DERIVATIVES OF INVARIANT POLYNOMIALS ON A SEMISIMPLE LIE ALGEBRA

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0. INTRODUCTION.

Let $\underline{\underline{g}}$ be a complex semisimple Lie algebra and let G be the adjoint group of $\underline{\underline{g}}$. It is known that the algebra $I(\underline{\underline{g}}) = \mathbf{C}[\underline{\underline{g}}]^G$ of G-invariant polynomial functions on g is generated by $n = rank(\underline{\underline{g}})$ algebraically independent homogeneous polynomials $P_1, \ldots, P_n$. Let $\pi : \underline{\underline{g}} \to \mathbf{C}^n$ be the polynomial map given by $\pi(x) = (P_1(x), \ldots, P_n(x))$; π is the *quotient morphism* of $\underline{\underline{g}}$. The geometry of the quotient morphism has been studied in detail by Kostant [9] and his results have played an important role in many problems in representation theory. Among other results, he shows that the differential $d\pi_x$ at a point $x \in \underline{\underline{g}}$ is of maximal rank n if and only if x is a regular element of $\underline{\underline{g}}$. Our goal is to compute rank$(d\pi_x)$ for *every* $x \in L(G)$. We have succeeded except for the case of one nilpotent conjugacy class in type E_8.

It clearly suffices to handle the case in which $\underline{\underline{g}}$ is simple. By using the Luna slice theorem, we can reduce to the case when x is a nilpotent element of $\underline{\underline{g}}$. For the classical simple Lie algebras, we can give a reasonably straightforward computation of the ranks. For the exceptional Lie algebras, we need to use the classification of nilpotent conjugacy classes and detailed information on the closures of nilpotent classes.

It is convenient to reinterpret our results in terms of G-invariant (polynomial) vector fields on $\underline{\underline{g}}$. A G-invariant vector field on $\underline{\underline{g}}$ is just a G-morphism $\varphi : \underline{\underline{g}} \to \underline{\underline{g}}$. For each $j = 1, \ldots, n$, let $\varphi_j = \text{grad}(P_j)$. It is known that $\varphi_1, \ldots, \varphi_n$ are a basis for the $I(\underline{\underline{g}})$-module of G-invariant vector fields. Using this result, it is easy to show that the following three numbers are equal for every $x \in \underline{\underline{g}}$: (i) rank$(d\pi_x)$; (ii) the dimension of the vector subspace of $\underline{\underline{g}}$ spanned by $\{\varphi_1(x), \ldots, \varphi_n(x)\}$; and (iii) the multiplicity of the adjoint representation of G in the G-module $\mathbf{C}[\overline{G \cdot x}]$ of regular functions on $\overline{G \cdot x}$, the closure of the orbit $G \cdot x$.

By using the result above and a theorem of Borho and Kraft [3], we obtain a number of new examples of non-normal nilpotent orbit closures in the exceptional Lie algebras of types E_6, E_7 and E_8. For example, we show that (at least) seven of the twenty-one nilpotent orbits in type E_6 have closures which are not normal varieties.

This paper contains only a statement of results, with outlines of the proofs. A detailed exposition of these results will appear elsewhere.

1. PRELIMINARIES.

Let $\underline{\underline{g}}$ be a complex reductive Lie algebra of rank n and let G be the adjoint group of $\underline{\underline{g}}$. Let $I(\underline{\underline{g}}) = \mathbf{C}[\underline{\underline{g}}]^G$ be the algebra of G-invariant polynomial functions on $\underline{\underline{g}}$. Then $I(\underline{\underline{g}})$ is generated by n algebraically independent homogeneous polynomials $P_1, \ldots, P_n$. Let $\pi : \underline{\underline{g}} \to \mathbf{C}^n$ denote the morphism $(P_1, \ldots, P_n)$. Let $d_j = \text{degree} P_j$ and let $m_j = d_j - 1$. The d_j's are the *fundamental degrees* of $\underline{\underline{g}}$ and the m_j's are the *exponents* of $\underline{\underline{g}}$. The generators $P_1, \ldots, P_n$ are not uniquely determined by $\underline{\underline{g}}$, but the fundamental degrees and the exponents are independent of the choice of $P_1, \ldots, P_n$. We always assume $d_1 \leq \cdots \leq d_n$. (If $\underline{\underline{g}}$ is semisimple, $d_1 = 2$ and, if $\underline{\underline{g}}$ is simple, $d_2 > 2$.) Let $\underline{\underline{t}}$ be a Cartan subalgebra of $\underline{\underline{g}}$ and let $W = W(\underline{\underline{g}}, \underline{\underline{t}})$ be the corresponding Weyl group. Let $I(\underline{\underline{t}}) = \mathbf{C}[\underline{\underline{t}}]^W$ be the invariant algebra of W. Then the restriction homomorphism $\rho : \mathbf{C}[\underline{\underline{g}}] \to \mathbf{C}[\underline{\underline{t}}]$ maps $I(\underline{\underline{g}})$ isomorphically onto $I(\underline{\underline{t}})$. For all of this material, see [4], Chap. 8.

Let β be a G-invariant, non-degenerate, symmetric bilinear form on $\underline{\underline{g}}$; if $\underline{\underline{g}}$ is semisimple, we may take β to be the Cartan-Killing form. Let $\sigma : \underline{\underline{g}}* \to \underline{\underline{g}}$ be the isomorphism corresponding to β. Then σ determines an isomorphism between differential one-forms on $\underline{\underline{g}}$ and vector fields on $\underline{\underline{g}}$. If f belongs to $\mathbf{C}[\underline{\underline{g}}]$, the algebra of polynomial functions on $\underline{\underline{g}}$, we define the vector field $\text{grad}(f)$ on $\underline{\underline{g}}$ by $\text{grad}(f)(x) = \sigma(df_x)$. If $f \in I(\underline{\underline{g}})$, then $\text{grad}(f)$ is a G-invariant vector field.

1.1. *Let $\varphi_j = \text{grad}(P_j)$, $j = 1, \ldots, n$. Then $\varphi_1, \ldots, \varphi_n$ are a basis for the $I(\underline{\underline{g}})$-module of G-invariant vector fields on $\underline{\underline{g}}$.*

See [12], Example 7.8, for the proof.

The G-invariant vector field $\varphi_j : \underline{\underline{g}} \to \underline{\underline{g}}$ is a homogeneous polynomial map of degree m_j.

For $x \in \underline{\underline{g}}$ let $\underline{\underline{g}}_x$ denote the centralizer of x in $\underline{\underline{g}}$ and let G_x denote the isotropy subgroup of x in G. Clearly $\underline{\underline{g}}_x$ is equal to $L(G_x)$, the Lie algebra of the isotropy subgroup G_x. We let $\underline{\underline{d}}(x)$ denote the centre of the centralizer $\underline{\underline{g}}_x$; $\underline{\underline{d}}(x)$ is the *double centralizer* of x ; $\underline{\underline{d}}(x)$ is an abelian subalgebra of $\underline{\underline{g}}$. We set

$$\underline{\underline{a}}(x) = \underline{\underline{g}}^{G_x} = \{y \in \underline{\underline{g}} | G_y \supset G_x\} \quad .$$

Then $\underline{\underline{a}}(x) \subset \underline{\underline{d}}(x)$, with equality if G_x is connected.

The following result is an easy consequence of the Luna slice theorem [11] :

PROPOSITION 1.2. *Let $x \in \underline{\underline{g}}$ have Jordan decomposition $x = h + v$, with h semisimple and v nilpotent. Let $\underline{\underline{s}} = [\underline{\underline{g}}_h\ ,\ \underline{\underline{g}}_h]$, let $r = \text{rank}(\underline{\underline{s}})$ and let $\pi_1 : \underline{\underline{s}} \to \mathbf{C}^r$ be the quotient morphism of $\underline{\underline{s}}$. Then*

$$\text{rank}(d\pi_x) = \dim \underline{\underline{d}}(h) + \text{rank}((d\pi_1)_v) \quad .$$

Proposition 1.2 reduces our problem to the case when x is nilpotent. It is also clear that we may reduce to the case when $\underline{\underline{g}}$ is a simple Lie algebra.

2. REINTERPRETATION IN TERMS OF G-INVARIANT VECTOR FIELDS

We continue with the notation of Section 1. *From now on we assume that* $\underline{\underline{g}}$ *is a simple Lie algebra.* Let Ψ denote the $I(\underline{\underline{g}})$ module of the G-invariant (polynomial) vector fields.

For $x \in \underline{\underline{g}}$, let $\underline{\underline{e}}(x)$ denote the linear span of $\{\varphi_1(x), \ldots, \varphi_n(x)\}$ in $\underline{\underline{g}}$. It follows from 1.1 that

$$\underline{\underline{e}}(x) = \{\varphi(x) | \varphi \in \Psi\} \quad .$$

LEMMA 2.1. *If* $x \in \underline{\underline{g}}$, *then*

$$\underline{\underline{e}}(x) \subset \underline{\underline{a}}(x) \subset \underline{\underline{d}}(x) \quad .$$

Moreover $\underline{\underline{e}}(x)^\perp$ *is the kernel of* $d\pi_x$. *Hence* $\dim \underline{\underline{e}}(x) = \operatorname{rank}(d\pi_x)$.

Here $\underline{\underline{e}}(x)^\perp$ denotes the subspace of $\underline{\underline{g}}$ orthogonal to $\underline{\underline{e}}(x)$. We note that it is not necessarily true that $\underline{\underline{e}}(x) \cap \underline{\underline{e}}(x)^\perp = \{0\}$.

Let $x \in \underline{\underline{g}}$ be nilpotent and let $\{x, h, y\}$ be an $\underline{\underline{s}}\,\underline{\underline{\ell}}_2(\mathbb{C})$-triple in $\underline{\underline{g}}$ containing x. It is known that the eigenvalues of $ad\ h$ on $\underline{\underline{g}}$ are integers. For each $m \in \mathbb{Z}$, let $\underline{\underline{g}}(h, m)$ denote the m-eigenspace of $ad\ h$ on $\underline{\underline{g}}$. (See [4], Chap. 8, for $\underline{\underline{s}}\,\underline{\underline{\ell}}_2(\mathbb{C})$-triples.)

PROPOSITION 2.2. *Let the notation be as above. Then* $\varphi_j(x) \in \underline{\underline{g}}(h, 2m_j)$, $j = 1, \ldots, n$.

COROLLARY 2.3. *Assume that the exponents* $m_1, \ldots, m_n$ *are pairwise distinct. Then*

$$\operatorname{rank}(d\pi_x) = \#\{j = 1, \ldots, n | \varphi_j(x) \neq 0\} \quad .$$

We note that if $\underline{\underline{g}}$ is not of type D_{2r}, then the exponents of $\underline{\underline{g}}$ are pairwise distinct.

If x is a nilpotent element of $\underline{\underline{g}}$, then we define the *exponents* of $(\underline{\underline{g}}, x)$ as follows : if $m \in \mathbb{N}$, then the multiplicity of m as an exponent of $(\underline{\underline{g}}, x)$ is the dimension of the linear span of $\{\varphi_j(x) | m_j = m\}$. We arrange the exponents of $(\underline{\underline{g}}, x)$ in a non-decreasing sequence, with multiple exponents repeated accordingly to their multiplicity. Clearly the number of exponents (counted according to their multiplicity) is equal to $\operatorname{rank}(d\pi_x)$. If x is a regular nilpotent element of $\underline{\underline{g}}$, then it follows from the results of Kostant [9] that the exponents of $(\underline{\underline{g}}, x)$ are just the usual exponents of $\underline{\underline{g}}$. For a general nilpotent $x \in \underline{\underline{g}}$, the sequence of exponents of $(\underline{\underline{g}}, x)$ is a subsequence of the sequence of exponents of $\underline{\underline{g}}$. If $\underline{\underline{g}}$ is not of type D_{2r}, then each exponent of $(\underline{\underline{g}}, x)$ is of multiplicity one.

LEMMA 2.4. *Let* $x \in \underline{\underline{g}}$ *be nilpotent,* $x \neq 0$. *Then* 1 *is an exponent of* $(\underline{\underline{g}}, x)$.

Proof. We may take $P_1(y) = \beta(y, y)$. Hence $\varphi_1(y) = 2y$.

Let $x \in \underline{g}$. Then $\mathrm{C}[\overline{G \cdot x}]$, the algebra of regular functions on the orbit closure $\overline{G \cdot x}$ is a rational G-module.

PROPOSITION 2.5. *Let $x \in \underline{g}$. Then* $\operatorname{rank}(d\pi_x)$ *is equal to the multiplicity of the adjoint representation $\underline{g}$ in the G-module $\mathrm{C}[\overline{G \cdot x}]$.*

In Proposition 2.5, we do not require that x be nilpotent.

COROLLARY 2.6. *Let $x \in \underline{g}$ be nilpotent and let $\mathrm{C}[\overline{G \cdot x}] = \oplus_{m \geq 0} \mathrm{C}[\overline{G \cdot x}]_m$ denote the graded structure on $\mathrm{C}[\overline{G \cdot x}]$. Then the multiplicity of $\underline{g}$ in $\mathrm{C}[\overline{G \cdot x}]_m$ is equal to the multiplicity of m as an exponent of $(\underline{g}, x)$.*

3. REDUCTIVE SUBALGEBRAS OF MAXIMAL RANK

In this section $\underline{r}$ will denote a reductive subalgebra of $\underline{g}$ of maximal rank and $\underline{s} = [\underline{r}, \underline{r}]$ is the commutator subalgebra of $\underline{r}$. The subalgebra $\underline{s}$ is a semisimple subalgebra of $\underline{g}$. A semisimple subalgebra of $\underline{g}$ which can be obtained in this manner is said to be a *regular semisimple subalgebra* of $\underline{g}$.

PROPOSITION 3.1. *Let $x \in \underline{s}$ be nilpotent. If $\varphi_j(x) \neq 0$, then m_j is an exponent of $(\underline{s}, x)$, and hence an exponent of $\underline{s}$. The sequence of exponents of $(\underline{g}, x)$ is a subsequence of the sequence of exponents of $\underline{s}$.*

Let R be the connected algebraic subgroup of G such that $L(R)$, the Lie algebra of R, is equal to $\underline{r}$. The algebra of invariants $I(\underline{r}) = \mathrm{C}[\underline{r}]^R$ is a graded polynomial algebra. Let $Q_1, \ldots, Q_n$ be a minimal set of homogeneous generators of $I(\underline{r})$ and let $\pi_0 : \underline{r} \to \mathrm{C}^n$ denote the polynomial map $(Q_1, \ldots, Q_n)$. Then there exists a unique polynomial map $\nu : \mathrm{C}^n \to \mathrm{C}^n$ such that the diagram

$$\begin{array}{ccc} \underline{r} & \hookrightarrow & \underline{g} \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi} \\ \mathrm{C}^n & \xrightarrow{\nu} & \mathrm{C}^n \end{array}$$

is commutative. More concretely, there exist polynomials $F_j \in \mathrm{C}[X_1, \ldots, X_n]$, $j = 1, \ldots, n$, where $X_1, \ldots, X_n$ are indeterminates, such that $P_j = F_j(Q_1, \ldots, Q_n)$, $j = 1, \ldots, \ell$. The map ν is given by

$$\nu(x) = (F_1(x), \ldots, F_n(x)) \quad .$$

PROPOSITION 3.2. *Let x be a regular nilpotent element of $\underline{s}$. Then* $\operatorname{rank}(d\pi_x) = \operatorname{rank}(d\nu_0)$.

Now let $\underline{t}$ be a Cartan subalgebra of $\underline{r}$, hence a Cartan subalgebra of $\underline{g}$. Let $W = W(\underline{g}, \underline{t})$ and $W_0 = W(\underline{r}, \underline{t})$ be the corresponding Weyl groups. The restriction homomorphisms $\mathbf{C}[\underline{g}] \to \mathbf{C}[\underline{t}]$ and $\mathbf{C}[\underline{r}] \to \mathbf{C}[\underline{t}]$ map $I(\underline{g})$ isomorphically onto $\mathbf{C}[\underline{t}]^W$ and $I(\underline{r})$ isomorphically onto $\mathbf{C}[\underline{t}]^{W_0}$. Therefore we obtain a commutative diagram

(3.3)

$$\begin{array}{ccc} \underline{t} & \xrightarrow{Id} & \underline{t} \\ \downarrow & & \downarrow \\ \mathbf{C}^n & \xrightarrow{\nu} & \mathbf{C}^n \end{array}$$

where the vertical maps are the restrictions to $\underline{t}$ of π_0 and π. *Thus we see from Proposition* 3.1 *that, if x is a regular nilpotent element of $\underline{s}$, then $rank(d\pi_x) = rank(d\nu_0)$ can be computed by a computation involving Weyl group invariants.*

4. COMPUTATIONS FOR THE CLASSICAL LIE ALGEBRAS

4.1 *Nilpotent classes in the classical Lie algebras.* (See [5] and [6]).

(a) Type A_n. Let $\underline{g} = \underline{s\ell}_{n+1}(\mathbf{C})$.

To each nilpotent element $x \in \underline{g}$, we can canonically assign a partition $\lambda(x) = (\lambda_1, \ldots, \lambda_r)$ of $n+1$; the λ_i's are the sizes of the Jordan blocks in the Jordan canonical form of x. We always assume $\lambda_1 \geq \lambda_2 \geq \ \ldots \ \geq \lambda_r > 0$. We set $r = \ell(\lambda(x))$; r is the *length* of the partition $\lambda(x)$. The map $x \to \lambda(x)$ determines a bijective correspondence between nilpotent classes in $\underline{g}$ and partitions of $n+1$.

(b) Types B_n and D_n. Here $\underline{g}$ is the Lie algebra $\underline{o}_m(\mathbf{C})$ of all skew symmetric complex $m \times m$ matrices; for type B_n (resp. type D_n) $m = 2n+1$ (resp. $m = 2n$). We consider $\underline{g}$ as a subalgebra of $\underline{s\ell}_m(\mathbf{C})$ and, if $x \in \underline{g}$ is nilpotent, we let $\lambda(x)$ be the corresponding partition of m. A partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of m corresponds to a nilpotent element $x \in \underline{g}$ if and only if λ satisfies the following condition :

(∗) *each even part of λ occurs an even number of times.*

Two nilpotent elements of $\underline{g}$ are conjugate under $SL_m(\mathbf{C})$ if and only if they are conjugate under the full orthogonal group $O_m(\mathbf{C})$. Thus there is a bijective correspondence between partitions λ of m satisfying (∗) and $O_m(\mathbf{C})$-classes of nilpotent elements in $\underline{o}_m(\mathbf{C})$. If all λ_i are even, then the $O_m(\mathbf{C})$-class corresponding to λ splits into two $SO_m(\mathbf{C})$-classes. Otherwise the $O_m(\mathbf{C})$-class corresponding to λ is a single $SO_m(\mathbf{C})$-class. In particular, if m is odd, there is a bijective correspondence between partitions of n satisfying (∗) and nilpotent conjugacy classes (under $SO_m(\mathbf{C})$) in $\underline{o}_m(\mathbf{C})$.

(c) Type C_n. Here $\underline{g} = \underline{sp}_n(\mathbf{C})$, which is a subalgebra of $\underline{s\ell}_{2n}(\mathbf{C})$. The map $x \to \lambda(x)$ determines a bijective correspondence between nilpotent classes in $\underline{g}$ and partitions λ of $2n$ which satisfy the following condition :

(∗∗) *every odd part of λ occurs an even number of times.*

4.2. *Computations for type* A_n. Section 3 gives a method by which, for certain nilpotent elements $x \in \underline{\underline{g}}$, rank$(d\pi_x)$ can be computed by a calculation involving Weyl group invariants. For type A_n, this method works for *every* nilpotent element. The Weyl group invariants are easy to describe in this case and one can compute explicitly. We obtain the following result :

THEOREM 4.2.1. (Type A_n) *Let* $x \in \underline{\underline{g}} = \underline{\underline{s\ell}}_{n+1}(\mathbf{C})$ *be nilpotent and let* $\lambda(x) = (\lambda_1, \ldots, \lambda_r)$ *be the corresponding partition of* $n+1$. *Then* $rank(d\pi_x) = \lambda_1 - 1$ *and the exponents of* $(\underline{\underline{g}}, x)$ *are* $1, 2, \ldots, \lambda_1 - 1$.

4.3. Types B_n and C_n. Let $N = 2n+1$ for type B_n and $N = 2n$ for type C_n. We consider $\underline{\underline{g}}$ as a subalgebra of $\underline{\underline{g}}_1 = \underline{\underline{s\ell}}_N(\mathbf{C})$. In both cases there exists an involution θ of $\underline{\underline{g}}_1$ such that $\underline{\underline{g}} = \underline{\underline{g}}_1^{\ \theta}$ is the fixed point subalgebra of $\underline{\underline{g}}_1$. Let $\underline{\underline{t}}_1$ be a θ-stable Cartan subalgebra of $\underline{\underline{g}}$ which includes a Cartan subalgebra $\underline{\underline{t}}$ of $\underline{\underline{g}}$; then $\underline{\underline{t}} = \underline{\underline{t}}_1^{\ \theta}$. Let $W = W(\underline{\underline{g}}, \underline{\underline{t}})$ and $W_1 = W(\underline{\underline{g}}_1, \underline{\underline{t}}_1)$ be the corresponding Weyl groups. Then one can identify W with the normalizer of $\underline{\underline{t}}$ in W_1. Moreover, the restriction homomorphism $\mathbf{C}[\underline{\underline{t}}_1] \to \mathbf{C}[\underline{\underline{t}}]$ maps $\mathbf{C}[t_1]^{W_1}$ surjectively onto $\mathbf{C}[\underline{\underline{t}}]^W$. (This is just a lucky coincidence.) Thus the inclusion $\underline{\underline{t}} \hookrightarrow \underline{\underline{t}}_1$ induces a morphism of orbit spaces $\underline{\underline{t}}/W \to \underline{\underline{t}}_1/W_1$, which identifies $\underline{\underline{t}}/W$ with a closed subvariety of $\underline{\underline{t}}_1/W_1$. We obtain a commutative diagram

(4.3.1)

$$\begin{array}{ccccccc}
\underline{\underline{t}} & \hookrightarrow & \underline{\underline{g}} & \hookrightarrow & \underline{\underline{g}}_1 & \hookleftarrow & \underline{\underline{t}}_1 \\
\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} & & \downarrow \\
\underline{\underline{t}}/W & \approx & \mathbf{C}^n & \xrightarrow{\gamma} & \mathbf{C}^N & \approx & t_1/W_1
\end{array}$$

The action of θ on $\underline{\underline{t}}_1$ induces an action of θ on $\underline{\underline{t}}_1/W_1 \approx \mathbf{C}^N$. If we identify $\underline{\underline{t}}$ with $\mathbf{C}^n$ in the appropriate manner, then $\mathbf{C}[\underline{\underline{t}}]^W$ is freely generated by $P_j(y_1^2, \ldots, y_n^2)$, $j = 1, \ldots, n$, where $y_1, \ldots, y_n$ are coordinates on $\mathbf{C}^n$ and the P_j's are the elementary symmetric functions. This allows us to give an explicit description of the morphism γ in 4.3.1. Using this description we can prove :

PROPOSITION 4.3.2. *Let* $x \in \underline{\underline{g}}$ *be nilpotent.*
Then $d\gamma_0(d\pi_x(\underline{\underline{g}})) = ((d\pi_1)_x\ (\underline{\underline{g}}_1))^\theta$. *Moreover* $d\gamma_0$ *is an injection.*

Proposition 4.3.2 allows us to use the explicit description of $(d\pi_1)_x$ to give precise results on image $(d\pi_x)$ for *every* nilpotent element $x \in \underline{\underline{g}}$. We obtain the following result :

THEOREM 4.3.3. (*Types* B_n *and* C_n). *Let* $x \in \underline{\underline{g}}$ *be nilpotent and let* $\lambda(x) = (\lambda_1, \ldots, \lambda_r)$ *be the corresponding partition of* N. *Then* rank$(d\pi_x)$ *is equal to* $[\lambda_1/2]$, *the integral part of* $\lambda_1/2$, *and the exponents of* $(\underline{\underline{g}}, x)$ *are* $1, 3, \ldots, 2[\lambda_1/2] - 1$.

4.4. *Type D_n.* Here $\underline{\underline{g}} = \underline{\underline{o}}_{2n}(\mathbf{C})$. We consider $\underline{\underline{g}}$ as a subalgebra of $\underline{\underline{g}}_1 = \underline{\underline{s\ell}}_{2n}(\mathbf{C})$. We have $\underline{\underline{g}} = \underline{\underline{g}}_1^{\theta}$, where θ is an involution of $\underline{\underline{g}}_1$. We argue as in 4.3. In this case, however, there is an "extra" polynomial invariant, the Pfaffian, which is not the restriction to $\underline{\underline{g}}$ of an element of $I(\underline{\underline{g}}_1)$. We need the following result :

LEMMA 4.4.1. *Let $x \in \underline{\underline{g}}$ be nilpotent. Then x is a critical point of the Pfaffian if and only if $\ell(\lambda(x)) > 2$.*

This result, and computations similar to those of 4.3, gives the following :

THEOREM 4.4.2. (Type D_n). *Let $x \in \underline{\underline{g}}$ and let $\lambda(x) = (\lambda_1, \ldots, \lambda_r)$.* (a) *If $\ell(\lambda(x)) > 2$, then* $\operatorname{rank}(d\pi_x) = [\lambda_1/2]$ *and the exponents of $(\underline{\underline{g}}\,, x)$ are $1, 3, \ldots, 2[\lambda_1/2]-1$.* (b) *Assume $\lambda(x) = (2n-i, i)$ with i odd. Then* $\operatorname{rank}(d\pi_x) = (2n-i+1)/2$ *and the exponents of x are $1, 3, \ldots, 2n-i-2$ and $n-1$.* (c) *Assume $n = 2m$ is even and $\lambda(x) = (n, n)$. Then* $\operatorname{rank}(d\pi_x) = m$ *and the exponents of $(\underline{\underline{g}}\,, x)$ are $1, 3, \ldots, 2m-1$.*

5. TWO SPECIAL RESULTS

We continue with the notation of Section 2. We recall that a nilpotent element $x \in \underline{\underline{g}}$ is *subregular* if $\dim G_x = n+2$. (Here $n = \operatorname{rank}(G)$.) The subregular nilpotent elements of $\underline{\underline{g}}$ form a single conjugacy class which is dense in the set of non-regular nilpotent elements.

LEMMA 5.1. *Let x be a subregular nilpotent element of $\underline{\underline{g}}$. Then* $\operatorname{rank}(d\pi_x) = n-1$ *and the exponents of $(\underline{\underline{g}}, x)$ are $m_1, \ldots, m_{n-1}$.*

The proof that $\operatorname{rank}(d\pi_x) = n-1$ is given in [13]. Using this, an easy argument involving Proposition 2.2 shows that the exponents of $(\underline{\underline{g}}, x)$ are $m_1, \ldots, m_{n-1}$.

PROPOSITION 5.2. *Let $\underline{\underline{s}}$ be a regular semisimple subalgebra of $\underline{\underline{g}}$ and let x be a regular nilpotent element of $\underline{\underline{s}}$. Assume that m_j is an exponent of $\underline{\underline{s}}$ and that d_j does not divide d_i for $i \neq j$. Then m_j is an exponent of $(\underline{\underline{g}}, x)$.*

It follows from Section 3 that Proposition 5.3 is equivalent to a result on Weyl group invariants. Our proof of the corresponding result on Weyl group invariants was suggested by T. A. Springer and uses his results [15] on regular eigenvectors of Weyl groups.

6. EXPLICIT RESULTS FOR THE EXCEPTIONAL LIE ALGEBRAS

PROPOSITION 6.1. *Let $\underline{\underline{g}}$ be an exceptional simple Lie algebra and let $\underline{\underline{s}}$ be a regular semisimple subalgebra of $\underline{\underline{g}}$. If m is an exponent of $\underline{\underline{g}}$ which is also an exponent of $\underline{\underline{s}}$, then m is an exponent of $\underline{\underline{g}}, x)$.*

Example 6.2. Let $\underline{\underline{g}}$ be of type E_8, let $\underline{\underline{s}}$ be a regular semisimple subalgebra of $\underline{\underline{g}}$ of type E_7 and let x be a regular nilpotent element of $\underline{\underline{s}}$. The exponents of E_8 which are also exponents of E_7 are 1, 7, 11, 13 and 17. It follows from Proposition 6.1 that the exponents of $(\underline{\underline{g}}, x)$ are 1, 7, 11, 13 and 17 and that $\mathrm{rank}(d\pi_x) = 5$.

Discussion of proof of Proposition 6.1. If $x \neq 0$, then 1 is an exponent of $(\underline{\underline{g}}, x)$ by Lemma 2.3. For most other exponents, the proof follows from Proposition 5.2. The exponents which cause trouble are as follows : (a) Types F_4, E_6 and E_7, $m = 6$; (b) Type E_8, $m = 8, 12$. For all cases except type E_7, $m = 6$, one can get the result by a refinement of the proof of Proposition 5.2, using the detailed information on regular eigenvectors of Weyl groups which is contained in Springer's paper [15]. The case of type E_7, $m = 6$, requires a special argument.

If $\underline{\underline{g}}$ is an exceptional simple Lie algebra and if $x \in \underline{\underline{g}}$ is a regular nilpotent element of a regular semisimple subalgebra $\underline{\underline{s}}$ of $\underline{\underline{g}}$, then Proposition 6.1 gives precise information on the exponents of $(\underline{\underline{g}}, x)$. In order to obtain the exponents for the other nilpotent elements, we need to use the order relation on nilpotent orbits given by the orbit closures. If C_1 and C_2 are nilpotent orbits in $\underline{\underline{g}}$, then we say that $C_1 < C_2$ if $C_1 \subset \overline{C}_2$. One has exact information on this order relation between nilpotent orbits. See, for example the tables in Carter's book [5], pp. 433 - 446.

The following results are elementary :

6.2. *If* $x \in C_1$, $y \in C_2$ *and* $C_1 < C_2$, *then the exponents of* $(\underline{\underline{g}}, x)$ *are a subsequence of the exponents of* $(\underline{\underline{g}}, y)$.

6.3. *Let* r *be the number of nodes with non-zero weight in the weighted Dynkin diagram associated to a nilpotent element* $x \in \underline{\underline{g}}$. *Then* $\mathrm{rank}(d\pi_x) \leq r$.

6.4. *Let* $\underline{\underline{s}}$ *be a regular semisimple subalgebra of* $\underline{\underline{g}}$, *let* x *be a nilpotent element of* $\underline{\underline{s}}$ *and let* m *be an exponent of* $\underline{\underline{s}}, x)$. [*We assume* $\underline{\underline{g}}$ *is not of type* D_{2r}. *If* $\underline{\underline{s}}$ *has a direct factor of type* D_{2r}, *then we assume that* $m \neq 2k - 1$]. *If* m *is also an exponent of* $\underline{\underline{g}}$, *then* m *is an exponent of* $(\underline{\underline{g}}, x)$.

Let $\underline{\underline{g}}$ be an exceptional Lie algebra. By using the results 5.1 and 6.1 - 6.4, one can get precise information on the exponents for every nilpotent orbit in $\underline{\underline{g}}$ with the following exception : $\underline{\underline{g}}$ of type E_8, C is the class $E_8(a_2)$ (notation as in [5]) and $m = 19$. In this case we expect that 19 is an exponent of C, but have not been able to prove it.

A list of the exponents of the nilpotent orbits in the exceptional simple Lie algebras is given in the Appendix to this paper.

7. NON-NORMAL ORBIT CLOSURES IN EXCEPTIONAL LIE ALGEBRAS.

The problem of whether the closure of a nilpotent orbit in $\underline{\underline{g}}$ is a normal variety is of interest in representation theory (see, e.g. [2], Thm. 5.6). For the classical Lie algebras, Kraft and Procesi [10] have obtained detailed, although not quite complete, information on this problem. For the exceptional Lie algebras, some results can be obtained from the (very difficult) calculations of Benyon and Spaltenstein [1] on "Green's functions" for the corresponding finite Chevalley groups, but the general problem is still open. We indicate below how the calculations of the previous sections lead to a number of new examples in the exceptional algebras of nilpotent orbits whose closures are not normal varieties.

For each integer $j > 0$, let $\underline{g}^{(j)} = \{x \in \underline{g} \mid \dim G \cdot x = j\}$. Each $\underline{g}^{(j)}$ is a locally closed subvariety of $\underline{g}$. A *sheet* in $\underline{g}$ is an irreducible component of some $\underline{g}^{(j)}$; a sheet S is a *Dixmier sheet* if S contains a semisimple orbit. It is known that each sheet of $\underline{g}$ contains a unique nilpotent orbit.

The following two results are due to Borho and Kraft [3] :

7.1. *If S is a sheet in $\underline{g}$, then there exists a parabolic subgroup P of G and a solvable ideal $\underline{r}$ of $\underline{p} = \mathrm{Lie(P)}$ such that $S = \mathrm{G} \cdot \underline{r}^{reg}$.*

See [3] for the definition of $\underline{r}^{reg}$.

7.2. *Let S, P, $\underline{r}$ be as in 7.1 and let $G \cdot x$ denote the unique nilpotent orbit in S. Assume that $G_x = P_x$ and that the orbit closure $\overline{G \cdot x}$ is a normal variety. Then for each simple G-module V, the function $y \mapsto \mathrm{mult}_V(\mathbb{C}[\overline{G \cdot y}])$ is constant on the sheet S. Moreover, for every $y \in S$, the orbit closure $\overline{G \cdot y}$ is a normal variety.*

Here $mult_V(\mathbb{C}[\overline{G \cdot y}])$ is the multiplicity of V in the G-module $\mathbb{C}[\overline{G \cdot y}]$.

As an immediate consequence of 7.2, we have :

7.3. *Let $\underline{p}$ be a parabolic subalgebra of $\underline{g}$, let $\underline{h}$ be a Levi subalgebra of $\underline{p}$, let $\underline{s} = [\underline{h}\ ,\underline{h}]$ and let $\underline{z}$ denote the centre of $\underline{h}$. Let S be the unique connected algebraic subgroup of G such that $\mathrm{Lie}(S) = \underline{s}$ and let $C = S \cdot u$ be a nilpotent orbit in $\underline{s}$. Let $C_1 = \mathrm{Ind}_{\underline{h}}^{\underline{g}} C$ be the induced nilpotent orbit in $\underline{g}$ and let $v \in C_1$. Let $\pi_1 : \underline{s} \to \mathbb{C}^r$ denote the quotient morphism for $\underline{s}$. Assume that the stabilizer G_v is connected.* (1) *If*

$$\text{(7.3.1)} \quad \dim \underline{z} + \mathrm{rank}(d\pi_1)_u > \mathrm{rank}(d\pi_v),$$

then $\overline{G \cdot v}$ is not a normal variety. (2) *If $\overline{S \cdot u}$ is not a normal variety, then $\overline{G \cdot v}$ is not a normal variety.*

Proof. (1) Let $h \in \underline{z}^{reg}$. Then the left hand side of (7.3.1) is equal to

$$\mathrm{rank}(d\pi)_{h+u} = \mathrm{mult}_{\underline{g}}(\overline{G \cdot (h+u)})$$

and the right hand side equals $\mathrm{mult}_{\underline{g}}(\overline{G \cdot v})$. Thus (1) follows from 7.2. (2) It is known that $\overline{G \cdot (h+u)}$ is a normal variety if and only if $\overline{S \cdot u}$ is normal. Thus (2) follows from 7.2.

Example. Let C denote the class in (the Lie algebra of type) B_3 corresponding to the partition $(3,2,2)$. By the results of Kraft-Procesi [10], the orbit closure $\overline{C}$ is not normal. Let $C' = \mathrm{Ind}_{B_3}^{F_4}(C)$. Then C' is the class in F_4 denoted by C_3. If $v \in C'$, then the stabilizer of v in G (the adjoint group of type F_4) is connected. Thus, by 7.3.(2), $\overline{C}\,'$ is not a normal variety.

There is a recipe which assigns to each nilpotent element v of $\underline{\underline{g}}$ a weighted Dynkin diagram (abbreviated W.D.D.); this is a function which assigns to each node a of the Dynkin diagram of $\underline{\underline{g}}$ an integer $n(a)$, the *weight* of a; the weights $n(a)$ are either 0, 1, or 2. The conjugacy class $G \cdot v$ is uniquely determined by the W.D.D. If all weights of the W.D.D. of v are even, then v is an *even nilpotent element* and $G \cdot v$ is an *even nilpotent class.*

Let $C = G\cdot v$ be an even nilpotent class of $\underline{\underline{g}}$. Let P denote the "standard parabolic subgroup" associated to the set of nodes of the W.D.D. of C which have weight zero and let $\underline{\underline{p}} = \mathrm{Lie(P)}$. Let $\underline{\underline{r}}$ resp $\underline{\underline{v}}$ be the solvable (resp nilpotent) radical of $\underline{\underline{p}}$. Then $\mathcal{S} = G \cdot \underline{\underline{r}}^{reg}$ is a sheet of $\underline{\underline{g}}$ (a Dixmier sheet) and $C = G \cdot \underline{\underline{v}}^{reg}$ is the unique nilpotent orbit of $\mathcal{S}$. In particular we may assume that $v \in \underline{\underline{v}}$.

7.4. *Let $C = G \cdot v$, $\underline{\underline{p}}$, P and $\underline{\underline{v}}$ be as above and assume that $v \in \underline{\underline{v}}$. Then $P_v = G_v$.*

This follows from results of Hesselink and Kraft (see [7], Thm. 11.3 and [8], Thm. 4.7).

7.5. *Let $C = G\cdot v$ be an even nilpotent class in $\underline{\underline{g}}$ and let d denote the number of $2's$ in the weighted Dynkin diagram of C. Then* $\mathrm{rank}(d\pi_v) \leqq d$ *and, if* $\mathrm{rank}(d\pi_v) < d$, *then the orbit closure $\overline{C}$ is not a normal variety.*

Proof. Let $\mathcal{S}$, $\underline{\underline{r}}$, $\underline{\underline{p}}$ be as in 7.3. If $h \in \underline{\underline{r}}^{reg}$ is semisimple, then $\mathrm{rank}(d\pi_h) = d$. Thus 7.4 follows from 7.2.

Remark 7.5. In 7.4, if G_v is connected, then the result follows from 7.2.(1).

Example 7.6. Let $\underline{\underline{g}}$ be of type E_8 and let $C = G \cdot v$ be the nilpotent orbit in $\underline{\underline{g}}$ denoted by $D_7(a_1)$. Then C is an even nilpotent class and we see from Carter's tables, [5], p. 407, that the W.D.D. has three $2's$. From the Appendix, we see that $\mathrm{rank}(d\pi_v) = 2$. Thus $\overline{C}$ is not normal. We note from the tables in [5] that G_v is not connected.

Using the results 7.2 and 7.4, one can obtain a number of examples of nilpotent orbit closures which are not normal in the exceptional Lie algebras of types E_6, E_7, E_8. It is a matter of checking our tables in the Appendix against the tables of Carter [5], pp. 402 - 407, and also the tables of Spaltenstein [14], pp. 174 - 175 on induced orbits.

In the appendix we have indicated the non-normal nilpotent orbit closures which we have detected in this way.

8. CONCLUDING REMARKS

The two results below can be checked case by case from our calculations :

8.1. *Let* $\underline{\underline{g}}$ *be a simple Lie algebra, let* $\underline{\underline{h}}$ *be a reductive subalgebra of* $\underline{\underline{g}}$ *of maximal rank and let* $\underline{\underline{s}} = [\underline{\underline{h}}, \underline{\underline{h}}]$. *Let* v *be a regular nilpotent element of* $\underline{\underline{s}}$. *For each integer* $m > 0$, *let* a_m (resp. b_m) *be the multiplicity of* m *as an exponent of the semisimple Lie algebra* $\underline{\underline{g}}$ (resp. $\underline{\underline{s}}$). *Then the multiplicity of* m *as an exponent of* $(\underline{\underline{g}}, v)$ *is equal to* $\min(a_m, b_m)$.

8.2. *Let* v *be a distinguished nilpotent element of the simple Lie algebra* $\underline{\underline{g}}$ *and let* d *denote the number of* $2'$*s in the* W.D.D. *of* $\underline{\underline{g}}$. *Assume* $G \cdot v$ *is not the class* $E_8(a_z)$. *Then* $\text{rank}(d\pi_v) = d$.

See [3] for the definition of distinguished nilpotent elements.

We have not been able to give direct proofs of either 8.1 or 8.2. If one could strengthen 8.2 by giving a procedure for getting the exponents of a distinguished nilpotent element, then 8.1 and 8.2 together would give an easy algorithm for reading off the exponents of an arbitrary nilpotent element of $\underline{\underline{g}}$ in terms of the Carter-Bala classification of nilpotent elements.

Our computations also give the following result :

8.3. *Let* $\underline{\underline{g}}$ *be a simple Lie algebra of type* A_n, B_n *or* C_n *and let* $\mathcal{S}$ *be a Dixmier sheet in* $\underline{\underline{g}}$. *Then the function*

$$x \mapsto rank(d\pi_x)$$

is constant on $\mathcal{S}$.

Let $\underline{\underline{g}}$ be a simple Lie algebra and let C be a nilpotent orbit in $\underline{\underline{g}}$. Let $\eta : \widetilde{\overline{C}} \to \overline{C}$ denote the normalization of $\overline{C}$. Then one can detect those orbits $G \cdot a$ in $\overline{C}$ for which $\#\eta^{-1}(a) > 1$ by means of results on Green's functions (See [1], p. 595). This allows one to detect all nilpotent orbits C such that η is not a bijection (assuming one can calculate the appropriate Green's functions!); in these cases, the non-normality of $\overline{C}$ is due to "branching". For $\underline{\underline{g}}$ of type E_6, it is shown in [1] that there are precisely three such orbits. However our results give seven nilpotent orbits with non-normal orbit closures. For the four new non-normal orbit closures in type E_6, the normalization map η is a bijection. For the classical groups this does not happen (except perhaps for a few special cases which Kraft-Procesi cannot handle). See [10], p. 543, Thm. 1.

APPENDIX. *Exponents and non-normal orbit closures for nilpotent orbits in exceptional simple Lie algebras.*

We list below the exponents for each nilpotent orbit in the exceptional simple Lie algebras. We follow the notation of Carter's book [5] for the nilpotent classes. An entry of the form A_5 - 1, 4, 5 ; under type E_6 means that if $\underline{\underline{g}}$ is of type E_6 and $x \in \underline{\underline{g}}$ is nilpotent, with the orbit of x denoted by A_5, then the exponents of $(\underline{\underline{g}}, x)$ are 1, 4, 5.

We also list those nilpotent orbits, *for which we can prove by our methods that the orbit closures are not normal.* Presumably, there are other non-normal nilpotent orbit closures.

1. Type E_6. *Exponents.*

E_6 - 1, 4, 5, 7, 8, 11 ; $E_6(a_1)$ - 1, 4, 5, 7, 8 ; D_5 - 1, 4, 5, 7 ; $E_6(a_3)$ - 1, 4, 5; $D_5(a_1)$ - 1, 4, 5 ; A_5 - 1, 4, 5, ; $A_4 + A_1$ - 1, 4 ; D_4 - 1, 5 ; A_4 - 1, 4 ;
all other (non-zero) nilpotent orbits - 1.

Non-normal nilpotent orbit closures: $A_4 + A_1$, A_4, $A_3 + A_1$, A_3, $A_2 + 2A_1$, $2A_2$, $A_2 + A_1$.

2. Type E_7. *Exponents.*

E_7 - 1, 5, 7, 9, 11, 13, 17 ; $E_7(a_1)$ - 1, 5, 7, 9, 11, 13 ; $E_7(a_2)$ - 1, 5, 7, 9, 11 ; $E_7(a_3)$ - 1, 5, 7, 9 ; E_6 - 1, 5, 7, 11 ; D_6 - 1, 5, 7, 9 ; $E_7(a_4)$ - 1, 5, 7 ; $D_6(a_1)$ - 1, 5, 7 ; $E_6(a_1)$ - 1, 5, 7 ; $D_5 + A_1$ - 1, 5, 7 ; A_6 - 1, 5 ; $E_7(a_5)$ - 1, 5 ;
D_5 - 1, 5, 7 ; $E_6(a_3)$ - 1, 5 ; $D_6(a_2)$ - 1, 5 ; $D_5(a_1) + A_1$ - 1, 5 ; $A_5 + A_1$ - 1, 5 ;
$(A_5)'$ - 1, 5 ; $D_5(a_1)$ - 1, 5 ; $D_4 + A_1$ - 1, 5 ; $(A_5)''$ - 1, 5 ; D_4 - 1, 5 ;
all other nilpotent orbits - 1.

Non-normal nilpotent orbit closures : $D_6(a_1)$, $D_6(a_2)$, $(A_5)''$, A_4, $A_3{+}2A_1$, $A_3{+}A_1$, A_3.

Type E_8. *Exponents*

E_8 - 1, 7, 11, 13, 17, 19, 23, 27 ; $E_8(a_1)$ - 1, 7, 9, 11, 13, 17, 19, 23 ; $E_8(a_2)$ - 1, 7, 11, 13, 17, (19?) ; $E_8(a_3)$ - 1, 7, 11, 13, 17 ; $E_8(a_4)$ - 1, 7, 11, 13 ; E_7 - 1, 7, 11, 13, 17 ; $E_8(b_4)$ - 1, 7, 11, 13 ; $E_8(a_5)$ - 1, 7, 11 ; $E_7(a_1)$ - 1, 7, 11, 13 ; $E_8(b_5)$ - 1, 7, 11 ; D_7 - 1, 7, 11 ; $E_8(a_6)$ - 1, 7 ; $E_7(a_2)$ - 1, 7, 11 ; $E_6 + A_1$ - 1, 7, 11 ;
the following classes have exponents 1 and 7 : $E_8(a_6)$, $D_7(a_1)$, $E_8(b_6)$, $E_7(a_3)$, $E_6(a_1) + A_1$, A_7, $D_7(a_2)$, D_6, $D_5 + A_2$, $E_6(a_1)$, $E_7(a_4)$, $D_6(a_1)$, $D_5 + A_1$, D_5 ;
all other nilpotent orbits - 1.

Non-normal nilpotent orbit closures : $E_7(a_1)$, $E_7(a_2)$, $D_7(a_1)$, E_6, D_6, $E_6(a_1)$, A_6, D_5+A_1, D_5, $E_6(a_3)$, D_4+A_2, $D_5(a_1)+A_1$, A_5, D_4+A_1, A_4, D_4, A_3.

4. Type F_4. *Exponents*

F_4 - 1, 5, 7, 11 ; $F_4(a_1)$ - 1, 5, 7 ; $F_4(a_2)$ - 1, 5 ; B_3 - 1, 5 ; C_3 - 1, 5 ; all other nilpotent orbits - 1.

Non-normal nilpotent orbit closures : C_3.

5. Type G_2. *Exponents*

G_2 - 1, 5 ; all other nilpotent orbits - 1.

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