ERGODIC MEASURES FOR THE ACTIONS OF DENSE SUBGROUPS

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1. INTRODUCTION

A construction for singular measures quasi-invariant and ergodic for the irrational rotation T_α by α on the circle was first given by Keane in [5]. It involved using the continued fraction expansion of α to identify T_α with the odometer action on a subset of an infinite product space. Subsequently Katznelson and Weiss [4] obtained a general method for constructing uncountably many mutually singular quasi-invariant measures ergodic with respect to a general homeomorphism T on a compact metric space, provided that T has a recurrent point.

In a recent paper [6], we have shown that it is possible to use a Riesz product technique to obtain measures of the kind that Keane produced; though, being Riesz products these measures are more susceptible to control of their Fourier-Stieltjes transform. In particular such measures can be chosen to have their Fourier-Stieltjes transform vanishing at infinity. This property was required in connection with the study of non-monomial representations of the discrete Heisenberg group (cf. [1]).

The idea of using Riesz products to obtain ergodicity was first introduced by Gavin Brown in [2]. He was, however, only able to deal with subgroups in which every element has finite order. Nevertheless it was a modification of his technique which was used in [6] to obtain measures ergodic for the irrational rotation. In this paper we generalise the technique to apply to any countable dense subgroup of a compact metric abelian group.

It turns out that in many circumstances these measures possess a uniqueness property. In the case of T_α, the measures μ constructed in [6] are the unique quasi-invariant measures with given Radon-Nikodym derivative $\frac{d(\mu\circ T_\alpha)}{d\mu}$. Here we show that the same property can be obtained in the more general setting.

We begin by exploring the basic ideas of the proof in somewhat more generality than is absolutely necessary in an attempt to clarify the underlying mechanism involved.

2. STABILITY

Fix a compact metric abelian group G and let B be a countable dense subgroup of G. We use Γ to denote the Pontryagin dual of G. Fix also an increasing sequence (A_n) of finite subsets of Γ whose union is the whole of Γ and such that $A_n^{-1} = A_n$. For a measure μ on G we write $S_n(\mu)$ for the nth partial sum of its Fourier Series; explicitly,

$$(1)\qquad S_n(\mu)(t) = \sum_{\alpha\in A_n} \mu^\wedge(\alpha)\alpha(t).$$

For a probability measure μ on G we introduce the concept of a <u>B-stabilizing sequence</u>; this is a sequence (λ_n) of probability measures on B satisfying

$$(2)\qquad \sum_{\alpha\in A_n^2\setminus\{1\}} |\lambda_n^\wedge(\alpha)| \to 0;$$

$$(3)\qquad \|S_n(\mu).(\lambda_n * \mu) - \mu\| \to 0$$

A measure μ for which there exists such a sequence is said to be <u>B-stable</u>. The next lemma, the key to the proof of ergodicity, is based on a result of Brown ([2], Theorem 5).

LEMMA 1. Every B-stable measure μ is B-ergodic.

Proof. Fix a Borel set E invariant for the action of B. We need to show that $\mu(E)$ is 0 or 1. Write μ_E for the "restriction" of μ to E. If (λ_n) is a B-stabilizing sequence for μ then, for $\gamma\in\Gamma$,

$$\int_E \overline{\gamma}(t).S_n(\mu)(t).d\lambda_n*\mu(t)$$

$$(4) \qquad = \sum_{\alpha\in A_n} \mu^\wedge(\alpha) \int_E (\alpha\overline{\gamma})(t)\ d(\lambda_N*\mu)(t)$$

$$= \sum_{\alpha\in A_n} \mu^\wedge(\alpha)\lambda_n^\wedge(\overline{\alpha}\gamma)\mu_E{}^\wedge(\overline{\alpha}\gamma)$$

by the B-invariance of E. This is equal to

$$\mu^\wedge(\gamma)\mu(E) + \sum_{\alpha\in A_n\setminus\{\gamma\}} \mu^\wedge(\alpha)\lambda_n^-(\overline{\alpha}\gamma)\mu_E{}^\wedge(\overline{\alpha}\gamma)$$

and the second term tends to 0 as $n\to\infty$ by (2) since $\gamma.A_n \subset A_n^2$ eventually. On the other hand, by (3),

$$(5) \qquad \int_E \overline{\gamma}(t).S_n(\mu)(t)d\lambda_n*\mu(t) \to \int_E \overline{\gamma}(t)d\mu(t) = \mu_E{}^\wedge(\gamma)$$

Combining these, we see that $\mu_E{}^\wedge(\gamma) = \mu^\wedge(\gamma)\mu(E)$ for all $\gamma\in\Gamma$ and hence that $\mu_E = \mu(E).\mu$ which gives the required conclusion.

There is already a sense in which B-stable measures exhibit a high degree of uniqueness. The following simple observation is an important step in the ultimate proof of unique ergodicity for appropriate measures.

LEMMA 2. If (λ_n) is a B-stabilizing sequence for μ and ν is a probability measure such that

$$S_n(\mu).(\lambda_n*\nu) \to \nu$$

in the weak* topology, then $\mu = \nu$.

Proof. It is enough to show that

$$S_n(\mu).(\lambda_n * \nu)^\wedge(\gamma) \to \mu^\wedge(\gamma) \qquad (\gamma \in \Gamma).$$

This follows at once from the following equation:

$$S_n(\mu).(\lambda_n * \nu)^\wedge(\gamma) = \sum_{\alpha \in A_n} \mu^\wedge(\alpha)(\lambda_n * \nu)^\wedge(\bar{\alpha}\gamma)$$

$$= \mu^\wedge(\gamma) + \sum_{\alpha \in A_n \setminus \{\gamma\}} \lambda_n^\wedge(\bar{\alpha}\gamma)\mu^\wedge(\gamma)\nu^\wedge(\bar{\alpha}\gamma)$$

and (2), as in Lemma 1.

However B-stability does not appear to be sufficient to guarantee uniqueness of μ in the sense stated in the Introduction; that is, that μ is the unique probability measure satisfying $\frac{d(\delta(b)*\mu)}{d\mu} = Q(b,t)$ where $Q: B \times G \to \mathbb{R}^+$ is some fixed cocycle:
$Q(b'b,t) = Q(b',bt)Q(b',t)$, μ a.e. for all $b, b' \in B$.

What we shall assume is that there is some continuous function $P: B \times G \to \mathbb{R}^+$ (where B has the discrete topology) such that

$$(6) \qquad S_n(\mu)(t)P(b,t) - S_n(\mu)(t.b^{-1}) \to 0$$

uniformly in $t \in G$.

First we show that P is just the Radon-Nikodym derivative.

LEMMA 3. $P(b,t) = \frac{d(\delta(b)*\mu)}{d\mu}$ (μ a.e.) for all $b \in B$.

Proof. Fix $\gamma \in \Gamma$. Then, on the one hand,

$$(7) \qquad \int \bar{\gamma}(t)P(b,t)S_n(\mu)(t)d(\lambda_n * \mu)(t) \to \int \bar{\gamma}(t)P(b,t)d\mu(t)$$

by (3). On the other,

$$\int \bar{\gamma}(t) S_n(\mu)(tb^{-1}) d(\lambda_n * \mu)(t)$$

(8) $$= \sum_{\alpha \in A_n} \mu^\wedge(\alpha) \int \bar{\gamma}(t)\alpha(t)\bar{\alpha}(b) d(\lambda_n * \mu)(t)$$

$$= \mu^\wedge(\gamma)\bar{\gamma}(b) + \sum_{\alpha \in A_n \setminus \{\gamma\}} \bar{\alpha}(b)\mu^\wedge(\alpha)\lambda_n^\wedge(\gamma\bar{\alpha})\mu^\wedge(\alpha\bar{\gamma})$$

and, as usual, the second term tends to 0 by (2). Now combining (6), (7) and (8) we produce

$$\int \bar{\gamma}(t) P(b,t) d\mu(t) = (\delta(b) * \mu)^\wedge(\gamma)$$

for all $\gamma \in \Gamma$, $b \in B$, whence the result.

Observe that we have, in proving the lemma, obtained a slightly stronger conclusion.

COROLLARY. If μ is a B-stable measure satisfying (6) then μ is quasi-invariant.

Now we turn the problem of unique ergodicity. The following lemma is what we require.

LEMMA 4. Let μ be a B-stable measure satisfying (6) and suppose that ν is a probability measure quasi-invariant for the action of B and satisfying $\dfrac{d(\delta(b) * \nu)}{d\nu} = P(b,t)$ (ν a.e.). Then $\nu = \mu$.

Proof. In view of Lemma 2, it is enough to show that

$$(S_n(\mu).(\lambda_n * \nu))^\wedge(\gamma) \to \nu^\wedge(\gamma) \qquad (\gamma \in \Gamma).$$

Now

$$(S_n(\mu).(\lambda_n * \nu))^\wedge(\gamma)$$

(9) $$= \sum_{b \in B} \lambda_n(b)(S_n(\mu).(\delta(b) * \nu))^\wedge(\gamma)$$

$$= \sum_{b \in B} \lambda_n(b) \int S_n(\mu)(t)\bar{\gamma}(t) P(b,t) d\nu(t)$$

By (6), noting that $\sum_{b\in B} \lambda_n(b) < \infty$, we observe that (9) has the same limit as $n\to\infty$, as

$$\sum_{b\in B} \lambda_n(b) \int S_n(\mu)(tb^{-1})\overline{\gamma}(t)d\nu(t)$$

$$= \sum_{b\in B} \sum_{\alpha\in A_n} \lambda_n(b)\mu^\wedge(\alpha)\overline{\alpha}(b) \int \alpha(t)\overline{\gamma}(t)d\nu(t)$$

$$= \sum_{\alpha\in A_n} \lambda_n^\wedge(\alpha)\mu^\wedge(\alpha)\nu^\wedge(\overline{\alpha}\gamma)$$

which tends to $\nu^\wedge(\gamma)$ by (2). This proves the result.

3. RIESZ PRODUCTS

A sequence (χ_n) of elements of Γ is <u>dissociate</u> if each element γ of Γ can be expressed in <u>at most one</u> way as a product

$$\gamma = \chi_{i_1}^{\varepsilon_1} \chi_{i_2}^{\varepsilon_2} \cdots \chi_{i_k}^{\varepsilon_k}$$

where $\varepsilon_i = \pm 1$. (If χ_k has order 2, then we formally identify χ_k^1 with χ_k^{-1}.) Such sequences were introduced by Hewitt and Zuckermann in their paper [3] generalising the concept of Riesz product to arbitrary compact abelian groups. Choose a sequence (a_n) of real numbers in the interval [0,1] and define

$$q_n = 1 + \frac{1}{2}a_n(\chi_n + \overline{\chi}_n).$$

Let $p_N = \prod_{n<N} q_n$ and observe that if m denotes the Haar measure of G, then $p_N.m$ is a probability measure (call it μ_N). By calculating the Fourier-Stieltjes coefficients of μ_N, we see quickly that the sequence (μ_N) converges weak* to a probability measure μ - the <u>Riesz product generated by</u> (χ_n) <u>with coefficients</u> (a_n). These measures have had wide application in harmonic analysis and, in particular, have been studied by Hewitt and Zuckermann (loc.cit.) and Gavin Brown [2].

For our purposes, it will be enough to note a few simple properties. First observe that the Fourier-Stieltjes transform of μ is given by

$$\mu^{\wedge}(\gamma) = \prod_{i=1}^{k} a_i^{|\varepsilon_i|} \quad \text{if } \gamma = \chi_1^{\varepsilon_1}\chi_2^{\varepsilon_2} \cdots \chi_k^{\varepsilon_k} \; (\varepsilon_i = 0,\pm 1) \tag{10}$$

so that $\mu^{\wedge}$ vanishes at infinity if and only if $a_n \to 0$ as $n\to\infty$. Furthermore μ is absolutely continuous if and only if $\sum_{n=1}^{\infty} a_n^2 < \infty$ ([2], Proposition 2). There are corresponding mutual singularity results, which would allow us to pick uncountably many mutually singular measures which are quasi-invariant and ergodic for a dense subgroup. We refrain from giving the details, save to mention that only the sequence (χ_n) plays a role in the construction of these measures; the coefficients may be chosen freely from an interval $[0,\rho]$ where $\rho < 1$.

4. THE CONSTRUCTION

Fix now the group G with dual Γ, the countable dense subgroup B of G, and a real number $\rho \in (0,1)$. Choose also a sequence (ε_n) of real numbers in [0,1] and tending to 0. We shall construct simultaneously by induction four sequences (λ_n), (χ_n), (D_n), (A_n), where λ_n is a probability measure on B, $\chi_n \in \Gamma$, D_n is a subset of B and $A_n \subset \Gamma$. To describe the properties which we shall impose on these objects, we need to introduce the notation

$$\Omega_n = \{\chi_1^{\varepsilon_1}\chi_2^{\varepsilon_2} \cdots \chi_n^{\varepsilon_n} \; ; \; \varepsilon_i = 0,\pm 1\}.$$

Let us write (β_n) and (b_n) for enumerations of the countable groups Γ and P with $\beta_1 = 1$. We shall write $|C|$ for the cardinality of the set C. The sequences will then be defined inductively to satisfy:

(a) $\Omega_n \subset A_n$, $\chi_{n+1} \notin \Omega_n A_n$, $A_n = \overline{A}_n$;

(b) $\beta_n \in A_n$, $b_n \in D_n$; $A_n \subset A_{n+1}$, $D_n \subset D_{n+1}$;

(c) $|\lambda_n^\wedge(\zeta)| < \dfrac{1}{|A_n|}\dfrac{\varepsilon_n}{2^{n+1}} \quad (\zeta \in A_n^2 \setminus \{1\})$;

(d) $|\chi_n(b)-1| < \dfrac{(1-\rho)}{2|D_n|} \cdot \dfrac{1}{6^{n+1}}\, \varepsilon_r \quad (b \in D_r,\ r < n)$;

(e) $D_n \supset \operatorname{supp}\lambda_n$;

for $n = 1,2,3,\ldots$.

Before giving the proof of existence of sequences with these properties, we show that (a), (b), (c), (d), (e) together lead to a B-stable measure satisfying (6). Observe first that by (a) $\chi_{n+1} \notin \Omega_n^2$, for all n, so that the sequence (χ_n) is dissociate. Now, for any sequence of real numbers (a_n) in the interval $(0,\rho)$, we may form the Riesz product generated by (χ_n) with coefficients (a_n). It follows also from (a) that

$$S_n(\mu) = \sum_{\alpha \in A_n} \mu^\wedge(\alpha)\alpha = P_n$$

and, of course, from (b) that $\cup A_n = \Gamma$. Observe too, that (c) implies that

$$\sum_{\alpha \in A_n^2 \setminus \{1\}} |\lambda_n^\wedge(\alpha)| < \varepsilon_n \to 0$$

to that (2) is satisfied. To achieve the B-stability of μ, therefore, it is enough to show that if $\nu_n = P_n.(\lambda_n * \mu) - \mu$ then $\|\nu_n\| \to 0$. We first establish the property (6), from this we can deduce the remaining conclusions. Define

(11) $$V(b,t) = \prod_{k=n+1}^{\infty} q_k(tb^{-1})q_k(t)^{-1} .$$

LEMMA 6. For $b \in D_n$, the infinite product (11) converges uniformly and

$$|V_n(b,t) - 1| < \frac{1}{2|D_n|} \varepsilon_n \cdot \frac{1}{6^n}.$$

Proof. First observe that, by (d), for $b \in D_n$

$$|q_k(tb^{-1})q_k(t)^{-1} - 1| \leq \frac{1}{1-\rho} |q_k(tb^{-1}) - q_k(t)|$$

$$\leq \frac{2}{1-\rho} |\chi_k(b) - 1| < \frac{1}{6|D_n|} \varepsilon_n \cdot \frac{1}{6^k}$$

provided $k \geq n+1$. Now the result follows from the inequality

$$\left|\prod_{k=1}^{\infty} (1+c_k) - 1\right| \leq \exp\left(\sum_{k=1}^{\infty} |c_R|\right) - 1.$$

Lemma 6 implies, in particular, that

$$|P_n(tb^{-1}) - V_o(b,t)p_n(t)|$$

$$= |p_n(tb^{-1})| \ |1 - V_n(b,t)| \leq \frac{1}{2|D_n|} \frac{\varepsilon_n}{3^n}$$

so that (6) follows from the remark that each $b \in B$ is eventually in some B_n.

The final step in the proof of B-stability is the next result.

LEMMA 7. $\|\nu_n\| = \|S_n(\mu).(\lambda_n * \mu) - \mu\| < \varepsilon_n$ for all n.

Proof. Write $\mu_n = S_n(\mu)^{-1}.\mu$ and $\tau_n = \lambda_n * \mu - \mu_n$. Then

$$\tau_n = \left(\sum_{\alpha \in \Omega_n} \mu^\wedge(\alpha)\lambda_n * (\alpha.\mu_n)\right) - \mu_n$$

$$= \left(\sum_{\alpha \in \Omega_n} \lambda^\wedge(\alpha) \sum_{b \in D_n} \lambda_n(b)\overline{\alpha}(b)\alpha.(\delta(b) * \mu_n)\right) - \mu_n,$$

so that

$$\|\tau_n\| \le \left\| \left(\sum_{\alpha\in\Omega_n} \mu^\wedge(\alpha) \sum_{b\in D_n} \lambda_n(b)\bar{\alpha}(b)\alpha\cdot\mu_n \right) - \mu_n \right\|$$

$$+ \left\| \sum_{\alpha\in\Omega_n} \mu^\wedge(\alpha) \sum_{b\in D_n} \lambda_n(b)\bar{\alpha}(b)\alpha\cdot(\delta(b)*\mu_n - \mu_n) \right\|$$

(12)

$$\le \left\| \sum_{\alpha\in\Omega_n} \mu^\wedge(\alpha)\lambda_n^\wedge(\alpha)(\alpha\cdot\mu_n) - \mu_n \right\|$$

$$+ 3^n |D_n| \sup_{b\in D_n} \|\delta(b)*\mu_n - \mu_n\|$$

$$\le \left\| \sum_{\substack{\alpha\in\Omega_n \\ \alpha\ne 1}} \mu^\wedge(\alpha)\lambda_n^\wedge(\alpha)\mu_n \right\| + 3^n |D_n| \sup_{b\in D_n} \|\delta(b)*\mu_n - \mu_n\|$$

The first term is less than

(13)
$$\sum_{\alpha\in A_n\setminus\{1\}} |\lambda_n^\wedge(\alpha)| < \frac{\varepsilon_n}{2^{n+1}}$$

by property (c). To estimate the second term, observe that

(14)
$$\|\delta(b)*\mu_n - \mu_n\| \le \int |V_n(b,t) - 1| d\mu_n(t)$$

$$\le \frac{1}{2|D_n|} \cdot \frac{n}{6^n}$$

Now by (12), (13) and (14), $\|\tau_n\| < 2^{-n}\varepsilon_n$ and since $|p_n(t)| \le 2^n$ for all $t\in G$, $\|\nu_n\| < \varepsilon_n$.

It only remains to indicate how the four sequences (λ_n), (χ_n), (D_n) and (A_n) may be obtained to satisfy (a), (b), (c), (d), (e). Assume then that the first n terms in each sequence have been defined and satisfy the required conditions. The first task is to find a suitable χ_{n+1}. D_{n-1} is a finite subset of B, so that it is possible to find χ_{n+1} to satisfy (d), since Γ is dense in $\hat{B}$. Furthermore, because $\Omega_n^2 A_n$ is also finite it is possible to arrange matters so that (a) is satisfied. Choose $A_{n+1} = \overline{A}_{n+1}$ to contain β_{n+1}, A_n and Ω_{n+1}. Next select λ_{n+1} to be a measure finitely supported on B which is a weak* approximation to Haar measure m on G so that (c) holds. Finally define D_{n+1} to be a finite subset of B containing b_n, D_n and the support of λ_{n+1}. This completes the construction. We now state the main theorem.

THEOREM. The Riesz product measure μ constructed above is quasi-invariant and ergodic for the action of B. If ν is any other B-quasi-invariant probability measure with the same Radon-Nokodym derivative then $\nu = \mu$.

As we have already indicated, by varying the coefficients it is possible to obtain uncountably many mutually singular ergodic measures for B.

ACKNOWLEDGEMENTS. This work was done, in part, while the author was on secondment to the Centre for Mathematical Analysis at the Australian National University from May to August 1987. I wish to thank the Director, Professor Neil Trudinger, and his staff for their help and hospitality.

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