We continue our study of central sequences in, and automorphisms of separable $C^*$-algebras begun in [7]. We would like to attempt, as far as possible, to follow the plan of attack developed by A. Connes in [2,3] for von Neumann algebras. Unfortunately, very few of the general ideas survive in the $C^*$-algebra setting. The main reason for this is the overabundance of nontrivial central sequences in a general separable $C^*$-algebra. Despite this, we are able to sufficiently analyze some large classes of $C^*$-algebras so that those automorphisms which behave trivially on central sequences can be computed. Complete proofs and more detailed examples will appear elsewhere.

§ 1. Preliminaries

Let $A$ denote a separable unital $C^*$-algebra over the complex numbers. Let $\text{Aut}A$ denote the group of all $*$-automorphisms of $A$, and $\text{Inn}A$ the normal subgroup of all inner automorphisms. Let $\epsilon: \text{Aut}A \rightarrow \text{Aut}A/\text{Inn}A = \text{Out}A$ be the quotient.

A central sequence in $A$ is a bounded sequence $\{x_n\}$ of elements of $A$ with the property that $[x_n, a] = x_n a - a x_n \rightarrow 0$ in norm for each $a \in A$. A uniformly central sequence is a bounded sequence $\{x_n\}$ for which the operators (on $A$) $\text{ad} x_n(\cdot) = [x_n, \cdot]$ converge to 0 in norm. A central sequence $\{x_n\}$ is called hypercentral if $||[x_n, y_n]|| \rightarrow 0$ for every central sequence $\{y_n\}$ of $A$. A central sequence $\{x_n\}$ is called trivial if there is a sequence $\{\lambda_n\}$ of central elements in $A$ so that $||x_n - \lambda_n|| \rightarrow 0$. It is evident that any trivial sequence is uniformly central and any uniformly central
sequence is hypercentral. Two central sequences are called equivalent if their difference converges to zero in norm.

If \( \alpha \in \text{Aut} A \), then we say that \( \alpha \) is centrally trivial if

\[ ||(x_n) - x_n|| \to 0 \]

for every central sequence \( \{x_n\} \) of \( A \). We denote the normal subgroup of centrally trivial automorphism of \( A \) by \( \text{Ct} A \) and note that \( \text{Inn} A \leq \text{Ct} A \). We let \( \text{Inn} A^- \) denote the closure of \( \text{Inn} A \) in the topology of pointwise norm convergence. Following A. Connes we define

\[ \chi(A) = \varepsilon(\text{Ct} A \cap \text{Inn} A^-) \].

A straightforward adaptation of an argument of A. Connes gives us the following.

1.1 Proposition: If \( A \) is a separable unital C*-algebra then \( \chi(A) \) is abelian. In fact, \( \varepsilon(\text{Ct} A) \) commutes with \( \varepsilon(\text{Inn} A^-) \).

Now, if we let \( \text{Inn} A^*_{||-||} \) denote the uniform norm closure of \( \text{Inn} A \) in \( \text{Aut} A \) then a simple argument shows that \( \text{Inn} A^*_{||-||} \leq \text{Ct} A \). Since \( \text{Inn} A^*_{||-||} \leq \text{Inn} A^- \) we have \( \varepsilon(\text{Inn} A^*_{||-||}) \leq \chi(A) \). We have proved:

1.2 Corollary: \( \varepsilon(\text{Inn} A^*_{||-||}) \) is abelian!

Now, by an argument of C. Akemann and G. Pedersen [1], given any central element \( z \in A^{**} \) we can find a central sequence \( \{x_n\} \) in \( A \) so that \( x_n \to z \) strongly. We easily conclude that if \( \alpha \in \text{Ct} A \) then \( \alpha^{**} \in \text{Aut} A^{**} \) fixes the centre of \( A^{**} \). We have proved:

1.3 Proposition: Under the natural embedding \( \text{Aut} A \to \text{Aut} A^{**} \), \( \text{Ct} A \) gets sent into \( \text{Aut}_z A^{**} \), the subgroup of centre-fixing automorphisms.

1.4 Corollary: If \( A \) is type I, \( \text{Ct} A \leq \pi(A) \) the subgroup of \( \pi \)-inner automorphisms.
1.5 **Corollary:** If $A$ is simple, $\text{Ct} A = \text{Inn} A$.

Since $\alpha^{**}$ fixes the center of $A^{**}$ we see that $\alpha$ acts trivially on $\hat{A}$, the space of irreducible representations of $A$. Combining this with a result of G.A. Elliott's [4] gives us 1.4: combining it with A. Kishimoto's result [6] gives us 1.5. It is easy to produce type I examples, $A$, for which $\text{Ct} A \subset \pi(A)$. We have the following conjecture:

1.6 **Conjecture:** $\text{Ct} A \subset \pi(A)$ for all separable, unital $C^*$-algebras, $A$.

§2. **Primitive $C^*$-algebras**

2.1 **Conjecture:** If $A$ is a primitive, unital, separable $C^*$-algebra, then $\text{Ct} A = \text{Inn} A$.

2.2 **Theorem:** If $A$ is a primitive, unital, separable $A.F.$-algebra, then $\text{Ct} A = \text{Inn} A$.

The idea here is to take an $\alpha \in \text{Ct} A$ and show that given $\epsilon > 0$ there is a finite-dimensional subalgebra $B \subset A$ so that $\|\alpha - \text{id}\| \leq \epsilon$ on the relative commutant $B^C = B' \cap A$. By representing $A$ irreducibly on $H$, $B^C$ is strongly dense in $B'$ and so $\|\alpha - \text{id}\| \leq \epsilon$ on $B'$; $\alpha$ is spatial on $H$ by 1.3 and so $\alpha$ makes sense on $B'$. If $\alpha = \text{Adv}$ then carefully projecting $v$ into $B$ yields an $x \in B$ with $\|v-x\| \leq \epsilon$. Thus, $v \in A$. 
2.3 **Theorem.** If $A$ is an extension of a separable unital $C^*$-algebra by the compact operators, then $\text{Ct} A = \text{Inn} A$.

Again we represent $A$ irreducibly on $H$ and use 1.3 to obtain a unitary $v$ with $\alpha = \text{Adv}$. If $v \in A$ then by Voiculescu's double commutant theorem, $\pi(v) \in \pi(A)^{\text{CC}}$ where $\pi : B(H) \to Q(H)$ is the Calkin map and $(\cdot)^C$ denotes the commutant in $Q(H)$, the Calkin algebra. Then using two carefully chosen quasicentral approximate identities in $\mathcal{K}(H)$ to successively modify $v$, we obtain a central sequence $\{x_n\} \subseteq \mathcal{K}(H) \subseteq A$ so that $||\alpha(x_n) - x_n||$ is bounded away from 0, a contradiction.

2.4 **Remark:** Thus, if our primitive (separable, unital) $C^*$-algebra $A$ is either simple, A.F., or an extension by $\mathcal{K}(H)$ we have $\text{Ct} A = \text{Inn} A$. This is the evidence we offer for conjecture 2.1.

§3. **Hypercentral sequences**

Here we discuss the existence problem for nontrivial hypercentral sequences; i.e., those central sequences which commute asymptotically with all other central sequences. First we prove by a fairly direct but tricky argument the following theorem.

3.1 **Theorem:** If $A$ is a primitive (separable, unital) AF algebra then every hypercentral sequence is trivial.

If $A$ is, in fact, a UHF algebra then this conclusion is easy to prove.

Now, a straightforward argument using partitions of unity establishes the following.

3.2 **Proposition:** If $A$ is a separable, unital $C^*$-algebra with no nontrivial hypercentral sequences and $X$ is a compact separable space, then $C(X) \otimes A$ has no nontrivial hypercentral sequences either.

The next lemma is a simple adaptation to the $C^*$-algebra setting of a result of A. Connes.
3.3 **Lemma.** If $A$ is a separable, unital C*-algebra in which all central sequences are hypercentral, then $\text{Inn} A^* \subseteq \text{Ct} A$ so that $\varepsilon(\text{Inn} A^*)$ is abelian ($= \chi(A)$, in fact).

We then deduce:

3.4 **Theorem:** Let $A$ be a separable unital C*-algebra which has an infinite-dimensional primitive quotient which is either simple, A.F. or contains the compact operators. Then not all central sequences of $A$ are hypercentral.

Since central sequences can always be lifted from a quotient by [1] we can assume $A$ is primitive, infinite-dimensional and either simple, A.F., or an extension by the compact operators. Thus, if all central sequences were hypercentral we would have $\text{Inn} A^* \subseteq \text{Ct} A = \text{Inn} A$. However, this is not possible by [7].

3.5 **Example:** A separable, unital C*-algebra $A$, for which all central sequences are hypercentral but not necessarily uniformly central and hence not trivial. Define: $A = \{f: [0,1] \to M_2(\mathbb{C}) | f \text{ is continuous and } f(1) \text{ is diagonal}\}$. It is easy and instructive to verify that $A$ has the desired properties.

3.6 **Example:** A separable, unital A.F. algebra, $A$, such that

1. $Z(A) = \mathbb{C}$
2. $A$ has non-hypercentral central sequences and also nontrivial hypercentral sequences.
3. $\text{Inn} A \subset \text{Inn} A^* = \text{Ct} A \subset \text{Inn} A^*$
4. $\chi(A) \equiv \{(\lambda_n) \in \prod_{1}^{\infty} S^1 | \lim_{n \to \infty} \lambda_n = 1\}$

\[ \{(\lambda_n) \mid \lim_{n \to \infty} \lambda_n \text{ exists in } S^1\} \]
This example is a modification of example 6.3 of [5]. Although the construction and proofs are not terribly difficult they are rather lengthy and so we omit them here.

3.7 Theorem: Let \( X \) be a separable compact space and let \( A \) be a separable unital C*-algebra such that

1. \( Z(A) = \mathbb{C} \)
2. \( \text{Ct} A = \text{Inn} A \)
3. Every hypercentral sequence in \( A \) is uniformly central.

If \( B = C(X) \otimes A \) then we have an exact sequence

\[
0 \to \text{Inn} B \to \text{Ct} B \to H^2(X, \mathbb{Z}) \text{.}
\]

In fact, \( \text{Ct} B = \{ \alpha|_X \mapsto \alpha : X \to \text{Inn} A \text{ is uniformly continuous} \} \).

To see this last statement (which is the crux of the matter) let \( \alpha \in \text{Ct} B \) . Then \( \alpha \) preserves the centre of \( B \) and so gives rise to a map

\[
x \mapsto \alpha_x : X \to \text{Aut} A \text{ continuous in the point-norm topology. Since central sequences lift from } A \text{ to } B \text{ we see each } \alpha_x \in \text{Ct} A = \text{Inn} A \text{.}
\]

Now, if \( x \mapsto \alpha_x \) is not uniformly continuous one can (without loss of generality)

obtain \( \alpha_x \to \text{id} \) pointwise but not uniformly. Since \( \alpha_x = \text{Ad} u_n \) , \( \{ u_n \} \)

is a central, but not uniformly central, sequence and therefore not hypercentral. We thus choose \( \{ a_n \} \) central in \( A \) so that \( ||u_n a_n|| \to 0 \).

But, then \( \{ 1 \otimes a_n \} \) is a central sequence in \( B \) and \( ||\alpha(1 \otimes a_n) - 1 \otimes a_n|| \to 0 \)

, a contradiction.

We observe that by 2.2 and 3.1, every primitive (separable, unital) A.F. algebra satisfies the hypotheses of theorem 3.7.
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