) \quad \forall E \subset \Phi_{\mathcal{P}}^+(\ell),\ (\lambda + \rho_K - \langle E \rangle | \alpha_{12}) \geq 0$$

and

$$(ii) \quad \lambda - 2\rho_{\mathcal{P}}(\ell) \quad \text{is } \Phi_K^+ - \text{dominant.}$$

When these conditions are satisfied, the operator $\mathcal{D}_\ell$ is elliptic and one can estimate the multiplicities of $K-$types in $(\ker(\mathcal{D}_\ell))_K$, the $K-$finite part of the kernel of $\mathcal{D}_\ell$.

Following page 154 in [**HP**], if $\mu \in \widehat{T}$ and $\ell = 0, 1$, or 2, let $Q_j(\ell, \mu)$ denote the number of distinct ways in which μ can be written as a sum of exactly j elements of $\Phi_{\mathcal{P}}^+(\ell)$, with $Q_0(\ell, 0) = 1$, and $Q(\ell, \mu) = \sum_{j=0}^\infty Q_j(\ell, \mu)$. The Blattner number for λ and μ in $\widehat{T}$ is

$$b(\ell, \lambda, \mu) = \sum_{w \in W} sign(w) Q\left(\ell, w(\mu + \rho_K) - (\lambda + 2\rho_{\mathcal{P}}(\ell) + \rho_K)\right).$$

Theorem 1 on page 156 of [**HP**] provides the following upper bound on the multiplicity of K-types in the space $(\ker(\mathcal{D}_\ell))_K$, the K-finite elements of the kernel of the Schmid operator.

THEOREM (HOTTA AND PARTHASARATHY). *Suppose (λ, ℓ) satisfy condition $(\sharp)$. Then, for each dominant integral weight μ, the multiplicity of $(\tau_\mu, \mathcal{V}_\mu)$ in $(\ker(\mathcal{D}_\ell))_K$ is less than or equal to the Blattner number $b(\ell, \lambda, \mu)$. In particular, $(\tau_\lambda, \mathcal{V}_\lambda)$ occurs at most* **once** *in $(\ker(\mathcal{D}_\ell))_K$.*

In seeking irreducible $(\mathcal{G}, K)-$modules with λ as a minimal $K-$type, we should look at the invariant subspace of $(\ker(\mathcal{D}_\ell))_K$ generated by the $\lambda-$isotypic part. To show that it occurs at all uses the Cauchy-Szegö map methods, in which case this space will be a subquotient of a principal series representation. This was done in [**KW**] for discrete series representations.

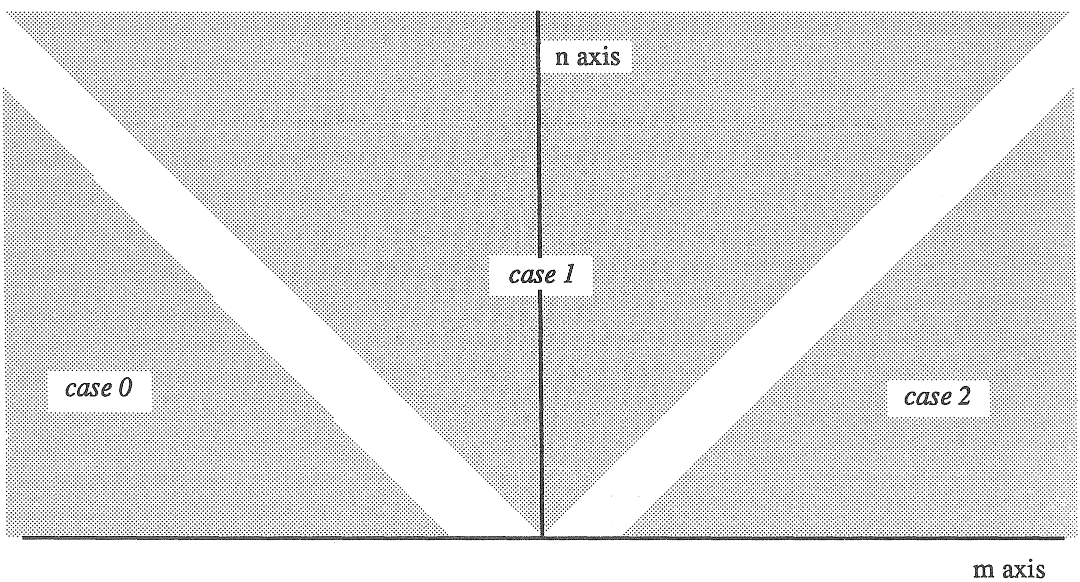
THE DISCRETE SERIES.

Among the highest weights for K there are those which are minimal $K-$types for the discrete series representations for G. Recall that an irreducible unitary representation of G is said to be in the discrete series if it occurs as an invariant subspace of $L^2(G)$. These were classified by Harish-Chandra. For a dominant integral weight λ of K and a system of positive roots $\Phi^+(\ell)$, there is the **Harish-Chandra parameter**

$$\lambda(\ell) \;=\; \lambda + \rho_{\mathcal{K}} - \rho_{\mathcal{P}}(\ell).$$

THEOREM (HARISH-CHANDRA'S CRITERION). *A dominant integral weight λ is a minimal $K-$type of a discrete series representation of G if and only if there is a system of positive roots $\Phi^+(\ell)$ such that $\lambda(\ell)$ is dominant and regular. In particular, if we parameterize the dominant integral weights of K by pairs of integers (m, n) as above, then these criteria are satisfied for:*

(1) $m < -n - 4$ *when* $\ell = 0$;
(2) $-n < m < n$ *when* $\ell = 1$;
(3) $n + 4 < m$ *when* $\ell = 2$.



The regions in the (m, n) plane for which Harish-Chandra's criterion applies.

Next, we examine some concrete realizations of the irreducible representations of K.

Spherical Harmonics.

Fix p and q, nonnegative integers. Denote a variable in $\mathbf{C}^2$ by $\xi = (\xi_1, \xi_2)$ and let $\mathcal{H}_{p,q}$ be the space of polynomials in ξ and ξ^* which are harmonic, homogeneous of degree p in ξ and homogeneous of degree q in ξ^*. For each integer N there is an action of K on $\mathcal{H}_{p,q}$ given by

$$\tau_{p,q,N}\begin{pmatrix} u & 0 \\ 0 & \det(u^{-1}) \end{pmatrix} f(\xi,\xi^*) \;=\; (\det(u^{-1}))^N f(\xi u, u^{-1}\xi^*),$$

where $u \in \mathrm{U}(2)$. When restricted to K_s, and using the ordering of compact roots described above, we can take $\psi_{p,q}(\xi,\xi^*) = \xi_1^p \overline{\xi_2}^q$ as a highest weight vector in $\mathcal{H}_{p,q}$. Also notice that

$$\tau_{p,q,N}\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix} \psi_{p,q} \;=\; e^{(p-q-2N)i\theta}\psi_{p,q}.$$

In the parameterization of $\widehat{K}$ based on equations (1) and (2), we take

$$\begin{aligned} m &= p - q - 2N \\ \text{and} \quad n &= p + q. \end{aligned} \tag{3}$$

We could also describe this in terms of functionals on $\mathbf{C}_0^3$, so that the highest weight of $(\tau_{p,q,N}, \mathcal{H}_{p,q})$ is

$$\lambda_{p,q,N} = \left(\frac{m+3n}{6}, \frac{m-3n}{6}, \frac{-m}{3}\right) = \left(\frac{2p+q-N}{3}, \frac{-p-2q-N}{3}, \frac{2N+q-p}{3}\right).$$

We then calculate the inner products which are used to test condition ($\sharp$).

Lemma. *For p,q,ℓ, and N as above, $(\lambda_{p,q,N}, \ell)$ satisfies condition ($\sharp$) if and only if $p+q \geq \ell$. In particular, if $n = p+q \geq \ell$, then $(\tau_{p,q}, \mathcal{H}_{p,q})$ is a representation of K for which the theorem of Hotta and Parthasarathy is valid.*

Principal Series.

Let $\mathcal{A} = \mathbf{R}(e_{13} + e_{31})$ and let M denote its centralizer in K, so that

$$M \;=\; \left\{ \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \;:\; 0 \leq \theta \leq 2\pi \right\}.$$

Furthermore, let $A = \{\mathbf{a}(t) = \exp(t(e_{13}+e_{31})) : t \in \mathbf{R}\}$. There is associated to this an Iwasawa decomposition $G = ANK$. We will identify $\mathcal{A}_{\mathbf{C}}^*$ with the set of complex numbers by identifying a functional ν with the complex number $\nu(e_{13}+e_{31})$. In particular, there is the special element $\rho_{\mathcal{A}}(e_{13}+e_{31}) = 2$. For each irreducible unitary representation (σ, H_σ) of M and $\nu \in \mathbf{C}$, there is the principal series representation of G acting by right translation on the space

$$\mathbf{I}_{\sigma,\nu} \;=\; \left\{ f : G \to H_\sigma \;:\; f \in C^\infty \text{ and } f(m\mathbf{a}(t)nx) = e^{(\nu+2)t}\sigma(m)f(x) \right\}.$$

Suppose (σ, H_σ) is a subrepresentation of $(\tau_\lambda|_M, \mathcal{V}_\lambda)$ and that R is an M-equivariant isometry mapping H_σ into $\mathcal{V}_\lambda$. The data $(\sigma, \nu, \lambda, R)$ determines the Cauchy-Szegö map

$$S : \mathbf{I}_{\sigma,\nu} \longrightarrow C^\infty(G, \tau_\lambda)$$

by the formula

$$Sf(x) \; = \; \int_K \tau_\lambda(k)^{-1} R f(kx) dk, \quad \forall f \in \mathbf{I}_{\sigma,\nu}.$$

We associate to this integral operator the function on $G \times K$ given by

$$\mathbf{S}(g,k) \; = \; e^{(2-\nu)(\mathbf{H}(kg^{-1}))} \tau_\lambda(\mathbf{K}(kg^{-1}))^{-1} \circ R,$$

where $\mathbf{H}$ and $\mathbf{K}$ are the Iwasawa projections of G to $\mathcal{A}$ and K, respectively. The image of the Cauchy-Szegö map includes the K-type $(\tau_\lambda, \mathcal{V}_\lambda)$. To see that its image could lie in the kernel of a Schmid operator, it suffices to show that there is a nonzero vector $\varphi \in H_\sigma$ such that

$$P_\ell(\; \mathbf{S}(1,g)\varphi)_{g=1} \; = \; 0.$$

LEMMA. *Maintaining the notation from above,*

$$\begin{aligned}(\; \mathbf{S}(1,g)\varphi)_{g=1} \; = \; & \frac{1}{2} R\varphi \otimes (e_{13} + e_{31}) + \tau_\lambda([e_{13} + e_{31}, e_{23}]) R\varphi \otimes e_{32} + \\ & + \tau_\lambda([e_{13} + e_{31}, e_{32}]) R\varphi \otimes e_{23} - \frac{1}{2}(\lambda|\alpha_{13}) R\varphi \otimes e_{31} + \frac{1}{2}(\lambda|\alpha_{13}) R\varphi \otimes e_{13}.\end{aligned}$$

For the details of this calculation in the general setting, see [**KW**], section 7. Note that $[e_{13} + e_{31}, e_{23}] = -e_{21}$ and $[e_{13} + e_{31}, e_{32}] = e_{12}$. The representation $(\mathrm{Ad}, \mathcal{P}_{\mathbf{C}})$ is the direct sum

$$(\tau_{1,0,-1}, \mathcal{H}_{1,0}) \oplus (\tau_{0,1,1}, \mathcal{H}_{0,1}),$$

with e_{13} corresponding to ξ_1, e_{23} to ξ_2, e_{31} to $\overline{\xi}_1$, and e_{32} to $\overline{\xi}_2$. The tensor product

$$(\tau_{p,q,N}, \mathcal{H}_{p,q}) \otimes (\mathrm{Ad}, \mathcal{P}_{\mathbf{C}})$$

is the direct sum of four spaces:

$$\begin{aligned}&(\tau_{p+1,q,N-1}, \mathcal{H}_{p+1,q}) \oplus (\tau_{p,q-1,N-1}, \mathcal{H}_{p,q-1}) \\ &\oplus (\tau_{p,q+1,N+1}, \mathcal{H}_{p,q+1}) \oplus (\tau_{p-1,q,N+1}, \mathcal{H}_{p-1,q})\end{aligned}$$

The corresponding highest weights are:

$$\begin{aligned}&\lambda_{p,q,N} + \alpha_{13}, \quad \lambda_{p,q,N} + \alpha_{23}, \\ &\lambda_{p,q,N} + \alpha_{32}, \quad \lambda_{p,q,N} + \alpha_{31}.\end{aligned}$$

We can realize the tensor products in a K-equivariant fashion by multiplication of polynomials, mapping into the direct sum of the spaces of polynomials which are bihomogeneous of degree $(p+1, q)$ and those which are bihomogeneous of degree $(p, q+1)$. The image is in

$$\mathcal{H}_{p+1,q} \oplus |\xi|^2 \mathcal{H}_{p,q-1} \oplus \mathcal{H}_{p,q+1} \oplus |\xi|^2 \mathcal{H}_{p-1,q}.$$

Knapp and Wallach [**KW**] deal with the case when φ is the highest weight vector $\psi_{p,q}$, and R is the M-equivariant map which intertwines the action of M on $\mathbf{C}$,via the character $\sigma : e^{i\theta} \mapsto e^{i(p-2q+N)\theta}$, and M acting on $\psi_{p,q}$ via $\tau_{p,q,N}|_M$.

THEOREM (KNAPP AND WALLACH). *For R, σ, p, q, and N as above, the Cauchy-Szegö map with this data takes $\mathbf{I}_{\sigma,\nu}$ into ker$(\mathcal{D}_1)$ when*

$$\nu = N - p = -(n+m)/2.$$

When $-n < m < n$, $(\ker(\mathcal{D}_1))_K$ is the $(\mathcal{G}, K)-$module of $K-$finite vectors in the discrete series representation with minimal $K-$type $\lambda_{p,q,N}$.

Remark. This parameter ν comes from the formula

$$\nu(e_{13} + e_{31}) = \frac{-2(\lambda_{p,q,N}(1)|\alpha_{13})}{(\alpha_{13}|\alpha_{13})},$$

derived from Theorem 6.1 and Lemma 8.5 in [**KW**]. Here $\lambda_{p,q,N}(1)$ is the Harish-Chandra parameter with respect to the system of positive roots $\Phi^+(1)$ and α_{13} is a simple noncompact root. There is another simple noncompact root in $\Phi^+(1)$, namely α_{32}, and this also leads to a quotient map from a principal series representation into the kernel of $\mathcal{D}_1$, again using the highest weight vector $\psi_{p,q}$ to set up a Cauchy-Szegö map. If we use $e_{23} + e_{32}$ to build an Iwasawa decomposition then the centralizer of this in K consists of the matrices

$$\begin{pmatrix} e^{-2i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}$$

and when acting on $\psi_{p,q}$, this yields the character $e^{i\theta} \mapsto e^{i(N-2p-q)\theta}$. This group is conjugate to M in K and we let σ denote the corresponding character on M.

THEOREM (KNAPP AND WALLACH). *The Cauchy-Szegö map based on this data maps $\mathbf{I}_{\sigma,\nu}$ into ker$(\mathcal{D}_1)$ when*

$$\nu = -(q+N) = (m-n)/2.$$

This formula is valid outside of the collection of dominant integral weights which satisfy Harish-Chandra's criterion. If we consider the pairs (m,n) with $m - n = 2$, then the value for ν is 1 and the character of M is $e^{i\theta} \mapsto e^{-i(2n+1)\theta}$. Thus, the ends of complementary series representations described in [**CK**] appear as subspaces of the kernel of a Schmid operator. When $\nu > 0$, $(\mathbf{I}_{\sigma,\nu})_K$ has a unique irreducible quotient and the Szegö map exhibits this as being the invariant $(\mathcal{G}, K)-$module of $(\ker(\mathcal{D}_1))_K$ generated by the $\lambda_{p,q,N}-$isotypic part.

ANOTHER POSSIBLE QUOTIENT MAP INTO KER$(\mathcal{D}_1)$

In addition to the highest weight vector $\psi_{p,q}$, we can consider the special vector

$$\varphi_{p,q}(\xi, \xi^*) = \xi_1^p \overline{\xi}_1^q F\left(-p, -q; 1; \frac{-|\xi_2|^2}{|\xi_1|^2}\right).$$

Under the action of M, we see that

$$\tau_{p,q,N}\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}\varphi_{p,q} = e^{i(p-q+N)\theta}\varphi_{p,q}$$

and so this generates a one-dimensional subrepresentation of M in $\mathcal{H}_{p,q}$.

Return to the Lemma, with the additional knowledge that:

$$\begin{aligned}\tau_{p,q,N}(e_{21})f &= \xi_2\partial_1 f - \overline{\xi}_1\overline{\partial}_2 f \\ \tau_{p,q,N}(e_{12})f &= \xi_1\partial_2 f - \overline{\xi}_2\overline{\partial}_1 f.\end{aligned}$$

Hence, in $\mathcal{H}_{p+1,q} + |\xi|^2\mathcal{H}_{p,q-1}$, the terms are

$$(I) = \frac{1}{2}(2-\nu)\varphi_{p,q}\xi_1 + (\xi_2\xi_1\partial_2\varphi_{p,q}) - |\xi_2|^2\overline{\partial}_1\varphi_{p,q} + \frac{1}{2}(p-q-N)\xi_1\varphi_{p,q}$$

and in $\mathcal{H}_{p,q+1} + |\xi|^2\mathcal{H}_{p-1,q}$ they are

$$(II) = \frac{1}{2}(2-\nu)\varphi_{p,q}\overline{\xi}_1 + (\overline{\xi}_2\overline{\xi}_1\overline{\partial}_2\varphi_{p,q}) - |\xi_2|^2\partial_1\varphi_{p,q} - \frac{1}{2}(p-q-N)\overline{\xi}_1\varphi_{p,q}.$$

We can rewrite these formulae as

$$(I) = \frac{(2-\nu+p-N+q)(1+p)}{2(1+p+q)}\varphi_{p+1,q} + \frac{q}{2(1+p+q)}(-\nu-p-q-N)|\xi|^2\varphi_{p,q-1}$$

and

$$(II) = \frac{(2-\nu+p+q+N)(1+q)}{2(1+p+q)}\varphi_{p,q+1} + \frac{p}{2(1+p+q)}(-\nu-p-q+N)|\xi|^2\varphi_{p-1,q}.$$

Since we are concentrating on $\mathcal{D}_1$ we are requiring that the terms in

$$|\xi|^2\mathcal{H}_{p-1,q} \oplus |\xi|^2\mathcal{H}_{p,q-1}$$

are zero, which means either solving the simultaneous equations:

$$\begin{aligned}\nu &= -p-q+N \\ \nu &= -p-q-N,\end{aligned} \tag{4}$$

or setting $q=0$ and $\nu = N-p$,or $p=0$ and $\nu=-N-q$. The solution of (4) is when $N=0$ and $\nu=-(p+q)$. The other two cases are the parameters given by Knapp and Wallach. In addition, recall that we must have $0 \le p \le n$ and $0 \le n-p = q \le n$.

THEOREM. *Suppose $N=0$ and that R is the $M-$equivariant map which takes $\mathbf{C}$ into $\mathbf{C}\varphi_{(m+n)/2,(n-m)/2} \subset \mathcal{H}_{(m+n)/2,(n-m)/2}$ Let σ denote the one-dimensional representation of M coming from the action of M on $\varphi_{(m+n)/2,(n-m)/2}$. If we identify M with the unit circle, then σ is the character $e^{i\theta} \mapsto e^{im\theta}$. The Cauchy-Szegö map with data $(\sigma, \nu, \lambda_{(m+n)/2,(n-m)/2,0}, R)$ takes $\mathbf{I}_{\sigma,\nu}$ into the kernel of $\mathcal{D}_1$ when*

$$-n \le m \le n$$

and $\nu = -(p+q) = -n$.

For the range $-n < m < n$ this theorem completes the list of realizations of the discrete series of $\mathrm{SU}(2,1)$ as quotients of principal series, a matter raised on page 164 of [**KW**]. The cases when $m=0$ are the quotients of spherical principal series which are in the discrete series, as described in [**JW**]. That is, the quotients of $\mathbf{I}_{1,-n}$, where n is a positive integer. The cases when $m=-n$ are the limits of discrete series representations in the quotients of $\mathbf{I}_{\sigma,0}$, where σ corresponds to $-n$. In a recent preprint R. C. Fabec has described the unitary structure on subquotients of the principal series of $\mathrm{SU}(2,1)$ in terms of derivatives of intertwining operators, see section 5 in [**Fb**].

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