HANKEL OPERATORS ON THE PALEY-WIENER SPACE IN $\mathbb{R}^d$

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Let $I^d = (-\pi,\pi)^d = \{\xi \in \mathbb{R}^d : -\pi < \xi_j < \pi \ , \ i = 1,\dots,d\}$ and let χ_{I^d} denote the characteristic function of I^d. Denote the Fourier transform of g by $F(g) = \hat{g}$ and the inverse Fourier transform of f by $F^{-1}(f) = \check{f}$:

$$(1) \qquad \hat{g}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} g(x) e^{-i\xi\cdot x} dx$$

and

$$(2) \qquad \check{f}(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\xi) e^{i\xi\cdot x} d\xi \ .$$

The Paley-Wiener Space on I^d, $PW(D^d)$, is defined to be the image of $L^2(I^d)$ under F^{-1}, i.e.

$$(3) \qquad PW(I^d) = \{F^{-1}(\chi_{I^d} f) : f \in L^2(I^d)\} \ .$$

As is well known, f is in $PW(I^d)$ if and only if it is the restriction to $\mathbb{R}^d$ of an entire function of exponential type at most $(\pi+\epsilon,\dots,\pi+\epsilon)$ in $\mathbb{C}^d$ which satisfies $\|f\|_2 = \left[\int_{\mathbb{R}^d} |f(x)|^2 dx\right]^{1/2} < \infty$.

Let P_{I^d} denote the projection defined by $(P_{I^d} g)^\wedge = \chi_{I^d}\hat{g}$. The Toeplitz operator on $PW(I^d)$ with symbol b is defined by

$$(4) \qquad T_b(f) = P_{I^d}(bf) \ , \quad \text{for} \quad f \in PW(I^d) \ .$$

And the Hankel operator on $PW(I^d)$ with symbol b is defined by

$$(5) \qquad H_b(f) = P_{I^d}(b\bar{f}) \ , \quad \text{for} \quad f \in PW(I^d) \ .$$

Because $PW(I^d)$ is preserved when taking complex conjugates, these two operators on $PW(I^d)$ are unitary equivalent. But as they have properties similar to those of classical Hankel operators (see below), we prefer the name Hankel operators in both cases.

In [7], Rochberg has studied the Hankel operators on $PW(I)$, i.e. the case of one dimension, the results he obtained are as follows:

Let $\theta(x) = e^{i\pi x}$, $\psi_L, \psi_R \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} \psi_L \subset [-4\pi, -\pi/2]$, $\psi_L(\xi) = 1$ on $[-3\pi, -\pi]$, $\psi_R(\xi) = \psi_L(-\xi)$, $\psi_c = 1 - \psi_R - J_L$, $(P_-b)\hat{} = \chi_{(-\infty,0]}\hat{b}$, $(P_+b)\hat{} = \chi_{[0,+\infty)}\hat{b}$.

THEOREM A (Rochberg [7])

1. $\|T_b\| \cong \|P_-(\bar{\theta}^2 b * \check{\psi}_R)\|_{BMO} + \|P_+(\theta^2 b * \check{\psi}_L)\|_{BMO} + \|b * \check{\psi}_c\|_\infty$;

2. T_b *is compact if and only if* $P_-(\bar{\theta}^2 b * \check{\psi}_R)$ *and* $P_+(\theta^2 b * \check{\psi}_L)$ *are in CMO and* $\lim_{|x|\to\infty} b * \check{\psi}_c(x) = 0$;

3. $\|T_b\|_{S_p} = \|b\|_{\mathcal{B}_p} \cong \|P_-(\bar{\theta}^2 * \check{\psi}_R)\|_{B_p} + \|P_+(\theta^2 b * \check{\psi}_L)\|_{B_p} + \|b * \check{\psi}_c\|_{L^p}$, *for* $1 \le p < \infty$, *where* S_p *are Schatten-von Neumann ideals,* B_p *are classical Besov spaces* $B_p^{1/p,p}(\mathbb{R})$.

He also gives a characterization of $\mathcal{B}_p$.

THEOREM B (Rochberg [7]) *Suppose that* $\operatorname{supp} \hat{b} \subset 2I$, $1 \le p < \infty$, *then*

$$\|b\|_{\mathcal{B}_p}^p \cong \sum_{j\in\mathbb{Z}} \operatorname{dist}_j \|\check{\varphi}_j * b\|_{L^p}^p ,$$

where $\{\varphi_j\}_{j\in\mathbb{Z}}$ *is a partition of unity for* $2I$ *with respect to the singular points* -2π *and* 2π, dist_j = *the distance from the centre of support set of* φ_j *to the complement of* $2I$.

In the end of [7], Rochberg proposed several open questions, e.g. do the S_p criteria in Theorem A and B extend to all positive p ? what are the analogs of above results in $\mathbb{R}^d$? what are the basic functional anlytic results for the spaces $\mathcal{B}_p$?

We study the Hankel operators on $PW(I^d)$, i.e., the case of d-dimension, and answer the questions.

Taking Fourier transform, we get

$$T_b(f)^\wedge(\xi) = \int_{\mathbb{R}^d} \hat{b}(\xi-\upsilon)\chi_{I^d}(\xi)\chi_{I^d}(\eta)\hat{f}(\eta)d\eta \ . \tag{6}$$

This turns out to be a paracommutator with symbol b , Fourier kernel $\chi_{I^d}(\xi)\chi_{I^d}(\eta)$ and index $s = t = 0$, see Janson and Peetre [3]. But now the Fourier kernel $\chi_{I^d}(\xi)\chi_{I^d}(\eta)$ does not satisfy the conditions in Janson and Peetre [3], and so it cannot be dealt with in the framework of [3]. We have to look for a new approach.

Note that $T_b = T_{P_2 b}$, where $(P_2 b)^\wedge = \chi_{(2I)^d}\hat{b}$. We assume that $\operatorname{supp}\hat{b} \subset (2I)^d$ throughout this paper.

As is well known, there exist two important decompositions in Harmonic analysis: the Whitney decomposition for open set of $\mathbb{R}^d_x$ and the Littlewood-Paley decomposition for $\mathbb{R}^d_\xi$. When $d = 1$, the Littlewood-Paley decomposition for $\mathbb{R}^1_\xi$ can be regarded as a Whitney decomposition of the open set $\mathbb{R}^1_\xi$ with boundary $\pm\infty$. We refine this idea and give an appropriate decomposition for I^d such that it possesses the properties of both the Whitney decomposition and the Littlewood-Paley decomposition. Using this decomposition we define a new kind of Besov spaces $B^{s,q}_p$ for $s \in \mathbb{R}^d$, $0 < p , q \leq \infty$. It turns out to be quite a success to characterize the Schatten-von Neumann ideal criteria for Hankel operators acting on $PW(I^d)$.

Let $\Delta_j = \emptyset$, for $j = 1,2,\ldots$, $\Delta_0 = \{\xi \in I : |\xi| \le \frac{\pi}{4}\}$, $\Delta_j = \{\xi \in I : 2^{j-1}\pi \le \pi-\xi \le 2^j\pi\} \cup \{\xi \in I : 2^{j-1}\pi \le \xi + \pi \le 2^j\pi\}$, for $j = -1,-2,\ldots$, $\bar{\Delta}_j = \Delta_{j-1} \cup \Delta_j \cup \Delta_{j+1}$. Thus $I = \bigcup_{j=-\infty}^{0} \Delta_j$.

Let Z_- denote the set $\{0,-1,-2,\ldots\}$, for $\underline{j} \in \mathbb{Z}_-^d$, $\Delta_{\underline{j}} = \Delta_{j_1}\times\ldots\times\Delta_{j_d}$, $\bar{\Delta}_{\underline{j}} = \bar{\Delta}_{j_1}\times\ldots\times\bar{\Delta}_{j_d}$, then we have

(7) $$I^d = \bigcup_{j=\mathbb{Z}_-^d} \Delta_j .$$

This gives a decomposition of I^d.

Take $\hat{\varphi}_0 \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} \hat{\varphi}_0 \subset \{\xi : |\xi| \le \frac{3}{4}\pi\}$, $\hat{\varphi}_0(\xi) \ge C$ on $\{\xi : |\xi| \le \frac{\pi}{2}\}$, $\hat{\varphi}_0(\xi) = \hat{\varphi}_0(-\xi)$. Let $\hat{\varphi}_j(\xi)$ $= \hat{\varphi}_0(2^{-j}(|\xi| - \pi + \frac{3}{2^{2-j}}\pi))$, for $j = -1,-2,\ldots$, and let $\hat{\varphi}_{\underline{j}}(\xi) = \prod_{i=1}^{d} \hat{\varphi}_{j_i}(\xi_i)$, for $\underline{j} \in \mathbb{Z}_-^d$, moreover we can also require that $\sum_{\underline{j}\in\mathbb{Z}_-^d} \hat{\varphi}_{\underline{j}}(\xi) \equiv 1$ if $\xi \in I^d$, i.e., it is a partition of unity for I^d.

DEFINITION 1 For $1 \le p \le \infty$, $s \in \mathbb{R}^d$,

(8) $$H_p^s(I^d) = \Big\{f \in S'(\mathbb{R}^d) : \operatorname{supp} \hat{f} \subset I^d ,\ \|f\|_{H_p^s(I^d)} = \Big\|\Big[\prod_{i=1}^{d} (\pi-|\xi_i|)^{s_i}\hat{f}(\xi)\Big]^\vee\Big\|_p < \infty\Big\} .$$

It is obvious that $PW(I^d) = H_2^0(I^d)$.

DEFINITION 2 For $s \in \mathbb{R}^d$, $0 < p,q \le \infty$.

(9) $$B_p^{s,q}(I^d) = \Big\{f \in S'(\mathbb{R}^d) : \operatorname{supp} \hat{f} \subset I^d ,\ \|f\|_{B_p^{s,q}(I^d)} = \Big[\sum_{\underline{j}\in\mathbb{Z}^d} (2^{s\cdot\underline{j}}\|f * \varphi_{\underline{j}}\|_p)^q\Big]^{1/q} < \infty\Big\} .$$

It is obvious that $B_p^{1/p,p}(2I) = \mathcal{B}_p$ for $d = 1$, $1 \le p < \infty$.

For $\sigma \in \mathbb{R}^d$, the operator I^σ is defined by

$$(I^\sigma f)\hat{}(\xi) = \prod_{i=1}^{d} (\pi - |\xi_i|)^{\sigma_i} \hat{f}(\xi) , \quad \text{for} \quad f \in S'_{I^d} = \left\{ f \in S'(\mathbb{R}^d) : \text{supp } \hat{f} \subset I^d \right\} .$$

We obtain the basic functional analytic results for $B_p^{s,q}(I^d)$ as follows.

THEOREM 1

(i) $B_p^{s,q}(I^d)$ *is a quasi-Banach space if* $s \in \mathbb{R}^d$, $0 < p,q \le \infty$ (Banach space if $1 \le p,q \le \infty$), and the quasi-norms $\|f\|^\varphi_{B_p^{s,q}(I^d)}$ with different choices $\{\varphi_j\}$ are equivalent.

(ii) $B_2^{(1/2,\ldots,1/2),2}(I^d) = H_2^{(1/2,\ldots,1/2)}(I^d)$.

(iii) $S_{I^d} \subset B_p^{s,q}(I^d) \subset S'_{I^d}$.

(iv) *If* $p,q < \infty$, S_{I^d} *is dense in* $B_p^{s,q}(I^d)$.

(v) $B_p^{s,q_0}(I^d) \subset B_p^{s,q_1}(I^d)$, *if* $q_0 \le q_1$.

(vi) $\forall \sigma \in \mathbb{R}^d$, I^σ *maps* $B_p^{s,q}(I^d)$ *isomorphically onto* $B_p^{s-\sigma,q}(I^d)$.

THEOREM 2 *Let* $s \in \mathbb{R}^d$, $1 \le p,q < \infty$, *then*

(10)
$$\left[B_p^{s,q}(I^d)\right]' = B_p^{-s,q'}(I^d)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$.

THEOREM 3 *Let* $s^0, s^1 \in \mathbb{R}^d$, $1 \le p_0, p_1, q_0, q_1 \le \infty$, $0 < \theta < 1$,

$s^* = (1-\theta)s_1^0 + \theta s^1$ (i.e. $s_1^* = (1-\theta)s_1^0 + \theta s_1^1$, $\ldots$, $s_d^* = 1-\theta)s_d^0 + \theta s_d^1$),

$$\frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} ,$$

$$\frac{1}{q^*} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} .$$

Then

$$(11) \qquad \left[B_{p_0}^{s^0,q_0}(I^d), B_{p_1}^{s^1,p_1}(I^d)\right]_{[\theta]} = B_{p*}^{s*,q*}(I^d) .$$

Extending the definition of T_b , we consider $T_b^{s,t}$ defined by

$$(12) \quad (T_b^{s,t}f)\hat{}(\xi) = \int \hat{b}(\xi-\eta) \prod_{i=1}^{d} (\pi-|\xi_i|)^{s_i} \prod_{i=1}^{d} (\pi-|\eta_i|)^{t_i})\chi_{I^d}(\xi)\chi_{I^d}(\eta)\hat{f}(\eta)d\eta$$

where $x,t \in \mathbb{R}^d$. We obtain a characterization of the Schatten-von Neumann ideal S_p of $T_b^{s,t}$ in terms of $b \in B_p^{s+t+(1/p,\ldots,1/p),p}((2I)^d)$.

THEOREM 4 *Suppose that* $0 < p \leq \infty$, $s,t \in \mathbb{R}^d$ *with* $s_i,t_i > \max(-1/2,-1/p)$. *Then* $T_b^{s,t} \in S_p$ *if and only if* $b \in B_p^{s+t+(1/p,\ldots,1p),p}((2I)^d)$, *and*

$$(13) \qquad \|T_b^{s,t}\|_{S_p} \cong \|b\|_{B_p^{s+t+(1/p,\ldots,1/p),p}((2I)^d)} .$$

To prove it, for $1 \leq p \leq \infty$ we follow the procedure of Janson and Peetre [3], for $0 < p < 1$ we follow the procedure of Peng [5].

Note that if $p = \infty$, Theorem 4 does not contain the result of $T_b = T_b^{0,0}$. We have the following direct result.

Let $\hat{\psi}_0 = \hat{\varphi}_{\underline{0}}$, $\hat{\psi}_1(\xi) = \chi_{(2I)^d}(\xi) - \hat{\psi}_0(\xi)$. Then $b = b_1 + b_0$, where $b_0 = b * \psi_0$, $b_1 = b * \psi_1$.

THEOREM 5 *If* $b_1 \in BMO$, $b_0 \in L_\infty$. *Then* $T_b \in S_\infty$ *and*

$$(14) \qquad \|T_b\| \leq C(\|b_1\|_{BMO} + \|b_0\|_{L^\infty}) .$$

For the converse result, we get it only for $d = 1$.

THEOREM 6 *If* $d = 1$, T_b *is bounded on* PW(I) , *then* $B_1 \in BMO$, $b_0 \in L_\infty$ *and*

(15) $$\|b_1\|_{BMO} + \|b_0\|_{L^\infty} \le C\|T_b\| .$$

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REFERENCES

[1] Bergh, J. and Löfström, J., *Interpolation spaces*, Grundlehren Math. Wiss. **223**, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[2] Higgins, J.R., *Five short stories about the cardinal series*. Bull. Amere. Math. Soc. **12**(1985), 45-90.

[3] Janson, S. and Peetre, J., *Paracommutators-Boundedness and Schatten-von Neumann properties*, Report of Dep. of Math., Stockholm, **15**(1985).

[4] Peetre, J., *New thoughts on Besov Spaces*, Duke Uni. Press, Durham, 1976.

[5] Peng, L.ZH., Paracommutators of Schatten-von Neumann class $S_p, 0 < p < 1$, to appear in Scand. Mat.

[6] Peng, L.Zh., Multilinear singular integrals of Schatten-von Neumann class S_p , to appear in Approximation Theory and its Applications.

[7] Rochberg, R., *Toeplitz and Hankel Operators on the Paley-Wiener Space*, Integral Equations and Operator Theory, **10**(1987), 186-235.

[8] Triebel, H., *Theory of functions spaces*, Birkhauser Verlag, 1983.

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